PROBABILITY AND STATISTICAL INFERENCE

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PROBABILITY

Chapter 1

1.1 PROPERTIES OF PROBABILITY

Exercises

1.1-1. Of a group of patients having injuries, 28% visit both a physical therapist and a chiropractor and 8% visit neither. Say that the probability of visiting a physical therapist exceeds the probability of visiting a chiropractor by 16%. What is the probability of a randomly selected person from this group visiting a physical therapist?

1.1-2. An insurance company looks at its auto insurance customers and finds that (a) all insure at least one car, (b) 85% insure more than one car, (c) 23% insure a sports car, and (d) 17% insure more than one car, including a sports car. Find the probability that a customer selected at random insures exactly one car and it is not a sports car.

1.1-3. Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns 1/52 to each of the 52 outcomes. Let

$$A = \{x\colon x \text{ is a jack, queen, or king}\},$$

$$B = \{x\colon x \text{ is a 9, 10, or jack and } x \text{ is red}\},$$

$$C = \{x\colon x \text{ is a club}\},$$

$$D = \{x\colon x \text{ is a diamond, a heart, or a spade}\}.$$

Find **(a)** $P(A)$, **(b)** $P(A \cap B)$, **(c)** $P(A \cup B)$, **(d)** $P(C \cup D)$, and **(e)** $P(C \cap D)$.

1.1-4. A fair coin is tossed four times, and the sequence of heads and tails is observed.

(a) List each of the 16 sequences in the sample space S.

(b) Let events A, B, C, and D be given by $A = \{\text{at least 3 heads}\}$, $B = \{\text{at most 2 heads}\}$, $C = \{\text{heads on the third toss}\}$, and $D = \{\text{1 head and 3 tails}\}$. If the probability set function assigns 1/16 to each outcome in the sample space, find **(i)** $P(A)$, **(ii)** $P(A \cap B)$, **(iii)** $P(B)$, **(iv)** $P(A \cap C)$, **(v)** $P(D)$, **(vi)** $P(A \cup C)$, and **(vii)** $P(B \cap D)$.

1.1-5. Consider the trial on which a 3 is first observed in successive rolls of a six-sided die. Let A be the event that 3 is observed on the first trial. Let B be the event that at least two trials are required to observe a 3. Assuming that each side has probability 1/6, find **(a)** $P(A)$, **(b)** $P(B)$, and **(c)** $P(A \cup B)$.

1.1-6. If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, find **(a)** $P(A \cup B)$, **(b)** $P(A \cap B')$, and **(c)** $P(A' \cup B')$.

1.1-7. Given that $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$.

1.1-8. During a visit to a primary care physician's office, the probability of having neither lab work nor referral to a specialist is 0.21. Of those coming to that office, the probability of having lab work is 0.41 and the probability of having a referral is 0.53. What is the probability of having both lab work and a referral?

1.1-9. Roll a fair six-sided die three times. Let $A_1 = \{1 \text{ or } 2 \text{ on the first roll}\}$, $A_2 = \{3 \text{ or } 4 \text{ on the second roll}\}$, and $A_3 = \{5 \text{ or } 6 \text{ on the third roll}\}$. It is given that $P(A_i) = 1/3$, $i = 1, 2, 3$; $P(A_i \cap A_j) = (1/3)^2$, $i \neq j$; and $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.

(a) Use Theorem 1.1-6 to find $P(A_1 \cup A_2 \cup A_3)$.

(b) Show that $P(A_1 \cup A_2 \cup A_3) = 1 - (1 - 1/3)^3$.

1.1-10. Prove Theorem 1.1-6.

1.1-11. A typical roulette wheel used in a casino has 38 slots that are numbered $1, 2, 3, \ldots, 36, 0, 00$, respectively. The 0 and 00 slots are colored green. Half of the remaining slots are red and half are black. Also, half of the integers between 1 and 36 inclusive are odd, half are even, and 0 and 00 are defined to be neither odd nor even. A ball is rolled around the wheel and ends up in one of the slots; we assume that each slot has equal probability of 1/38, and we are interested in the number of the slot into which the ball falls.

(a) Define the sample space S.

(b) Let $A = \{0, 00\}$. Give the value of $P(A)$.

(c) Let $B = \{14, 15, 17, 18\}$. Give the value of $P(B)$.

(d) Let $D = \{x : x \text{ is odd}\}$. Give the value of $P(D)$.

1.1-12. Let x equal a number that is selected randomly from the closed interval from zero to one, $[0, 1]$. Use your intuition to assign values to

(a) $P(\{x\colon 0 \le x \le 1/3\})$.

(b) $P(\{x\colon 1/3 \le x \le 1\})$.

(c) $P(\{x\colon x = 1/3\})$.

(d) $P(\{x\colon 1/2 < x < 5\})$.

1.1-13. Divide a line segment into two parts by selecting a point at random. Use your intuition to assign a probability to the event that the longer segment is at least two times longer than the shorter segment.

1.1-14. Let the interval $[-r, r]$ be the base of a semicircle. If a point is selected at random from this interval, assign a probability to the event that the length of the perpendicular segment from the point to the semicircle is less than $r/2$.

1.1-15. Let $S = A_1 \cup A_2 \cup \cdots \cup A_m$, where events $A_1, A_2, \ldots, A_m$ are mutually exclusive and exhaustive.

(a) If $P(A_1) = P(A_2) = \cdots = P(A_m)$, show that $P(A_i) = 1/m$, $i = 1, 2, \ldots, m$.

(b) If $A = A_1 \cup A_2 \cup \cdots \cup A_h$, where $h < m$, and (a) holds, prove that $P(A) = h/m$.

1.1-16. Let p_n, $n = 0, 1, 2, \ldots$, be the probability that an automobile policyholder will file for n claims in a five-year period. The actuary involved makes the assumption that $p_{n+1} = (1/4)p_n$. What is the probability that the holder will file two or more claims during this period?

1.1 Properties of Probability

1.1-2 Sketch a figure and fill in the probabilities of each of the disjoint sets.

Let $A = \{\text{insure more than one car}\}$, $P(A) = 0.85$.

Let $B = \{\text{insure a sports car}\}$, $P(B) = 0.23$.

Let $C = \{\text{insure exactly one car}\}$, $P(C) = 0.15$.

It is also given that $P(A \cap B) = 0.17$. Since $A \cap C = \phi$, $P(A \cap C) = 0$. It follows that $P(A \cap B \cap C') = 0.17$. Thus $P(A' \cap B \cap C') = 0.06$ and $P(A' \cap B' \cap C) = 0.09$.

1.1-4 **(a)** $S = \{$HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, TTHH, HTHT, THTH, THHT, HTTT, THTT, TTHT, TTTH, TTTT$\}$;

(b) **(i)** $5/16$, **(ii)** 0, **(iii)** $11/16$, **(iv)** $4/16$, **(v)** $4/16$, **(vi)** $9/16$, **(vii)** $4/16$.

1.1-6 **(a)** $P(A \cup B) = 0.4 + 0.5 - 0.3 = 0.6$;

(b)
$$\begin{aligned} A &= (A \cap B') \cup (A \cap B) \\ P(A) &= P(A \cap B') + P(A \cap B) \\ 0.4 &= P(A \cap B') + 0.3 \\ P(A \cap B') &= 0.1; \end{aligned}$$

(c) $P(A' \cup B') = P[(A \cap B)'] = 1 - P(A \cap B) = 1 - 0.3 = 0.7$.

1.1-8 Let $A =$\{lab work done\}, $B =$\{referral to a specialist\},

$P(A) = 0.41$, $P(B) = 0.53$, $P([A \cup B]') = 0.21$.
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.79 &= 0.41 + 0.53 - P(A \cap B) \\ P(A \cap B) &= 0.41 + 0.53 - 0.79 \ = \ 0.15. \end{aligned}$$

1.1-10
$$\begin{aligned} A \cup B \cup C &= A \cup (B \cup C) \\ P(A \cup B \cup C) &= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup (A \cap C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) \\ &\quad + P(A \cap B \cap C). \end{aligned}$$

1.1-12 **(a)** $1/3$; **(b)** $2/3$; **(c)** 0; **(d)** $1/2$.

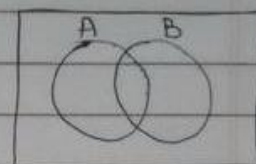
1.1-14 $P(A) = \dfrac{2[r - r(\sqrt{3}/2)]}{2r} = 1 - \dfrac{\sqrt{3}}{2}.$

1.1-16 Note that the respective probabilities are $p_0,\ p_1 = p_0/4,\ p_2 = p_0/4^2, \cdots$.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{p_0}{4^k} &= 1 \\ \frac{p_0}{1 - 1/4} &= 1 \\ p_0 &= \frac{3}{4} \end{aligned}$$

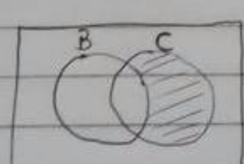
$$1 - p_0 - p_1 = 1 - \frac{15}{16} = \frac{1}{16}.$$

1. Let, A = visit physical therapist
B = visit chiropractor



$P(A \cap B) = 0.28$
$P(\text{neither}) = 1 - P(A \cup B) = 0.08$
$P(A) = P(B) + 0.16$
Now, we know that $P(A \cup B) = -P(A \cap B) + P(A) + P(B)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$0.92 = 2P(B) + 0.16 - 0.28$
$P(B) = 0.52 \quad P(A) = 0.68$
68% visit physical therapist (Ans)

2. A = at least one car is insured
B = more than one car is insured
C = a sport car is insured
$P(A) = 1$, $P(B) = 0.85$ $P(C) = 0.23$ $P(B \cap C) = 0.17$
P(insure exactly one car and that is not sport car) = $P(B \cup C) - P(B)$
$= P(B) + P(C) - P(B \cap C) - P(B)$
$= P(C) - P(B \cap C)$
$= 0.06$ (Ans)



It seems this solution has found the probability that only one, sports car is insured. The question is asking for the probability of only one, non-sports car being insured.

I am not certain that this is the case. However, here is my solution:

Let X be the event that more than one car is insured. Let Y be the event that a sports car is insured.

$$P(X)=0.85 \Longrightarrow P(X^c)=0.15$$
$P(X)=0.85 \Longrightarrow P(X^c)=0.15$

$$P(Y)=0.23\ ,\ P(Y\cap X)=0.17$$
$P(Y)=0.23\ ,\ P(Y\cap X)=0.17$

$$\Longrightarrow P(Y\cap X^c)=0.06$$
$\Longrightarrow P(Y\cap X^c)=0.06$

$$P(Y^c\cap X^c)=P(X^c)-P(Y\cap X^c)=0.15-0.06=0.09$$

3. a) $P(A) = \frac{\text{Possible Favourable Outcome}}{\text{Total outcome}} = \frac{12}{52} = 0.230$

b) $P(A \cap B) = \frac{2}{52} = 0.038$ (Fav. outcomes are two red jack (heart and diamond))

c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{12}{52} + \frac{6}{52} - \frac{2}{52} = \frac{16}{52} = 0.3077$

d) $P(C \cup D) = 1$ (cover all cards)

e) $P(C \cap D) = 0$ (no cards among them is common

4 a) Sample space S = { HHHH, HHHT, HHTH, HHTT, HTHH, HTTH, HTTT, HTHT, THHH, TTHH, THTH, THHT, TTTH, TTHT, THTT, TTTT }

b) $P(A) = \frac{\text{Fav. outcome}}{\text{Total outcome}} = \frac{5}{16} = 0.3125$

$P(B) = P(\text{atmost 2 Heads}) = \frac{11}{16} = 0.6875$

$P(C) = \frac{\text{Fav outcome}}{\text{Totle outcome}} = \frac{2\times2\times2\times2}{16} = 0.5$

$P(D) = \frac{4}{16} = 0.25$

i) $P(A) = 0.3125$

ii) $P(A \cap B) = 0$ (No possible case)

iii) $P(B) = 0.6875$

iv) $P(A \cap C) = \frac{4}{16} = 0.25$ (Possible cases HHHH, HHHT, HTHH,)

v) $P(D) = 0.25$

vi) $P(A \cup C) = P(A) + P(C) - P(A \cap C)$

$= 0.3125 + 0.5 - 0.25 = 0.5625$

vii) $P(B \cap D) = P(D) = 0.25$

1

$P(A) = 0.4$, $P(B) = 0.5$, $P(A \cap B) = 0.3$,

GIVEN

2

$P(AUB) = P(A) + P(B) - P(A \cap B)$

FROM THE FORMULA

3

$P(AUB) = 0.4 + 0.5 - 0.3 = 0.6$

PUTTING THE VALUES

4

$P(A \cap B') = P(A) - P(A \cap B)$

FROM THE FORMULA

5

$P(A \cap B') = 0.4 - 0.3 = 0.1$

PUTTING THE VALUES

6

$P((A \cap B)') = P(A' \cup B')$

HENCE $P(A' \cup B') = 1 - P(A \cap B)$

FROM THE FORMULA

7

$P(A' \cup B') = 1 - 0.3 = 0.7$

PUTTING THE VALUES

RESULT

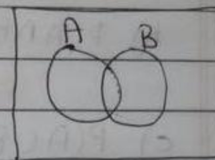
(A) $P(AUB) = 0.6$

(B) $P(A \cap B') = 0.1$

(C) $P(A' \cup B') = 0.7$

This is the wrong problem

6. $P(A) = 0.4 \quad P(B) = 0.5 \quad P(A \cap B) = 0.3$



a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.4 + 0.5 - 0.3 = 0.6$

b) $P(A \cup B') = P(A) + P(B') - P(A \cap B')$

$P(A \cap B') = P(A) - P(A \cap B)$

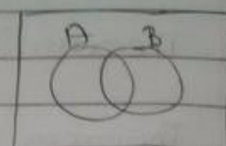
$P(A \cup B') = P(B') + P(A \cap B) = 1 - P(B) + P(A \cap B)$

$= 0.5 + 0.3 = 0.8$

c) $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B)$

$= 1 - 0.3 = 0.7$

7. $P(A \cup B) = 0.76 \quad P(A \cup B') = 0.87$



$P(A) = ?$

$P(A \cup B') = 1 - P(B) + P(A \cap B) = 0.87$

(from qust 6)

$P(B) - P(A \cap B) = 0.13$

$P(A) + P(B) - P(A \cap B) = P(A \cup B) = 0.13 + P(A)$

$P(A) = 0.76 - 0.13 = 0.63$ (As)

8. event A = lab work event B = having referral

$P(A' \cap B') = 0.21$

$P(A) = 0.41 \quad P(B) = 0.53$

$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 0.21$

$P(A \cup B) = 0.79 = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = 0.41 + 0.53 - 0.79$

9. Total possible case = $6 \times 6 \times 6 = 216$

$P(A_i) = \frac{1}{3}$, $i = 1,2,3$

$P(A_i \cap A_j) = \left(\frac{1}{3}\right)^2$ $i \neq j$ $P(A_1 \cap A_2 \cap A_3) = \left(\frac{1}{3}\right)^3$

a) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$

(a/c to Theorem)

$= 3 \times \frac{1}{3} - 3 \times \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{2}{3} + \frac{1}{3^3} = \frac{19}{27}$

$= 0.7037$ (Ans)

b) $P(A_1 \cup A_2 \cup A_3) = 3 \times \frac{1}{3} - 3 \times \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3$

$= 1 - 1 + 3 \times \frac{1}{3} - 3 \times \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3$

$= 1 - \left(1^3 - \left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)\right) = 1 - \left(1 - \frac{1}{3}\right)^3$

Identity: $(a-b)^3 = a^3 - b^3 + 3ab^2 - 3a^2b$

10. Theorem $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Proof: $P(A \cup B \cup C) = P(A \cup (B \cup C))$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (C \cap A))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(C \cap A) + P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$ (H.P.).

11 $0, 00 \rightarrow$ green slots

Half of 36 slots are red and other half is black

a) Sample space $S = \{0, 00, 1, 2, 3, \dots, 36\}$

b) $A = \{0, 00\}$ $P(A) = \frac{\text{Fav. Outcome}}{\text{Total outcome}} = \frac{2}{38} = 0.0526$

c) $P(B) = \frac{4}{38} = 0.10526$

d) $P(D) = \frac{18}{38} = 0.47368$

12 There are infinite number between the interval $[0,1]$. So let make probabilty function a continous which is uniform

~~$f(x) = p = f(x)$~~

$$F(x) = \int_{-\infty}^{x} f(x) \text{, where } f(x) = \begin{cases} P & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x) = \int_{0}^{1} P\,dx \Rightarrow P = 1$$

a) $P(\{x : 0 \le x \le 1/3\}) = \int_{0}^{1/3} f(x)\,dx = \int_{0}^{1/3} dx = \frac{1}{3}$

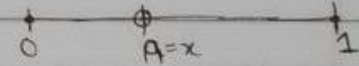
b) $P(\{x : 1/3 \le x \le 1\}) = \int_{1/3}^{1} f(x)\,dx = \int_{1/3}^{1} dx = 2/3$

c) $P(\{x : x = 1/3\}) = 1$, For continous function, it is wrong to give probabilty at definite point, it is given for the interval $f(x)$ is pdf.

d) $P(\{x : 1/2 < x < 5\}) = \int_{1/2}^{5} f(x)\,dx = \int_{1/2}^{1} dx + \int_{1}^{5} 0\,dx$

$= 1/2$ (Ans)

13. Let, consider the line as part of number line from 0 to 1, choosing a point same as choosing a number, so $f(x) = 1$ (from previous quest)



Let, choosen point A, correspond to number x

required cases if $x > 0.5$, $P_1 = P(\{x : x > 2(1-x)\})$

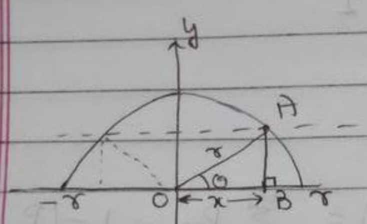
$x < 0.5$ $P_2 = P(\{x : (1-x) > 2x\})$

$P_1 = P(\{x : x > 2/3 \text{ and } x < 1\})$

$P_2 = P(\{x : x < 1/3 \text{ and } x > 0\})$

there $P(\text{required}) = P_1 + P_2 = \int_{2/3}^{1} 1\,dx + \int_{0}^{1/3} 1\,dx = \frac{2}{3}$

14.



$\underline{AB} = r\sin\theta$

for $AB < \frac{r}{2} \Rightarrow r\sin\theta < \frac{r}{2} \Rightarrow \sin\theta < \frac{1}{2}$

$\theta < 30^\circ$, similarly on the otherside of the y-axis.

Therefore, for $|x| > r\cos 30^\circ$, $AB < r/2$

Probability $= \frac{(r - r\cos 30^\circ) \times 2}{2r} = 1 - \frac{\sqrt{3}}{2} = 0.134$ (As)

15 $S = A_1 \cup A_2 \cup \dots \cup A_m$, since events are mutually exclusive. Therefore. $P(A_i \cap A_j) = 0 \quad \forall i \neq j$
and since exhaustive also so $P(S) = P(A_1 \cup A_2 \dots A_m) = 1$

a) $P(A_1 \cup A_2 \dots \cup A_m) = P(A_1) + P(A_2) + \dots P(A_m) = 1$
(bcoz Mutually exclusive)
Let $P(A_1) = P(A_2) = \dots P(A_m) = p$
then $mp = 1 \Rightarrow p = \frac{1}{m}$

b) $P(A) = P(A_1) + P(A_2) + \dots P(A_h)$, $h < m$
$= hp = \frac{h}{m}$ (from above part)

16. $P_{n+1} = \frac{P_n}{4}$, Let $P_0 = P$, $P_1 = \frac{P}{4}$, $P_2 = \frac{P}{16}$ ......

So, it would make an infinite G.P $P_0, P_1, P_2 \dots$
with $a = P$ and $r = \frac{1}{4}$

$$\sum_{n=0}^{\infty} P_n = \frac{a}{1-r} = \frac{P}{1-1/4} = 1$$

$$P = \frac{3}{4}$$

$P\left(\begin{matrix}\text{holder will file 2 or more} \\ \text{claims}\end{matrix}\right) = \sum_{n=2}^{\infty} P_n = 1 - (P_0 + P_1)$

$= 1 - \left(\frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4}\right)$

$= 0.4375 = 7/16$

1.2 METHODS OF ENUMERATION

Exercises

1.2-1. A boy found a bicycle lock for which the combination was unknown. The correct combination is a four-digit number, $d_1d_2d_3d_4$, where d_i, $i = 1, 2, 3, 4$, is selected from 1, 2, 3, 4, 5, 6, 7, and 8. How many different lock combinations are possible with such a lock?

1.2-2. In designing an experiment, the researcher can often choose many different levels of the various factors in order to try to find the best combination at which to operate. As an illustration, suppose the researcher is studying a certain chemical reaction and can choose four levels of temperature, five different pressures, and two different catalysts.

(a) To consider all possible combinations, how many experiments would need to be conducted?

(b) Often in preliminary experimentation, each factor is restricted to two levels. With the three factors noted, how many experiments would need to be run to cover all possible combinations with each of the three factors at two levels? (Note: This is often called a 2^3 design.)

1.2-3. How many different license plates are possible if a state uses

(a) Two letters followed by a four-digit integer (leading zeros are permissible and the letters and digits can be repeated)?

(b) Three letters followed by a three-digit integer? (In practice, it is possible that certain "spellings" are ruled out.)

1.2-4. The "eating club" is hosting a make-your-own sundae at which the following are provided:

Ice Cream Flavors	Toppings
Chocolate	Caramel
Cookies 'n' cream	Hot fudge
Strawberry	Marshmallow
Vanilla	M&M's
	Nuts
	Strawberries

(a) How many sundaes are possible using one flavor of ice cream and three different toppings?

(b) How many sundaes are possible using one flavor of ice cream and from zero to six toppings?

(c) How many different combinations of flavors of three scoops of ice cream are possible if it is permissible to make all three scoops the same flavor?

1.2-5. How many four-letter code words are possible using the letters in IOWA if

(a) The letters may not be repeated?

(b) The letters may be repeated?

1.2-6. Suppose that Novak Djokovic and Roger Federer are playing a tennis match in which the first player to win three sets wins the match. Using **D** and **F** for the winning player of a set, in how many ways could this tennis match end?

1.2-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

(a) 6, 7, 8, 9.

(b) 6, 7, 8, 8.

(c) 7, 7, 8, 8.

(d) 7, 8, 8, 8.

1.2-8. How many different varieties of pizza can be made if you have the following choice: small, medium, or large size; thin 'n' crispy, hand-tossed, or pan crust; and 12 toppings (cheese is automatic), from which you may select from 0 to 12?

1.2-9. The World Series in baseball continues until either the American League team or the National League team wins four games. How many different orders are possible (e.g., *ANNAAA* means the American League team wins in six games) if the series goes

(a) Four games?

(b) Five games?

(c) Six games?

(d) Seven games?

1.2-10. Pascal's triangle gives a method for calculating the binomial coefficients; it begins as follows:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

The nth row of this triangle gives the coefficients for $(a+b)^{n-1}$. To find an entry in the table other than a 1 on the boundary, add the two nearest numbers in the row directly above. The equation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1},$$

called **Pascal's equation**, explains why Pascal's triangle works. Prove that this equation is correct.

1.2-11. Three students (S) and six faculty members (F) are on a panel discussing a new college policy.

(a) In how many different ways can the nine participants be lined up at a table in the front of the auditorium?

(b) How many lineups are possible, considering only the labels S and F?

(c) For each of the nine participants, you are to decide whether the participant did a good job or a poor job stating his or her opinion of the new policy; that is, give each of the nine participants a grade of G or P. How many different "scorecards" are possible?

1.2-12. Prove

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0 \qquad \text{and} \qquad \sum_{r=0}^{n}\binom{n}{r} = 2^n.$$

HINT: Consider $(1-1)^n$ and $(1+1)^n$, or use Pascal's equation and proof by induction.

1.2-13. A bridge hand is found by taking 13 cards at random and without replacement from a deck of 52 playing cards. Find the probability of drawing each of the following hands.

(a) One in which there are 5 spades, 4 hearts, 3 diamonds, and 1 club.

(b) One in which there are 5 spades, 4 hearts, 2 diamonds, and 2 clubs.

(c) One in which there are 5 spades, 4 hearts, 1 diamond, and 3 clubs.

(d) Suppose you are dealt 5 cards of one suit, 4 cards of another. Would the probability of having the other suits split 3 and 1 be greater than the probability of having them split 2 and 2?

1.2-14. A bag of 36 dum-dum pops (suckers) contains up to 10 flavors. That is, there are from 0 to 36 suckers of each of 10 flavors in the bag. How many different flavor combinations are possible?

1.2-15. Prove Equation 1.2-2. HINT: First select n_1 positions in $\binom{n}{n_1}$ ways. Then select n_2 from the remaining $n-n_1$ positions in $\binom{n-n_1}{n_2}$ ways, and so on. Finally, use the multiplication rule.

1.2-16. A box of candy hearts contains 52 hearts, of which 19 are white, 10 are tan, 7 are pink, 3 are purple, 5 are yellow, 2 are orange, and 6 are green. If you select nine pieces of candy randomly from the box, without replacement, give the probability that

(a) Three of the hearts are white.

(b) Three are white, two are tan, one is pink, one is yellow, and two are green.

1.2-17. A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:

(a) Four of a kind (four cards of equal face value and one card of a different value).

(b) Full house (one pair and one triple of cards with equal face value).

(c) Three of a kind (three equal face values plus two cards of different values).

(d) Two pairs (two pairs of equal face value plus one card of a different value).

(e) One pair (one pair of equal face value plus three cards of different values).

1.2 Methods of Enumeration

1.2-2 **(a)** $(4)(5)(2) = 40$; **(b)** $(2)(2)(2) = 8$.

1.2-4 **(a)** $4\binom{6}{3} = 80$;

(b) $4(2^6) = 256$;

(c) $\dfrac{(4-1+3)!}{(4-1)!3!} = 20$.

1.2-6 $S =\{$ DDD, DDFD, DFDD, FDDD, DDFFD, DFDFD, FDDFD, DFFDD, FDFDD, FFDDD, FFF, FFDF, FDFF, DFFF FFDDF, FDFDF, DFFDF, FDDFF, DFDFF, DDFFF $\}$ so there are 20 possibilities.

1.2-8 $3 \cdot 3 \cdot 2^{12} = 36{,}864$.

1.2-10
$$\begin{aligned}\binom{n-1}{r} + \binom{n-1}{r-1} &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-r)(n-1)! + r(n-1)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.\end{aligned}$$

1.2-12
$$\begin{aligned}0 &= (1-1)^n = \sum_{r=0}^{n}\binom{n}{r}(-1)^r(1)^{n-r} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}. \\ 2^n &= (1+1)^n = \sum_{r=0}^{n}\binom{n}{r}(1)^r(1)^{n-r} = \sum_{r=0}^{n}\binom{n}{r}.\end{aligned}$$

1.2-14 $\dbinom{10-1+36}{36} = \dfrac{45!}{36!9!} = 886{,}163{,}135$.

1.2-16 **(a)** $\dfrac{\dbinom{19}{3}\dbinom{52-19}{6}}{\dbinom{52}{9}} = \dfrac{102{,}486}{351{,}325} = 0.2917;$

(b) $\dfrac{\dbinom{19}{3}\dbinom{10}{2}\dbinom{7}{1}\dbinom{3}{0}\dbinom{5}{1}\dbinom{2}{0}\dbinom{6}{2}}{\dbinom{52}{9}} = \dfrac{7{,}695}{1{,}236{,}664} = 0.00622.$

combination = | d_1 | d_2 | d_3 | d_4 |

We have 4 places, each has to be filled by one of 1, 2, 3, 4, 5, 6, 7 or 8.
Hence we have 8 different ways to fill each of the four places.

Hence total combinations possible $= 8 \times 8 \times 8 \times 8$

(d_1 can be replaced by 8 different digits) (d_2 can be replaced by 8 diff) (d_3 can be replaced by 8 diff digits) (d_4 can be replaced by 8 different digits)

$= 8^4$

2. $n_1 \rightarrow$ 4 levels of temperature
$n_2 \rightarrow$ 5 different pressure
$n_3 \rightarrow$ 2 different catalyst

a) all combination $= n_1 n_2 n_3 = 4 \times 5 \times 2 = 40$
40 experiment need to conduct all cases

b) all combination $= 2 \times 2 \times 2 = 2^3$

3. a) $\boxed{L}\ \boxed{L}\ \boxed{I}\ \boxed{I}\ \boxed{I}\ \boxed{I}$ L: Letter I: Integer

L = 26 I = 10

diff. license plate $= 26 \times 26 \times 10 \times 10 \times 10 \times 10$

$= 6760000$

b) LLLIII

since, total case $= 26 \times 26 \times 26 \times 10 \times 10 \times 10$

$= 17576000$

4. a) no. of sundaes possible $= 4 \times ({}^6C_3)$ [${}^6C_3$: no. of ways we can choose three toppings]

$= 4 \times \frac{6!}{3!\,3!}$

$= 80$ (Ans)

b) no. of sundaes possible = one ice zero topping + one ice one topping + ... + one ice six topping

$= 4 \times {}^6C_0 + 4 \times {}^6C_1 + 4 \times {}^6C_2 + \ldots\ldots + 4 \times {}^6C_6$

$= 4(1 + 6 + 15 + 20 + 15 + 6 + 1)$

$= 256$ (Ans)

c) no. of flavors of 3 scoops of ice-cream $= 4 \times 4 \times 4 = 64$ (Ans)

5. a) no. of four-letter code possible $= 4 \times 3 \times 2 \times 1 = 24$

explanation: since letter not allowed to repeat, so from 4 letter we have to make code $({}^4P_4)$

b) no. of 4-letter code possible repetion allowed $= 4 \times 4 \times 4 \times 4 = 256$

6. Here one assumption we should make that draw will not occur, to solve problem.
for win to someone atleast 3 games have to play, or at most 5 games.

Total cases in a way game end = 3 games cases + 4 games cases + 5 games cases

3 games cases = 2 (either D win or F win)

4 games cases = 2 × (way of winning)
= $2 \times ({}^4C_1 - 1)$ [As, 1 subtracted becase WWWE wouldnot be the case]
= 6

5 games cases = 2 × (way of winning)
= $2 \times ({}^4C_2)$ (same above reason) (at last place, winner should win the match for 5 games)
= 2 × 6
= 12

Total ways in game end = 2 + 6 + 12 = 20

7. Total integer that can possible = 1 + 9999 = 10,000
(1 for 0000 case)

a) 6, 7, 8, 9, ⇒ Possible case = ${}^4P_4 = 4! = 24$

$\text{Prob (Winning)} = \frac{24}{10{,}000} = 0.0024$

b) 6, 7, 8, 8 ⇒ Possible case = $\frac{4!}{2!} = 12$

$\text{Prob} = \frac{12}{10{,}000} = 0.0012$ (since, two 8 are same so divide by 2!)

c) 7, 7, 8, 8 $\text{Prob (winni)} = \frac{4!/(2! \times 2!)}{10{,}000} = 0.0006$

d) 7, 8, 8, 8 $\text{Prob (winni)} = \frac{4!/(3!)}{10{,}000} = 0.0004$

8. $n_1 \rightarrow$ 3 choice of size available
$n_2 \rightarrow$ 3 type of varity
$n_3 \rightarrow$ 12 toppings
Here, we make one assumption that order of topping will not matter.
So, ways of topping $= 2^{12}$ [as either topping will be taken or not]
So, total different Pizza $= 3 \times 3 \times 2^{12} = 36,864$

9. When any team American League (A) or National League (N) won 4 games, then game would be end
a) for 4 games;
only 2 order possible
b) 5 games $\Rightarrow$ order possible $= 2 \times$ (way of winning)
$= 2 \times$ (Permutation of LWWWW provided L won't be in end)
$= 2 \times {}^4C_1 = 8$ (2 for which team win)
c) 6 games $\Rightarrow$ order possible $= 2 \times ({}^5C_2)$
$= 20$
d) 7 games $\Rightarrow$ order possible $= 2 \times ({}^6C_3)$
$= 40$
same as above, but in question we make an assumption that match will not end in draw.

10. We have to prove ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$
R.H.S. $\Rightarrow {}^{n-1}C_r + {}^{n-1}C_{r-1} = \dfrac{(n-1)!}{r!(n-r-1)!} + \dfrac{(n-1)!}{(r-1)!(n-r)!}$
$= \dfrac{(n-1)!}{r(r-1)!(n-r)!}\left[(n-r)+r\right] = \dfrac{n!}{r!(n-r)!}$
$= {}^nC_r$

11. Student → 3 (S) Faculty → 6 (F)

a) No of ways = $9! = 362880$

b) if we consider only labels, ways $= \frac{9!}{6!\,3!} = 84$

(As all S and all F would be consider same)

c) No. of score card possible = $2 \times 2 \times \dots$ 9 times

(either one choise for either good or bad)

ways $= 2^9 = 512$

12. $\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0$

Proof: A/c to pascal equation

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \, a^r \, b^{n-r}$$

$$(-1+1)^n = \sum_{r=0}^{n} (-1)^r (1)^{n-r} \, {}^nC_r$$

$$0 = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \quad (\text{H.P.})$$

13. a) $P = \frac{\text{Fav. outcome}}{\text{Total outcome}} = \frac{{}^{13}C_5 \times {}^{13}C_4 \times {}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_{13}}$

$= 0.005388 \approx 0.00539$

b) $P = \frac{{}^{13}C_5 \times {}^{13}C_4 \times {}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_{13}} = 0.00882$

c) $P = \frac{{}^{13}C_5 \times {}^{13}C_4 \times {}^{13}C_1 \times {}^{13}C_3}{{}^{52}C_{13}} = 0.00539$

d) From solved part in (a) & (b) we can see.

Probability of having split them in 2 and 2 is more.

14. $n = 10$ flavors, $r = 36$ dum-dum

diff flavor combination Possible = ${}^{n+r-1}C_r = {}^{45}C_{36}$

$= 886163135$

15. $\binom{n}{n_1\, n_2 \ldots n_s} = \frac{n!}{n_1!\, n_2! \ldots n_s!}$

In this equation, we have to arranges n object in which $n_1, n_2, \ldots n_s$ are one type of object (same, indistinguishable)

~~no~~ ways in which n_1 can place $= {}^{n}C_{n_1}$
types of object

then remaining $(n-n_1)$ place $= {}^{n-n_1}C_{n_2}$
and we have to n_2 types of object place

Similarly : ${}^{n-n_1-n_2-\ldots-n_{s-1}}C_{n_s}$

$$\text{Ways} = {}^{n}C_{n_1} \times {}^{n-n_1}C_{n_2} \times {}^{n-n_1-n_2}C_{n_3} * \ldots$$

$$= \frac{n!}{n_1!\, n_2! \ldots n_s!} \quad \text{(on solving)}$$

16. 52 hearts = 19 white + 10 tan + 7 pink + 3 purple + 5 yellow + 2 orange + 6 green

a) Total ways of choosing 9 candy $= {}^{52}C_9$
Ways when 3 of them is white $= {}^{19}C_3 \times {}^{33}C_6$

$$\text{Prob.} = \frac{{}^{19}C_3 \times {}^{33}C_6}{{}^{52}C_9} = 0.2917$$

b) no. of ways 3 White, 2 tan, one pink, 1 yellow, 2 green $= {}^{19}C_3 \times {}^{10}C_2 \times {}^{7}C_1 \times {}^{5}C_1 \times {}^{6}C_2$

$$\text{Prob} = \frac{{}^{15}C_3 \times {}^{10}C_2 \times {}^{7}C_1 \times {}^{5}C_1 \times {}^{6}C_2}{{}^{52}C_9} = \frac{{}^{19}C_3 \times 45 \times 7 \times 5 \times 15}{{}^{52}C_9}$$

$$= 0.00622$$

17. a) Total no of ways, different Poker hand can possible $= {}^{52}C_5 =$

Ways when four of a kind $= {}^{13}C_1 \times {}^{48}C_1 = 624$

Prob. $= \frac{624}{{}^{52}C_5} = 0.00024$

b) Full house ways = (ways of choosing a pair) × (ways of choosing triple)

$= ({}^{13}C_1 \times {}^4C_2) \times ({}^{12}C_1 \times {}^4C_3)$

Prob $= \frac{{}^{13}C_1 \times {}^4C_2 \times {}^{12}C_1 \times {}^4C_3}{{}^{52}C_5} = 0.00144$

c) Ways when 3 of a kind $= ({}^{13}C_1 \times {}^4C_3) \times ({}^{12}C_1 \times {}^4C_1)({}^{11}C_1 \times {}^4C_1)$

Prob $= \frac{({}^{13}C_1 \times {}^4C_3 \times {}^{12}C_1 \times {}^4C_1 \times {}^{11}C_1 \times {}^4C_1)/2}{{}^{52}C_5}$

We divide by 2 because, when we choosing last 2 different card we inclded their order

Alter, Prob $= \frac{({}^{13}C_1 \times {}^4C_3) \times ({}^{12}C_2 \times {}^4C_1 \times {}^4C_1)}{{}^{52}C_5}$

$= 0.0211$

d) Prob $= \frac{({}^{13}C_2 \times {}^4C_2 \times {}^4C_2) \times ({}^{11}C_1 \times {}^4C_1)}{{}^{52}C_5} = 0.04754$

e) Prob $= \frac{({}^{13}C_1 \times {}^4C_2) \times ({}^{12}C_3 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1)}{{}^{52}C_5} = 0.42257$

1.3 CONDITIONAL PROBABILITY

Exercises

1.3-1. A common test for AIDS is called the ELISA (enzyme-linked immunosorbent assay) test. Among 1 million people who are given the ELISA test, we can expect results similar to those given in the following table:

	B_1: Carry AIDS Virus	B_2: Do Not Carry Aids Virus	Totals
A_1: Test Positive	4,885	73,630	78,515
A_2: Test Negative	115	921,370	921,485
Totals	5,000	995,000	1,000,000

If one of these 1 million people is selected randomly, find the following probabilities: **(a)** $P(B_1)$, **(b)** $P(A_1)$, **(c)** $P(A_1 \mid B_2)$, **(d)** $P(B_1 \mid A_1)$. **(e)** In words, what do parts (c) and (d) say?

1.3-2. The following table classifies 1456 people by their gender and by whether or not they favor a gun law.

	Male (S_1)	Female (S_2)	Totals
Favor (A_1)	392	649	1041
Oppose (A_2)	241	174	415
Totals	633	823	1456

Compute the following probabilities if one of these 1456 persons is selected randomly: **(a)** $P(A_1)$, **(b)** $P(A_1 \mid S_1)$, **(c)** $P(A_1 \mid S_2)$. **(d)** Interpret your answers to parts (b) and (c).

1.3-3. Let A_1 and A_2 be the events that a person is left-eye dominant or right-eye dominant, respectively. When a person folds his or her hands, let B_1 and B_2 be the events that the left thumb and right thumb, respectively, are on top. A survey in one statistics class yielded the following table:

	B_1	B_2	Totals
A_1	5	7	12
A_2	14	9	23
Totals	19	16	35

If a student is selected randomly, find the following probabilities: **(a)** $P(A_1 \cap B_1)$, **(b)** $P(A_1 \cup B_1)$, **(c)** $P(A_1 \mid B_1)$, **(d)** $P(B_2 \mid A_2)$. **(e)** If the students had their hands folded and you hoped to select a right-eye-dominant student, would you select a "right thumb on top" or a "left thumb on top" student? Why?

1.3-4. Two cards are drawn successively and without replacement from an ordinary deck of playing cards. Compute the probability of drawing

(a) Two hearts.

(b) A heart on the first draw and a club on the second draw.

(c) A heart on the first draw and an ace on the second draw.

HINT: In part (c), note that a heart can be drawn by getting the ace of hearts or one of the other 12 hearts.

1.3-5. Suppose that the alleles for eye color for a certain male fruit fly are (R, W) and the alleles for eye color for the mating female fruit fly are (R, W), where R and W represent red and white, respectively. Their offspring receive one allele for eye color from each parent.

(a) Define the sample space of the alleles for eye color for the offspring.

(b) Assume that each of the four possible outcomes has equal probability. If an offspring ends up with either two white alleles or one red and one white allele for eye color, its eyes will look white. Given that an offspring's eyes look white, what is the conditional probability that it has two white alleles for eye color?

1.3-6. A researcher finds that, of 982 men who died in 2002, 221 died from some heart disease. Also, of the 982 men, 334 had at least one parent who had some heart disease. Of the latter 334 men, 111 died from some heart disease. A man is selected from the group of 982. Given that neither of his parents had some heart disease, find the conditional probability that this man died of some heart disease.

1.3-7. An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

1.3-8. An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting a single ball at random from the urn without replacement.

The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls.

(a) If you draw first, find the probability that you win the game on your second draw.

(b) If you draw first, find the probability that your opponent wins the game on his second draw.

(c) If you draw first, what is the probability that you win? HINT: You could win on your second, third, fourth, …, or tenth draw, but not on your first.

(d) Would you prefer to draw first or second? Why?

1.3-9. An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the mth ball selected. Let the event A_i denote a match on the ith draw, $i = 1, 2, 3, 4$.

(a) Show that $P(A_i) = \dfrac{3!}{4!}$ for each i.

(b) Show that $P(A_i \cap A_j) = \dfrac{2!}{4!}, i \neq j$.

(c) Show that $P(A_i \cap A_j \cap A_k) = \dfrac{1!}{4!}, i \neq j, i \neq k, j \neq k$.

(d) Show that the probability of at least one match is

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}.$$

(e) Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is

$$\begin{aligned} &P(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!} \\ &= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right). \end{aligned}$$

(f) What is the limit of this probability as n increases without bound?

1.3-10. A single card is drawn at random from each of six well-shuffled decks of playing cards. Let A be the event that all six cards drawn are different.

(a) Find $P(A)$.

(b) Find the probability that at least two of the drawn cards match.

1.3-11. Consider the birthdays of the students in a class of size r. Assume that the year consists of 365 days.

(a) How many different ordered samples of birthdays are possible (r in sample) allowing repetitions (with replacement)?

(b) The same as part (a), except requiring that all the students have different birthdays (without replacement)?

(c) If we can assume that each ordered outcome in part (a) has the same probability, what is the probability that at least two students have the same birthday?

(d) For what value of r is the probability in part (c) about equal to 1/2? Is this number surprisingly small? HINT: Use a calculator or computer to find r.

1.3-12. You are a member of a class of 18 students. A bowl contains 18 chips: 1 blue and 17 red. Each student is to take 1 chip from the bowl without replacement. The student who draws the blue chip is guaranteed an A for the course.

(a) If you have a choice of drawing first, fifth, or last, which position would you choose? Justify your choice on the basis of probability.

(b) Suppose the bowl contains 2 blue and 16 red chips. What position would you now choose?

1.3-13. In the gambling game "craps," a pair of dice is rolled and the outcome of the experiment is the sum of the points on the up sides of the six-sided dice. The bettor wins on the first roll if the sum is 7 or 11. The bettor loses on the first roll if the sum is 2, 3, or 12. If the sum is 4, 5, 6, 8, 9, or 10, that number is called the bettor's "point." Once the point is established, the rule is as follows: If the bettor rolls a 7 before the point, the bettor loses; but if the point is rolled before a 7, the bettor wins.

(a) List the 36 outcomes in the sample space for the roll of a pair of dice. Assume that each of them has a probability of 1/36.

(b) Find the probability that the bettor wins on the first roll. That is, find the probability of rolling a 7 or 11, $P(7 \text{ or } 11)$.

(c) Given that 8 is the outcome on the first roll, find the probability that the bettor now rolls the point 8 before rolling a 7 and thus wins. Note that at this stage in the game the only outcomes of interest are 7 and 8. Thus find $P(8 \mid 7 \text{ or } 8)$.

(d) The probability that a bettor rolls an 8 on the first roll and then wins is given by $P(8)P(8 \mid 7 \text{ or } 8)$. Show that this probability is (5/36)(5/11).

(e) Show that the total probability that a bettor wins in the game of craps is 0.49293. HINT: Note that the bettor can win in one of several mutually exclusive ways: by rolling a 7 or an 11 on the first roll or by establishing one of the points 4, 5, 6, 8, 9, or 10 on the first roll and then obtaining that point on successive rolls before a 7 comes up.

1.3-14. Paper is often tested for "burst strength" and "tear strength." Say we classify these strengths as low, middle, and high. Then, after examining 100 pieces of paper, we find the following:

	Burst Strength		
Tear Strength	A_1 (low)	A_2 (middle)	A_3 (high)
B_1 (low)	7	11	13
B_2 (middle)	11	21	9
B_3 (high)	12	9	7

If we select one of the pieces at random, what are the probabilities that it has the following characteristics:

(a) A_1?

(b) $A_3 \cap B_2$?

(c) $A_2 \cup B_3$?

(d) A_1, given that it is B_2?

(e) B_1, given that it is A_3?

1.3-15. An urn contains eight red and seven blue balls. A second urn contains an unknown number of red balls and nine blue balls. A ball is drawn from each urn at random, and the probability of getting two balls of the same color is 151/300. How many red balls are in the second urn?

1.3-16. Bowl A contains three red and two white chips, and bowl B contains four red and three white chips. A chip is drawn at random from bowl A and transferred to bowl B. Compute the probability of then drawing a red chip from bowl B.

1.3 Conditional Probability

1.3-2 **(a)** $\dfrac{1041}{1456}$;

(b) $\dfrac{392}{633}$;

(c) $\dfrac{649}{823}$.

(d) The proportion of women who favor a gun law is greater than the proportion of men who favor a gun law.

1.3-4 **(a)** $P(\mathrm{HH}) = \dfrac{13}{52} \cdot \dfrac{12}{51} = \dfrac{1}{17}$;

(b) $P(\mathrm{HC}) = \dfrac{13}{52} \cdot \dfrac{13}{51} = \dfrac{13}{204}$;

(c) $P(\text{Non-Ace Heart, Ace}) + P(\text{Ace of Hearts, Non-Heart Ace})$

$$= \frac{12}{52} \cdot \frac{4}{51} + \frac{1}{52} \cdot \frac{3}{51} = \frac{51}{52 \cdot 51} = \frac{1}{52}.$$

1.3-6 Let H ={died from heart disease}; P ={at least one parent had heart disease}.

$$P(H \mid P') = \frac{N(H \cap P')}{N(P')} = \frac{110}{648}.$$

1.3-8 **(a)** $\dfrac{3}{20} \cdot \dfrac{2}{19} \cdot \dfrac{1}{18} = \dfrac{1}{1140}$;

(b) $\dfrac{\binom{3}{2}\binom{17}{1}}{\binom{20}{3}} \cdot \dfrac{1}{17} = \dfrac{1}{380}$;

(c) $\displaystyle\sum_{k=1}^{9} \frac{\binom{3}{2}\binom{17}{2k-2}}{\binom{20}{2k}} \cdot \frac{1}{20-2k} = \frac{35}{76} = 0.4605.$

(d) Draw second. The probability of winning is $1 - 0.4605 = 0.5395$.

1.3-10 **(a)** $P(A) = \dfrac{52}{52} \cdot \dfrac{51}{52} \cdot \dfrac{50}{52} \cdot \dfrac{49}{52} \cdot \dfrac{48}{52} \cdot \dfrac{47}{52} = \dfrac{8{,}808{,}975}{11{,}881{,}376} = 0.74141$;

(b) $P(A') = 1 - P(A) = 0.25859$.

1.3-12 **(a)** It doesn't matter because $P(B_1) = \dfrac{1}{18}$, $P(B_5) = \dfrac{1}{18}$, $P(B_{18}) = \dfrac{1}{18}$;

(b) $P(B) = \dfrac{2}{18} = \dfrac{1}{9}$ on each draw.

1.3-14 **(a)** $P(A_1) = 30/100$;

(b) $P(A_3 \cap B_2) = 9/100$;

(c) $P(A_2 \cup B_3) = 41/100 + 28/100 - 9/100 = 60/100$;

(d) $P(A_1 \mid B_2) = 11/41$;

(e) $P(B_1 \mid A_3) = 13/29$.

1.3-16 $\dfrac{3}{5} \cdot \dfrac{5}{8} + \dfrac{2}{5} \cdot \dfrac{4}{8} = \dfrac{23}{40}$.

1 a) $P(B_1) = \dfrac{4885+115}{1000000} = 5\times10^{-3} = 0.005$

b) $P(A_1) = \dfrac{78515}{1000000} = 0.078515$

c) $P(A_1 \mid B_2) = \dfrac{P(A_1 \cap B_2)}{P(B_2)} = \dfrac{73630/1000000}{995000/1000000} = \dfrac{73630}{995000}$

$= 0.074$

d) $P(B_1 \mid A_1) = \dfrac{P(A_1 \cap B_1)}{P(A_1)} = \dfrac{4885}{78515} = 0.0622$

e) parts (c) and (d) say, if event (say B_2 and A_1 respectively) happened then what is the probability of event A_1 and B_1 respectively, will happen

2 a) $P(A_1) = \dfrac{\text{Fav. outcome}}{\text{Total outcome}} = \dfrac{1041}{1456} = 0.715$

b) $P(A_1 \mid S_1) = \dfrac{P(A_1 \cap S_1)}{P(S_1)} = \dfrac{392}{633} = 0.6193$

c) $P(A_1 \mid S_2) = \dfrac{P(A_1 \cap S_2)}{P(S_2)} = \dfrac{649}{823} = 0.7886$

d) In part b) and c) we calculate the probabity that A_1 occur when event S_1 is already occured what, we get to know is proportion of woman is greater than man (in favoring a gun law)

3 a) $P(A_1 \cap B_1) = \frac{5}{35} = 0.143$

b) $P(A_1 \cup B_1) = P(A_1) + P(B_1) - P(A_1 \cap B_1)$

$= \frac{12}{35} + \frac{19}{35} - \frac{5}{35} = \frac{26}{35} = 0.743$

c) $P(A_1 \mid B_1) = \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{5}{19} = 0.263$

d) $P(B_2 \mid A_2) = \frac{P(B_2 \cap A_2)}{P(A_2)} = \frac{9}{23} = 0.3913$

e) Since $P(A_2 \mid B_1) = \frac{P(A_2 \cap B_1)}{P(B_1)} = \frac{14}{19} = 0.7368$

$P(A_2 \mid B_2) = \frac{P(A_2 \cap B_2)}{P(B_2)} = \frac{9}{16} = 0.5625$

$P(A_2 \mid B_1) > P(A_2 \mid B_2)$, therefore so select "left thumb on top"

4 a) $P(\text{Two hearts}) = P(HH) = \frac{13}{52} \times \frac{12}{51}$

as 2 cards drawn successively, that is why order matter

b) $P(HC) = \frac{13}{52} \times \frac{13}{51}$

c) $P(H1) = \frac{\text{Fav. outcome}}{\text{Total outcome}}$

Fav. outcome = when first card is any heart but not ace + when first card is heart and ace both

$= (13-1) \times 4 + 1 \times 3 = 51$

Date: / /

$P(H1) = \frac{51}{52 \times 51} = \frac{1}{52}$

5 d) $(R, W) \times (R, W) \rightarrow$ sample space S for their offspring

$S = \{(R,R), (R,W), (W,R), (W,W)\}$

b) event A : that eyes look white

event B : it has 2 white alleles for eye colour

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

6. from 982 men died,

221 died from heart disease

334 had atleast one parent who had heart disease

	at least one parent who had disease Yes	No	Total
Died	111	110	221
Not	223	538	761
Total	334	648	982

$$P(\text{Died} \mid \text{neither of his parent}) = \frac{110}{648} = 0.169753$$

7. Let Balls are O_1, O_2, B_1, B_2

Total ways of selecting 2 ball = ${}^4C_2 = 6$

$$P(\text{atleast one of them is orange}) = P(A) = 1 - \frac{{}^2C_2}{6} = \frac{5}{6}$$

$$P\left(\begin{array}{c}\text{both ball} \\ \text{orange}\end{array} \middle| \begin{array}{c}\text{atleast one of them} \\ \text{is orange}\end{array}\right) = P(B|A)$$

$$= \frac{P(A \cap B)}{P(A)} = \frac{1/6}{5/6} = \frac{1}{5}$$

8. a) Y: You O: Opponent

$$P(YOY) = P(WWW) = \frac{3}{20} \times \frac{2}{19} \times \frac{1}{18}$$

b) $P(YOYO) = P(\text{2 Win, 1 Lose, 1 Win})$

$$= \frac{{}^3C_2 \times {}^{17}C_1}{{}^{20}C_3} \times \frac{1}{17} = \frac{1}{760}$$

c) $P(\text{Win}) = P(YOY) + P(YOYOY) + \cdots$

$$= P(\text{3 Win}) + P(\text{2 Win, 2 Lose, 1 Win}) + \cdots$$

$$= \frac{{}^3C_3}{{}^{20}C_3} + \frac{{}^3C_2 \times {}^{17}C_2}{{}^{20}C_4} \times \frac{1}{16} + \frac{{}^3C_2 \times {}^{17}C_4}{{}^{20}C_6} \times \frac{1}{14}$$

$$+ \frac{{}^3C_2 \times {}^{17}C_6}{{}^{20}C_8} \times \frac{1}{12} + \frac{{}^3C_2 \times {}^{17}C_8}{{}^{20}C_{10}} \times \frac{1}{10} + \frac{{}^3C_2 \times {}^{17}C_{10}}{{}^{20}C_{12}} \times \frac{1}{8}$$

$$+ \frac{{}^3C_2 \times {}^{17}C_{12}}{{}^{20}C_{14}} \times \frac{1}{6} + \frac{{}^3C_2 \times {}^{17}C_{14}}{{}^{20}C_{16}} \times \frac{1}{4} + \frac{{}^3C_2 \times {}^{17}C_{16}}{{}^{20}C_{18}} \times \frac{1}{2}$$

d) Let's I draw first and win in my 2nd draw, Prob P_1

$$P_1 = \frac{{}^3C_3}{{}^{20}C_3} = \frac{1}{1140}$$

If I draw 2nd and win in my 2nd draw prob P_2

$$P(OYOY) = \frac{1}{760} = P_2$$

Since, $P_2 > P_1$, so I prefer to draw 2nd.

Similar for larger game.

9 Balls are B_1, B_2, B_3, B_4

a) $P(A_i) = 1 - P\left(\begin{array}{c}\text{"i" numbered ball is not selected} \\ \text{at ith position}\end{array}\right)$

$$= 1 - \frac{3 \times 3!}{4!} = \frac{3!}{4!}$$

b) $P(A_i \cap A_j) = P(A_i \mid A_j) \times P(A_j)$, $i \neq j$

$$= \left(1 - \frac{2 \times 2!}{3!}\right) \times \frac{3!}{4!} = \frac{2!}{3!} \cdot \frac{3!}{4!} = \frac{2!}{4!}$$

c) $P(A_i \cap A_j\, A_k) = \frac{1}{4!}$ $\left(\begin{array}{l}\text{only one Possible case} \\ B_1 B_2 B_3 B_4 \text{ drawn}\end{array}\right)$

where $i \neq j \neq k$, $i \neq k$

d) $P(\text{at least one match}) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$$\cancel{= \sum_{i=1}^{4} P(A_i) - \sum_{\substack{i \neq j \neq k \\ k \neq i \neq 1}} P(A_i \cap A_j \cap A_k) + \sum P(A_i \cap A_j}$$

$$= \sum_{i=1}^{4} P(A_i) - \sum_{\substack{i, j \\ i \neq j}} P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k)$$

$$- P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$= 4 \times \frac{3!}{4!} - 6 \times \frac{2!}{4!} + 4 \times \frac{1}{4!} - \frac{1}{4!}$$

$$= 1 - \frac{1}{2} + \frac{1}{8} = \frac{5}{8} \quad \left(\text{or } 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}\right)$$

e) $P(A_1 \cup A_2 \dots \cup A_n) = n \times P(A_i) - {}^nC_2 \times P(A_i \cap A_j)$

$+ {}^nC_3 \times P(A_i \cap A_j \cap A_k) - \dots$

$(-1)^{n+1}\, {}^nC_n\, P(A_i \cap A_j \dots \cap A_k)$

$= n \times \frac{(n-1)!}{n!} - \frac{n(n-1)}{2} \times \frac{(n-2)!}{n!} + \dots \frac{(-1)^{n+1}}{n!}$

$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \dots \frac{(-1)^{n+1}}{n!}$

$= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \frac{(-1)^n}{n!}\right)$

d) $\lim_{n \to \infty} P(A_1 \cup A_2 \cup \dots A_n) = 1 - \lim_{n \to \infty} \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \frac{(-1)^n}{n!}\right)$

$= 1 - e^{-1}$ $\left(e^{x} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots \infty\right)$

$= 1 - \frac{1}{e}$

10.a) $P\left(\begin{array}{c}\text{all 6 cards drawn}\\ \text{are different}\end{array}\right) = P(A) = \frac{52}{52} \times \frac{51}{52} \times \frac{50}{52} \times \frac{49}{52} \times \frac{48}{52} \times \frac{47}{52}$

$= 0.74141$

b) $P\left(\begin{array}{c}\text{at least 2 of cards}\\ \text{match}\end{array}\right) = 1 - P(A) = 0.25859$

11. (a) number of different ordered sample $= 365 \times 365 \times \cdots$ r times

$= (365)^r$

(b) number of different ordered sample $= 365 \times 364 \times \cdots (365 - r + 1)$

(without replacement) $\Rightarrow {}^{365}P_r$

(c) $P\left(\begin{array}{c}\text{at least 2 student}\\ \text{have same birthday}\end{array}\right) = 1 - P\left(\begin{array}{c}\text{none of student}\\ \text{have same birthday}\end{array}\right)$

$$= 1 - \frac{{}^{365}P_r}{(365)^r}$$

(d) $$1 - \frac{{}^{365}P_r}{(365)^r} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{(365)!}{(365-r)!} \times \frac{1}{(365)^r} = \frac{365 \times 364 \times \cdots (365-r+1)}{(365)^r}$$

$$\frac{1}{2} = \frac{364}{365} \times \frac{363}{365} \times \cdots \frac{(365-r+1)}{365}$$

on solving, $r = 23$

12 a) $P\left(\begin{array}{c}\text{blue chip draw}\\ \text{on first}\end{array}\right) = \frac{1}{18}$

$$P\left(\begin{array}{c}\text{blue chip draw}\\ \text{on 5th}\end{array}\right) = \frac{{}^{17}C_4}{{}^{18}C_4} \times \frac{1}{14} = \frac{17!}{13!} \times \frac{14!}{18!} \times \frac{1}{14}$$

$$= \frac{1}{18}$$

$$P\left(\begin{array}{c}\text{blue chip on}\\ \text{last draw}\end{array}\right) = \frac{{}^{17}C_{17}}{{}^{18}C_{17}} \times 1 = \frac{1}{18}$$

So, It doesn't matter, as probability is same

b) if two blue chip

$P(\text{first draw}) = \frac{2}{18}$ $\quad$ ~~$P(\text{5th draw}) = \frac{{}^{16}C_4}{{}^{18}C_4} \times \frac{2}{14} = \frac{13 \times 2}{18 \times 17}$~~

$$P(\text{last draw}) = \frac{({}^{16}C_{16})({}^{2}C_1)}{{}^{18}C_{17}} \times 1 = \frac{2}{18}$$

b) $P(\text{1st draw}) = \frac{{}^{2}C_1}{{}^{18}C_1} \times \frac{{}^{17}C_{17}}{{}^{17}C_{17}} = \frac{2}{18}$

$P(\text{5th draw}) = \frac{{}^{16}C_4}{{}^{18}C_4} \times \frac{2}{14} + \frac{{}^{16}C_3 \cdot {}^{2}C_1}{{}^{18}C_4} \times \frac{1}{14} = \frac{2}{18}$

$P(\text{last draw}) = \frac{{}^{16}C_{16} \cdot {}^{2}C_1}{{}^{18}C_{17}} \cdot \frac{1}{1} = \frac{2}{18}$

Now, also It doesn't matter

13. a) S = (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1)
(2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2)
(3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3)
(4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4)
(5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5)
(6,6)

b) $P\left(\begin{array}{c}\text{betters win on}\\ \text{first roll}\end{array}\right) = P(7 \text{ or } 11) = \frac{2+6}{36} = \frac{8}{36}$

c) $P(8 \mid 7 \text{ or } 8) = \frac{P(A \cap B)}{P(B)} = \frac{P(8)}{P(7 \text{ or } 8)} = \frac{5/36}{11/36} = \frac{5}{11}$

d) $P(8) \cdot P(8 \mid 7 \text{ or } 8) = \frac{5}{36} \times P(8 \mid 7 \text{ or } 8)$

$= \frac{5}{36} \times \frac{5}{11}$ (from previous part)

e) $P(\text{betters win}) = P(7 \text{ or } 11) + P(4)\, P(4 \mid 4 \text{ or } 7)$
$+ P(5)\, P(5 \mid 5 \text{ or } 7) + P(6 \mid 6 \text{ or } 7)$
$P(8)\, P(8 \mid 8 \text{ or } 7) + P(9)\, P(9 \mid 9 \text{ or } 7)$
$+ P(10)\, P(10 \mid 10 \text{ or } 7)$

$= \frac{8}{36} + \frac{3}{36} \times \frac{3}{9} + \frac{4}{36} \times \frac{4}{10} + \frac{5}{36} \cdot \frac{5}{11} + \frac{5}{36} \cdot \frac{5}{11}$

$+ \frac{4}{36} \times \frac{4}{16} + \frac{3}{36} \times \frac{3}{9}$

$= 0.49293$

14. a) $P(A_1) = \frac{7+11+12}{100} = \frac{30}{100} = 0.3$

b) $P(A_3 \cap B_2) = \frac{9}{100} = 0.09$

c) $P(A_2 \cup B_3) = P(A_2) + P(B_3) - P(A_2 \cap B_3)$

$= \frac{11+21+9}{100} + \frac{12+9+7}{100} - \frac{9}{100}$

$= 0.6$

d) $P(A_1 | B_2) = \frac{P(A_1 \cap B_2)}{P(B_2)} = \frac{11/100}{41/100} = \frac{11}{41}$

e) $P(B_1 | A_3) = \frac{P(B_1 \cap A_3)}{P(A_3)} = \frac{13/100}{29/100} = \frac{13}{29}$

15. 1st urn = 8R + 7B 2nd urn = xR + 9B

P(same color) = P(RR) + P(BB)

$\Rightarrow \frac{151}{300} = \frac{8}{15} \times \frac{x}{9+x} + \frac{7}{15} \times \frac{9}{9+x}$

$\Rightarrow \frac{151}{20} = \frac{8x + 63}{9+x}$

$\Rightarrow 9+x = \frac{160}{151}x + \frac{20}{151} \times 63 = 8.34437 + \frac{160}{151}x$

$\Rightarrow 0.65563 = \frac{9}{151}x$

$\Rightarrow x = 11$ (Ans)

16. Bowl A = 3R + 2W Bowl B = 4R + 3W

$P = P\left(\begin{array}{l}\text{red from Bowl B if}\\ \text{red chip is drawn from A}\end{array}\right) + P\left(\begin{array}{l}\text{red from B if}\\ \text{white chip is drawn from A}\end{array}\right)$

$= \frac{3}{5} \times \frac{5}{8} + \frac{2}{5} \times \frac{4}{8} = \frac{23}{40} = 0.575$

1.4 INDEPENDENT EVENTS

Exercises

1.4-1. Let A and B be independent events with $P(A) = 0.7$ and $P(B) = 0.2$. Compute **(a)** $P(A \cap B)$, **(b)** $P(A \cup B)$, and **(c)** $P(A' \cup B')$.

1.4-2. Let $P(A) = 0.3$ and $P(B) = 0.6$.

(a) Find $P(A \cup B)$ when A and B are independent.

(b) Find $P(A \mid B)$ when A and B are mutually exclusive.

1.4-3. Let A and B be independent events with $P(A) = 1/4$ and $P(B) = 2/3$. Compute **(a)** $P(A \cap B)$, **(b)** $P(A \cap B')$, **(c)** $P(A' \cap B')$, **(d)** $P[(A \cup B)']$, and **(e)** $P(A' \cap B)$.

1.4-4. Prove parts (b) and (c) of Theorem 1.4-1.

1.4-5. If $P(A) = 0.8$, $P(B) = 0.5$, and $P(A \cup B) = 0.9$, are A and B independent events? Why or why not?

1.4-6. Show that if A, B, and C are mutually independent, then the following pairs of events are independent: A and $(B \cap C)$, A and $(B \cup C)$, A' and $(B \cap C')$. Show also that A', B', and C' are mutually independent.

1.4-7. Each of three football players will attempt to kick a field goal from the 25-yard line. Let A_i denote the event that the field goal is made by player i, $i = 1, 2, 3$. Assume that A_1, A_2, A_3 are mutually independent and that $P(A_1) = 0.5$, $P(A_2) = 0.7$, $P(A_3) = 0.6$.

(a) Compute the probability that exactly one player is successful.

(b) Compute the probability that exactly two players make a field goal (i.e., one misses).

1.4-8. Die A has orange on one face and blue on five faces, Die B has orange on two faces and blue on four faces, Die C has orange on three faces and blue on three faces. All are fair dice. If the three dice are rolled, find the probability that exactly two of the three dice come up orange.

1.4-9. Suppose that A, B, and C are mutually independent events and that $P(A) = 0.5$, $P(B) = 0.8$, and $P(C) = 0.9$. Find the probabilities that **(a)** all three events occur, **(b)** exactly two of the three events occur, and **(c)** none of the events occurs.

1.4-10. Let D_1, D_2, D_3 be three four-sided dice whose sides have been labeled as follows:

$$D_1 : 0\,3\,3\,3 \qquad D_2 : 2\,2\,2\,5 \qquad D_3 : 1\,1\,4\,6$$

The three dice are rolled at random. Let A, B, and C be the events that the outcome on die D_1 is larger than the outcome on D_2, the outcome on D_2 is larger than the outcome on D_3, and the outcome on D_3 is larger than the outcome on D_1, respectively. Show that **(a)** $P(A) = 9/16$, **(b)** $P(B) = 9/16$, and **(c)** $P(C) = 10/16$. Do you find it interesting that each of the probabilities that D_1 "beats" D_2, D_2 "beats" D_3, and D_3 "beats" D_1 is greater than 1/2? Thus, it is difficult to determine the "best" die.

1.4-11. Let A and B be two events.

(a) If the events A and B are mutually exclusive, are A and B always independent? If the answer is no, can they ever be independent? Explain.

(b) If $A \subset B$, can A and B ever be independent events? Explain.

1.4-12. Flip an unbiased coin five independent times. Compute the probability of

(a) $HHTHT$.

(b) $THHHT$.

(c) $HTHTH$.

(d) Three heads occurring in the five trials.

1.4-13. An urn contains two red balls and four white balls. Sample successively five times at random and with replacement, so that the trials are independent. Compute the probability of each of the two sequences $WWRWR$ and $RWWWR$.

1.4-14. In Example 1.4-5, suppose that the probability of failure of a component is $p = 0.4$. Find the probability that the system does not fail if the number of redundant components is

(a) 3.

(b) 8.

1.4-15. An urn contains 10 red and 10 white balls. The balls are drawn from the urn at random, one at a time. Find the probabilities that the fourth white ball is the fourth, fifth, sixth, or seventh ball drawn if the sampling is done

(a) With replacement.

(b) Without replacement.

(c) In the World Series, the American League (red) and National League (white) teams play until one team wins four games. Do you think that the urn model presented in this exercise could be used to describe the probabilities of a 4-, 5-, 6-, or 7-game series? (Note that either "red" or "white" could win.) If your answer is yes, would you choose sampling with or without replacement in your model? (For your information, the numbers of 4-, 5-, 6-, and 7-game series, up to and including 2012, were 21, 24, 23, 36. This ignores games that ended in a tie, which occurred in 1907, 1912, and 1922. Also, it does not include the 1903 and 1919–1921 series, in which the winner had to take five out of nine games. The World Series was canceled in 1994.)

1.4-16. An urn contains five balls, one marked WIN and four marked LOSE. You and another player take turns selecting a ball at random from the urn, one at a time. The first person to select the WIN ball is the winner. If you draw first, find the probability that you will win if the sampling is done

(a) With replacement.

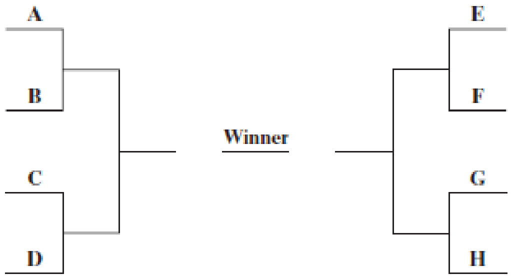
(b) Without replacement.

1.4-17. Each of the 12 students in a class is given a fair 12-sided die. In addition, each student is numbered from 1 to 12.

(a) If the students roll their dice, what is the probability that there is at least one "match" (e.g., student 4 rolls a 4)?

(b) If you are a member of this class, what is the probability that at least one of the other 11 students rolls the same number as you do?

1.4-18. An eight-team single-elimination tournament is set up as follows:



For example, eight students (called A–H) set up a tournament among themselves. The top-listed student in each bracket calls heads or tails when his or her opponent flips a coin. If the call is correct, the student moves on to the next bracket.

(a) How many coin flips are required to determine the tournament winner?

(b) What is the probability that you can predict all of the winners?

(c) In NCAA Division I basketball, after the "play-in" games, 64 teams participate in a single-elimination tournament to determine the national champion. Considering only the remaining 64 teams, how many games are required to determine the national champion?

(d) Assume that for any given game, either team has an equal chance of winning. (That is probably not true.) On page 43 of the March 22, 1999, issue, *Time* claimed that the "mathematical odds of predicting all 63 NCAA games correctly is 1 in 75 million." Do you agree with this statement? If not, why not?

1.4-19. Extend Example 1.4-6 to an n-sided die. That is, suppose that a fair n-sided die is rolled n independent times. A match occurs if side i is observed on the ith trial, $i = 1, 2, \ldots, n$.

(a) Show that the probability of at least one match is

$$1 - \left(\frac{n-1}{n}\right)^n = 1 - \left(1 - \frac{1}{n}\right)^n.$$

(b) Find the limit of this probability as n increases without bound.

1.4-20. Hunters A and B shoot at a target with probabilities of p_1 and p_2, respectively. Assuming independence, can p_1 and p_2 be selected so that $P(\text{zero hits}) = P(\text{one hit}) = P(\text{two hits})$?

.4 Independent Events

1.4-2 **(a)**
$$\begin{aligned} P(A\cap B) &= P(A)P(B) = (0.3)(0.6) = 0.18; \\ P(A\cup B) &= P(A)+P(B)-P(A\cap B) \\ &= 0.3+0.6-0.18 \\ &= 0.72. \end{aligned}$$

(b) $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{0}{0.6} = 0.$

1.4-4 Proof of **(b)**:
$$\begin{aligned} P(A'\cap B) &= P(B)P(A'|B) \\ &= P(B)[1-P(A|B)] \\ &= P(B)[1-P(A)] \\ &= P(B)P(A'). \end{aligned}$$

Proof of **(c)**:
$$\begin{aligned} P(A'\cap B') &= P[(A\cup B)'] \\ &= 1-P(A\cup B) \\ &= 1-P(A)-P(B)+P(A\cap B) \\ &= 1-P(A)-P(B)+P(A)P(B) \\ &= [1-P(A)][1-P(B)] \\ &= P(A')P(B'). \end{aligned}$$

1.4-6
$$\begin{aligned} P[A\cap(B\cap C)] &= P[A\cap B\cap C] \\ &= P(A)P(B)P(C) \\ &= P(A)P(B\cap C). \end{aligned}$$

$$\begin{aligned} P[A\cap(B\cup C)] &= P[(A\cap B)\cup(A\cap C)] \\ &= P(A\cap B)+P(A\cap C)-P(A\cap B\cap C) \\ &= P(A)P(B)+P(A)P(C)-P(A)P(B)P(C) \\ &= P(A)[P(B)+P(C)-P(B\cap C)] \\ &= P(A)P(B\cup C). \end{aligned}$$

$$\begin{aligned} P[A'\cap(B\cap C')] &= P(A'\cap C'\cap B) \\ &= P(B)[P(A'\cap C')\,|\,B] \\ &= P(B)[1-P(A\cup C\,|\,B)] \\ &= P(B)[1-P(A\cup C)] \\ &= P(B)P[(A\cup C)'] \\ &= P(B)P(A'\cap C') \\ &= P(B)P(A')P(C') \\ &= P(A')P(B)P(C') \\ &= P(A')P(B\cap C') \end{aligned}$$

$$\begin{aligned} P[A'\cap B'\cap C'] &= P[(A\cup B\cup C)'] \\ &= 1-P(A\cup B\cup C) \\ &= 1-P(A)-P(B)-P(C)+P(A)P(B)+P(A)P(C)+ \\ &\qquad P(B)P(C)-PA)P(B)P(C) \\ &= [1-P(A)][1-P(B)][1-P(C)] \\ &= P(A')P(B')P(C'). \end{aligned}$$

1.4-8 $\dfrac{1}{6}\cdot\dfrac{2}{6}\cdot\dfrac{3}{6}+\dfrac{1}{6}\cdot\dfrac{4}{6}\cdot\dfrac{3}{6}+\dfrac{5}{6}\cdot\dfrac{2}{6}\cdot\dfrac{3}{6}=\dfrac{2}{9}.$

1.4-10 **(a)** $\dfrac{3}{4}\cdot\dfrac{3}{4}=\dfrac{9}{16}$;

(b) $\dfrac{1}{4}\cdot\dfrac{3}{4}+\dfrac{3}{4}\cdot\dfrac{2}{4}=\dfrac{9}{16}$;

(c) $\dfrac{2}{4}\cdot\dfrac{1}{4}+\dfrac{2}{4}\cdot\dfrac{4}{4}=\dfrac{10}{16}$.

1.4-12 **(a)** $\left(\dfrac{1}{2}\right)^3\left(\dfrac{1}{2}\right)^2$;

(b) $\left(\dfrac{1}{2}\right)^3\left(\dfrac{1}{2}\right)^2$;

(c) $\left(\dfrac{1}{2}\right)^3\left(\dfrac{1}{2}\right)^2$;

(d) $\dfrac{5!}{3!\,2!}\left(\dfrac{1}{2}\right)^3\left(\dfrac{1}{2}\right)^2$.

1.4-14 **(a)** $1-(0.4)^3=1-0.064=0.936$;

(b) $1-(0.4)^8=1-0.00065536=0.99934464$.

1.4-16 **(a)** $\displaystyle\sum_{k=0}^{\infty}\frac{1}{5}\left(\frac{4}{5}\right)^{2k}=\frac{5}{9}$;

(b) $\dfrac{1}{5}+\dfrac{4}{5}\cdot\dfrac{3}{4}\cdot\dfrac{1}{3}+\dfrac{4}{5}\cdot\dfrac{3}{4}\cdot\dfrac{2}{3}\cdot\dfrac{1}{2}\cdot\dfrac{1}{1}=\dfrac{3}{5}$.

1.4-18 **(a)** 7; **(b)** $(1/2)^7$; **(c)** 63; **(d)** No! $(1/2)^{63}=1/9{,}223{,}372{,}036{,}854{,}775{,}808$.

1.4-20 No.

$P(A) = 0.7$, $P(B) = 0.2$

A and B are independent events

hence $P(A \cap B) = P(A) \cdot P(B)$

$= (0.7) \times (0.2)$

(a) $P(A \cap B) = 0.14$

And we know $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.7 + 0.2 - 0.14$

~~$P(A \cap B) =$~~

(b) $P(A \cup B) = 0.76$

intersection sign

$$P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$$

$$= (1 - P(A)) + (1 - P(B)) - P(A') \cdot P(B')$$

$$= (1 - P(A)) + (1 - P(B)) - (1 - P(A)) \cdot (1 - P(B))$$

$$= (1 - 0.7) + (1 - 0.2) - (1 - 0.7) \cdot (1 - 0.2)$$

$$P(A' \cup B') = 0.86$$

2 $P(A) = 0.3 \quad P(B) = 0.6$

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) \cdot P(B)$ [Because A & B are independent]

$P(A \cup B) = 0.3 + 0.6 - 0.3 \times 0.6 = 0.3 \times 0.4 + 0.6$

$= 0.3 \times 2.4 = 0.72$

b) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) \cdot P(B)}{P(B)} = P(A) = 0.3$

If events are independent

But, Here events are mutually exculsive, therefore

$P(A \cap B) = 0$,

so, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = 0$

Given that A and B are independent
and $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{3}$

Hence $P(A') = 1 - \frac{1}{4} = \frac{3}{4}$

and $P(B') = 1 - \frac{2}{3} = \frac{1}{3}$

(a) $P(A \cap B) = P(A) \cdot P(B)$ {for independent events}

$= \frac{1}{4} \times \frac{2}{3}$

$P(A \cap B) = \frac{1}{6}$

(b) $P(A \cap B') = P(A) \cdot P(B')$

$= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

(c) $P(A' \cap B') = P(A') \cdot P(B') = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$

(d) $P[(A \cup B)'] = 1 - P(A \cup B) = 1 - \{P(A) + P(B) - P(A \cap B)\}$

$\cancel{= 1 - \{\frac{2}{3} +}$ $= 1 - \{\frac{1}{4} + \frac{2}{3} - \frac{1}{6}\}$

$P[(A \cup B)'] = \frac{1}{4}$

(e) $P(A' \cap B) = P(A') \cdot P(B) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

4. It is given that A and B are independent events.

So $P(A \cap B) = P(A) \cdot P(B)$

To Prove: A' and B are independent

$$P(A' \cap B) = P(A' | B)\, P(B)$$
$$= \left[1 - P\left(\frac{A}{B}\right)\right] P(B) = P(B) - P(A \cap B)$$
$$= P(B) - P(A) \cdot P(B)$$
$$= P(B)\left[1 - P(A)\right] = P(B) \cdot P(A')$$

So, A' and B are independent event

To prove $P(A' \cap B') = P(A') \cdot P(B')$

$$P((A \cup B)') = P(A' \cap B') = 1 - P(A \cup B)$$
$$P(A' \cap B') = 1 - P(A) - P(B) + P(A) \cdot P(B)$$
$$= P(A') - P(B) \cdot P(A')$$
$$= P(A') \cdot P(B') \quad (\text{H.P.})$$

So, A' and B' are also independent event

5. $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 + 0.5 - 0.9$
$$= 0.4 = P(A) \cdot P(B)$$

Therefore, A and B are independent event because $P(A \cap B) = P(A) \cdot P(B)$

6 A, B, C are mutually independent event

i) To Prove $P(A \cap (B \cap C)) = P(A)\, P(B \cap C)$

$$= P(A)\, P(B)\, P(C)$$

$$P(A \cap (B \cap C)) = P(A) + P(B \cap C) - P(A \cup (B \cap C))$$

we know that $P(A \cup (B \cap C)) = P((A \cup B) \cap (A \cup C))$

$$= P(A \cup B) + P(A \cup C) - P(A \cup B \cup C)$$

$$= \cancel{P(A)} + \cancel{P(B)} - P(A \cap B) + P(A) + \cancel{P(C)} - P(A \cap C)$$

$$- \cancel{P(A)} - \cancel{P(B)} - P(C) + P(A \cap B) + P(B \cap C) + P(A \cap C)$$

$$- P(A \cap B \cap C)$$

$$= P(A) + P(B \cap C) - P(A \cap B \cap C)$$

$$\Rightarrow P(A \cap (B \cap C)) = P(A) + P(B \cap C) - \left[\begin{array}{c} P(A) + P(B \cap C) \\ - P(A \cap B \cap C) \end{array} \right]$$

$$= P(A \cap B \cap C)$$

$$= P(A)\, P(B)\, P(C) \quad (\text{H.P})$$

ii) To Prove $P(A \cap (B \cup C)) = P(A)\, P(B \cup C)$

$$\Rightarrow P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$

$$= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$

$$= P(A) P(B) + P(A) P(C) - P(A) P(B) P(C)$$

$$= P(A) P(B) + P(A) P(C)\, P(B')$$

$$P(A\cap(B\cup C)) = P(A)\left[P(B)+P(C)\left(1-P(B)\right)\right]$$
$$= P(A)\left[P(B)+P(C)-P(B\cap C)\right]$$
$$= P(A)\,P(B\cup C) \quad (H.P)$$

To prove: $P(A'\cap(B\cap C')) = P(A').\,P(B\cap C')$

Proof: $P(A'\cap(B\cap C')) = P(A')+P(B\cap C')-P(A'\cup(B\cap C'))$

L.H.S $\Rightarrow P(A')+P(B\cap C')-P((A'\cup B)\cap(A'\cup C'))$

$$P((A'\cup B)\cap(A'\cup C')) = P(A'\cup B)+P(A'\cup C')$$
$$-P(A'\cup B\cup C')$$
$$= P(A')+P(B)-P(A'\cap B)+P(A')+P(C')-P(A'\cap C')$$
$$-P(A')-P(B)-P(C')+P(A'\cap B)+P(B\cap C')$$
$$+P(A'\cap C')-P(A'\cap B\cap C')$$
$$= P(A')+P(B\cap C')-P(A'\cap B\cap C')$$

$$P(A'\cap(B\cap C')) = P(A'\cap B\cap C')$$
$$= P(A').\,P(B\cap C') \quad (H.P)$$

$$\Rightarrow P(A'\cap B'\cap C') = P(A').\,P(B').\,P(C')$$
$$P(A'\cap B'\cap C') = 1-P(A\cup B\cup C)$$
$$= 1-P(A)-P(B)-P(C)+P(A\cap B)+P(B\cap C)$$
$$+P(A\cap C)-P(A\cap B\cap C)$$
$$= P(A')-P(B)-P(C)+P(A).P(B)+P(B).P(C)$$
$$+P(A)P(C)-P(A)P(B)P(C)$$
$$= P(A')-P(C)-P(A')P(B)+P(B)P(C)$$
$$+P(A)P(B')P(C)$$
$$= P(A')P(B')+P(A)P(B')P(C)-P(B')P(C)$$
$$= P(A')P(B')-P(A')P(B')P(C)$$
$$= P(A')P(B')P(C') \quad (H.P)$$

7 a) $P\left(\begin{array}{c}\text{exactly one} \\ \text{Player succesful}\end{array}\right) = P(A_1 \cap A_2' \cap A_3') + P(A_1' \cap A_2 \cap A_3')$

$+ P(A_1' \cap A_2' \cap A_3)$

$= 0.5 \times 0.3 \times 0.4 + 0.5 \times 0.7 \times 0.4$

$+ 0.5 \times 0.3 \times 0.6$

$= 0.29$

b) $P\left(\begin{array}{c}\text{exactly one} \\ \text{missed goal}\end{array}\right) = P(A \cap A_2 \cap A_3') + P(A_1 \cap A_2' \cap A_3) +$

$P(A_1' \cap A_2 \cap A_3)$

$= 0.5 \times 0.7 \times 0.4 + 0.5 \times 0.3 \times 0.6 + 0.5 \times 0.7 \times 0.6$

$= 0.44$

Let $P(A) \to$ Orange comes up on die A
$P(B) \to$ Orange comes up on die B
$P(C) \to$ ———————— C

Hence $P(A) = \frac{1}{6}$; $P(A') = 1 - \frac{1}{6} = \frac{5}{6}$

$P(B) = \frac{2}{6} = \frac{1}{3}$; $P(B') = 1 - \frac{1}{3} = \frac{2}{3}$

$P(C) = \frac{3}{6} = \frac{1}{2}$; $P(C') = 1 - \frac{1}{2} = \frac{1}{2}$

Let $P(D)$ be the probability that exactly two of the three dice come up orange.

$$P(D) = P(A \cap B \cap C') + P(A \cap C \cap B') + P(B \cap C \cap A')$$

$$P(D) = P(A) \cdot P(B) \cdot P(C') + P(A) \cdot P(C) \cdot P(B') + P(B) \cdot P(C) \cdot P(A')$$

$$P(D) = \frac{1}{6} \times \frac{1}{3} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{2} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{2} \times \frac{5}{6}$$

$$\boxed{P(D) = \frac{2}{9}}$$

A, B and C are mutually independent events

$P(A) = 0.5$, $P(B) = 0.8$, $P(C) = 0.9$

(a) P(all three event occurs) = P(A occurs and B occurs and C occurs)

OR $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$P(A \cap B \cap C) = 0.5 \times 0.8 \times 0.9$

$= 0.36$

(b)

P(exactly two occurs) = P[(A, B occurs and C does not occur) OR (B, C occurs and A does not occur) OR (A, C occur and B does not)]

= P(A, B occur and C does not) + P(B, C occur and A does not) + P(A, C occur and B does not)

$= \cancel{P(A)}.$

$= P[(A \cap B) \cap C'] + P[(B \cap C) \cap A'] + P[(A \cap C) \cap B']$

$= P(A) \cdot P(B) \, P(C') + P(B) \cdot P(C) \cdot P(A') + P(A) \cdot P(C) \cdot P(B')$

$= 0.5 \times 0.8 \times 0.1 + 0.8 \times 0.9 \times 0.5 + 0.5 \times 0.9 \times 0.2$

$= 0.49$

(c) P(non of the events occur) = $P(A' \cap B' \cap C')$

$= P(A') \cdot P(B') \cdot P(C') = 0.5 \times 0.2 \times 0.1$

$= 0.01$

10. D_1: 0333 D_2: 2225 D_3: 1146

A: outcome on $D_1 > D_2$

B: outcome on $D_2 > D_3$

C: outcome on $D_3 > D_1$

a) $P(A) = ?$, Total cases when two dice roll $= 4 \times 4 = 16$

Possible required cases $(3,2) = (D_1, D_2)$

$P(A) = \frac{3 \times 3}{16} = \frac{9}{16}$

b) Required case $(D_2, D_3) = \{(2,1), (5,1), (5,4)\}$

$P(B) = \frac{3}{4} \times \frac{2}{4} + \frac{1}{4} \times \frac{2}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{9}{16} = \frac{9}{16}$

(c) Required case when $(D_3, D_1) = \{(1,0), (4,0), (4,3), (6,0)(6,3)\}$

$P(C) = \frac{2}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4}$

$= \frac{10}{16}$

11. a) If A & B are mutually exclusive event i.e $P(A \cap B) = 0$
No, its need not to compulsory that A & B are independent event
Yes, they can be ~~dep~~ independent if at least one of event ~~A~~ either A or B have 0 probability.

b) $A \subset B$

Since $A \subset B$, $P(A \cap B) = P(A)$, so when $P(A) = 0$, $P(B) = 1$, so A and B can be independent event

For unbiased coin

$P(H) = \frac{1}{2}$; $P(T) = \frac{1}{2}$

(a) $P(HHTHT) = P(H \cap H \cap T \cap H \cap T)$

$= P(H) \cdot P(H) \cdot P(T) \cdot P(H) \cdot P(T)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{32}$

(b) $P(THHHT) = P(T \cap H \cap H \cap H \cap T)$

$= P(T) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(T)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{32}$

(c) $P(HTHTH) = P(H \cap T \cap H \cap T \cap H)$

$= P(H) \cdot P(T) \cdot P(H) \cdot P(T) \cdot P(H)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{32}$

(d) Prob. of three heads occuring in five trials $= {}^5C_3 \{P(H)\}^3 \{P(T)\}^{5-3}$

$= {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2$

$= \frac{10}{32} = \frac{5}{16}$

$$\boxed{2R + 4W}$$

$$P(R) = \frac{2}{6} = \frac{1}{3}$$

$$P(W) = \frac{4}{6} = \frac{2}{3}$$

$$P(WWRWR) = P(W \cap W \cap R \cap W \cap R)$$

$$= P(W) \cdot P(W) \cdot P(R) \cdot P(W) \cdot P(R)$$

$$= \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3}$$

$$= \frac{8}{243}$$

$$P(RWWWR) = P(R \cap W \cap W \cap W \cap R)$$

$$= P(R) \cdot P(W) \cdot P(W) \cdot P(W) \cdot P(R)$$

$$= \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$$

$$= \frac{8}{243}$$

14 Prob of failure of component is $p = 0.4$

Let no of reductanct component is n

$P(\text{system doesn't fail}) = P[(A_1 \cap A_2 \cap A_3 \cdots)']$

$= 1 - P(A_1 \cap A_2 \cdots)$

$= 1 - p^n$

a) when $n = 3$,

$P(\text{system doesn't fail}) = 1 - (0.4)^3 = 0.936$

b) when $n = 8$

$P(\text{system doesn't fail}) = 1 - (0.4)^8 = 0.9993446$

15 Urn = 10R + 10W

a) with replacement

$P(\text{4th white ball is 4th drawn}) = \frac{10}{20} \times \frac{10}{20} \times \frac{10}{20} \times \frac{10}{20} = \frac{1}{16}$

$P(\text{4th white ball in 5th draw}) = P(WWWRW) + P(WWRWW)$

$P(WRWWW) + P(RWWWW)$

$= {}^4C_1 \times \left(\frac{10}{20}\right)^5 = \frac{1}{8}$

$P(\text{4th white ball in 6th draw}) = {}^5C_2 \times \left(\frac{10}{20}\right)^6 = \frac{5 \times 2}{2^6} = \frac{5}{32}$

$P(\text{4th white ball in 7th draw}) = {}^6C_3 \times \left(\frac{10}{20}\right)^7 = \frac{5 \times 4}{2^7} = \frac{5}{32}$

b) Without Replacement

$P(\text{4th W in 4th draw}) = \frac{10}{20} \times \frac{9}{19} \times \frac{8}{18} \times \frac{7}{17} = \frac{14}{323}$

$P(\text{4th W in 5th draw}) = {}^4C_1 \times \frac{10 \times 10 \times 9 \times 8 \times 7}{20 \times 19 \times 18 \times 17 \times 16} \Rightarrow$

$= \frac{35}{323}$ $\left(\text{Alter. } \frac{{}^4C_1 \times {}^{10}C_1 \times {}^{10}C_4}{{}^{20}C_5} \right.$

$P(\text{4th W in 6th draw}) = {}^5C_2 \times \frac{10 \times 9 \times 10 \times 9 \times 8 \times 7}{20 \times 19 \times 18 \times 17 \times 16 \times 15}$

$P(\text{4th W in 6th draw}) = \frac{105}{646}$

$P(\text{4th W in 7th draw}) = {}^{6}C_3 \times \frac{10 \times 9 \times 8 \times 10 \times 9 \times 8 \times 7}{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14}$

$= \frac{30}{323} \times 2 = \frac{60}{323}$

c) Neither model is very good.

5 5ball = 1Win + 4 Loss

a) With replacement

$P\{(W, LLW, LLLLW, \ldots)\} = \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} \ldots$

$= \frac{1}{5}\left[1 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^4 + \ldots\right]$

$= \frac{1}{5} \times \frac{1}{1 - \left(\frac{4}{5}\right)^2} = \frac{5}{9}$

b) Without Replacement:

$P(\{W, LLW, LLLLW\}) = \frac{1}{5} + \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1}$

$= \frac{3}{5}$

17 a) P(at least one "match") = 1 − P(none of them "match")

$= 1 - \left(\frac{11}{12}\right)^{12}$

b) P(at least one other student roll same number) = 1 − P(none of them (other student) roll the same number)

$= 1 - \frac{{}^{12}C_1}{{}^{12}C_1} \times \left(\frac{11}{12}\right)^{11} = 1 - \left(\frac{11}{12}\right)^{11}$

18. a. 7

b. $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$

c. $32+16+8+4+2+1=63$ games

d. No because $\left(\frac{1}{2}\right)^{63} < 1/75$ million

19. a) P(atleast one "match") = 1 − P(no "match")

$$= 1 - \left[\frac{(n-1)}{n} \times \frac{(n-1)}{n} \cdots \text{ n times}\right]$$

$$= 1 - \left(1 - \frac{1}{n}\right)^n$$

b) $$\lim_{n\to\infty} 1 - \left(1-\frac{1}{n}\right)^n = 1 - \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

$$= 1 - e^{\lim_{n\to\infty}\left(\frac{-1}{n}\right)\cdot n}$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e}$$

when $\lim_{x\to 0}(1+x)^{1/x} = e^{\lim_{x\to 0} x \cdot \frac{1}{x}}$

20. $P(\text{zero hits}) = (1-P_1)\times(1-P_2)$

P(one hits) = when A hit but B don't + when B hit but A don't

$$= P_1(1-P_2) + P_2(1-P_1)$$

$P(\text{two hits}) = P_1 P_2$

$P(0 \text{ hits}) = P(1 \text{ hits})$

$\Rightarrow$ $1 - P_1 - P_2 + P_1P_2 = P_1 + P_2 - 2P_1P_2$

$1 + 3P_1P_2 = 2(P_1+P_2)$ ...... i)

$P(0 \text{ hits}) = P(2 \text{ hits})$

$1 - P_1 - P_2 + P_1P_2 = P_1P_2$

$P_1 + P_2 = 1$ ...... ii)

from i) and ii) eqⁿ we get

$$1 + 3P_1(1 - P_1) = 2$$

$$3P_1 - 3P_1^2 = 1$$

$$3P_1^2 - 3P_1 + 1 = 0 \Rightarrow D = 9 - 12 < 0$$

P_1 have imaginary root,

Therefore, no P_1 and P_2 exist such that

$P(0 \text{ hits}) = P(\text{one hits}) = P(2 \text{ hits})$

1.5 BAYES' THEOREM

Exercises

1.5-1. Bowl B_1 contains two white chips, bowl B_2 contains two red chips, bowl B_3 contains two white and two red chips, and bowl B_4 contains three white chips and one red chip. The probabilities of selecting bowl B_1, B_2, B_3, or B_4 are 1/2, 1/4, 1/8, and 1/8, respectively. A bowl is selected using these probabilities and a chip is then drawn at random. Find

(a) $P(W)$, the probability of drawing a white chip.

(b) $P(B_1 \mid W)$, the conditional probability that bowl B_1 had been selected, given that a white chip was drawn.

1.5-2. Bean seeds from supplier A have an 85% germination rate and those from supplier B have a 75% germination rate. A seed-packaging company purchases 40% of its bean seeds from supplier A and 60% from supplier B and mixes these seeds together.

(a) Find the probability $P(G)$ that a seed selected at random from the mixed seeds will germinate.

(b) Given that a seed germinates, find the probability that the seed was purchased from supplier A.

1.5-3. A doctor is concerned about the relationship between blood pressure and irregular heartbeats. Among her patients, she classifies blood pressures as high, normal, or low and heartbeats as regular or irregular and finds that (a) 16% have high blood pressure; (b) 19% have low blood pressure; (c) 17% have an irregular heartbeat; (d) of those with an irregular heartbeat, 35% have high blood pressure; and (e) of those with normal blood pressure, 11% have an irregular heartbeat. What percentage of her patients have a regular heartbeat and low blood pressure?

1.5-4. Assume that an insurance company knows the following probabilities relating to automobile accidents (where the second column refers to the probability that the policyholder has at least one accident during the annual policy period):

Age of Driver	Probability of Accident	Fraction of Company's Insured Drivers
16–25	0.05	0.10
26–50	0.02	0.55
51–65	0.03	0.20
66–90	0.04	0.15

A randomly selected driver from the company's insured drivers has an accident. What is the conditional probability that the driver is in the 16–25 age group?

1.5-5. At a hospital's emergency room, patients are classified and 20% of them are critical, 30% are serious, and 50% are stable. Of the critical ones, 30% die; of the serious, 10% die; and of the stable, 1% die. Given that a patient dies, what is the conditional probability that the patient was classified as critical?

1.5-6. A life insurance company issues standard, preferred, and ultrapreferred policies. Of the company's policyholders of a certain age, 60% have standard policies and a probability of 0.01 of dying in the next year, 30% have preferred policies and a probability of 0.008 of dying in the next year, and 10% have ultrapreferred policies and a probability of 0.007 of dying in the next year. A policyholder of that age dies in the next year. What are the conditional probabilities of the deceased having had a standard, a preferred, and an ultrapreferred policy?

1.5-7. A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not about 5% of the time. The chemist produces compounds with the impurity about 20% of the time; that is, 80% do not have the impurity. A compound is selected at random from the chemist's output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has an impurity?

1.5-8. A store sells four brands of tablets. The least expensive brand, B_1, accounts for 40% of the sales. The other brands (in order of their price) have the following percentages of sales: B_2, 30%; B_3, 20%; and B_4, 10%. The respective probabilities of needing repair during warranty are 0.10 for B_1, 0.05 for B_2, 0.03 for B_3, and 0.02 for B_4. A randomly selected purchaser has a tablet that needs repair under warranty. What are the four conditional probabilities of being brand B_i, $i = 1, 2, 3, 4$?

1.5-9. There is a new diagnostic test for a disease that occurs in about 0.05% of the population. The test is not perfect, but will detect a person with the disease 99% of the time. It will, however, say that a person without the disease has the disease about 3% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What are the conditional probabilities that

(a) the person has the disease?

(b) the person does not have the disease?

Discuss. HINT: Note that the fraction 0.0005 of diseased persons in the population is much smaller than the error probabilities of 0.01 and 0.03.

1.5-10. Suppose we want to investigate the percentage of abused children in a certain population. To do this, doctors examine some of these children taken at random from that population. However, doctors are not perfect: They sometimes classify an abused child (A^+) as one not abused (D^-) or they classify a nonabused child (A^-) as one that is abused (D^+). Suppose these error rates are $P(D^- \mid A^+) = 0.08$ and $P(D^+ \mid A^-) = 0.05$, respectively; thus, $P(D^+ \mid A^+) = 0.92$ and $P(D^- \mid A^-) = 0.95$ are the probabilities of the correct decisions. Let us pretend that only 2% of all children are abused; that is, $P(A^+) = 0.02$ and $P(A^-) = 0.98$.

(a) Select a child at random. What is the probability that the doctor classifies this child as abused? That is, compute

$$P(D^+) = P(A^+)P(D^+ \mid A^+) + P(A^-)P(D^+ \mid A^-).$$

(b) Compute $P(A^- \mid D^+)$ and $P(A^+ \mid D^+)$.

(c) Compute $P(A^- \mid D^-)$ and $P(A^+ \mid D^-)$.

(d) Are the probabilities in (b) and (c) alarming? This happens because the error rates of 0.08 and 0.05 are high relative to the fraction 0.02 of abused children in the population.

1.5-11. At the beginning of a certain study of a group of persons, 15% were classified as heavy smokers, 30% as light smokers, and 55% as nonsmokers. In the five-year study, it was determined that the death rates of the heavy and light smokers were five and three times that of the nonsmokers, respectively. A randomly selected participant died over the five-year period; calculate the probability that the participant was a nonsmoker.

1.5-12. A test indicates the presence of a particular disease 90% of the time when the disease is present and the presence of the disease 2% of the time when the disease is not present. If 0.5% of the population has the disease, calculate the conditional probability that a person selected at random has the disease if the test indicates the presence of the disease.

1.5-13. A hospital receives two fifths of its flu vaccine from Company A and the remainder from Company B. Each shipment contains a large number of vials of vaccine. From Company A, 3% of the vials are ineffective; from Company B, 2% are ineffective. A hospital tests $n = 25$ randomly selected vials from one shipment and finds that 2 are ineffective. What is the conditional probability that this shipment came from Company A?

1.5-14. Two processes of a company produce rolls of materials: The rolls of Process I are 3% defective and the rolls of Process II are 1% defective. Process I produces 60% of the company's output, Process II 40%. A roll is selected at random from the total output. Given that this roll is defective, what is the conditional probability that it is from Process I?

1.5 Bayes' Theorem

1.5-2 **(a)**
$$\begin{aligned} P(G) &= P(A \cap G) + P(B \cap G) \\ &= P(A)P(G \mid A) + P(B)P(G \mid B) \\ &= (0.40)(0.85) + (0.60)(0.75) = 0.79; \end{aligned}$$

(b)
$$\begin{aligned} P(A \mid G) &= \frac{P(A \cap G)}{P(G)} \\ &= \frac{(0.40)(0.85)}{0.79} = 0.43. \end{aligned}$$

1.5-4 Let event B denote an accident and let A_1 be the event that age of the driver is 16–25. Then

$$\begin{aligned} P(A_1 \mid B) &= \frac{(0.1)(0.05)}{(0.1)(0.05) + (0.55)(0.02) + (0.20)(0.03) + (0.15)(0.04)} \\ &= \frac{50}{50 + 110 + 60 + 60} = \frac{50}{280} = 0.179. \end{aligned}$$

1.5-6 Let B be the event that the policyholder dies. Let A_1, A_2, A_3 be the events that the deceased is standard, preferred and ultra-preferred, respectively. Then

$$\begin{aligned}
P(A_1 \,|\, B) &= \frac{(0.60)(0.01)}{(0.60)(0.01)+(0.30)(0.008)+(0.10)(0.007)}\\
&= \frac{60}{60+24+7} = \frac{60}{91} = 0.659;\\
P(A_2 \,|\, B) &= \frac{24}{91} = 0.264;\\
P(A_3 \,|\, B) &= \frac{7}{91} = 0.077.
\end{aligned}$$

1.5-8 Let A be the event that the tablet is under warranty.

$$\begin{aligned}
P(B_1 \,|\, A) &= \frac{(0.40)(0.10)}{(0.40)(0.10)+(0.30)(0.05)+(0.20)(0.03)+(0.10)(0.02)}\\
&= \frac{40}{40+15+6+2} = \frac{40}{63} = 0.635;\\
P(B_2 \,|\, A) &= \frac{15}{63} = 0.238;\\
P(B_3 \,|\, A) &= \frac{6}{63} = 0.095;\\
P(B_4 \,|\, A) &= \frac{2}{63} = 0.032.
\end{aligned}$$

1.5-10 **(a)** $P(D^+) = (0.02)(0.92)+(0.98)(0.05) = 0.0184+0.0490 = 0.0674;$

(b) $P(A^- \,|\, D^+) = \dfrac{0.0490}{0.0674} = 0.727;\ P(A^+ \,|\, D^+) = \dfrac{0.0184}{0.0674} = 0.273;$

(c) $P(A^- \,|\, D^-) = \dfrac{(0.98)(0.95)}{(0.02)(0.08)+(0.98)(0.95)} = \dfrac{9310}{16+9310} = 0.998;$

$P(A^+ \,|\, D^-) = 0.002.$

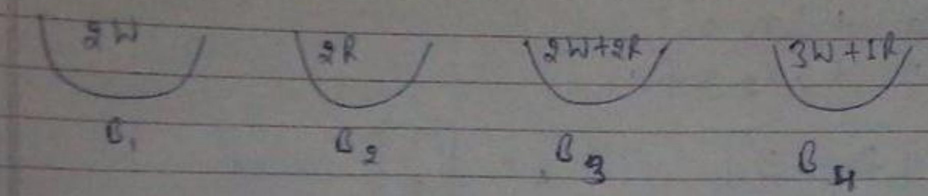
(d) Yes, particularly those in part (b).

1.5-12 Let $D = \{\text{has the disease}\}$, $DP = \{\text{detects presence of disease}\}$. Then

$$\begin{aligned}
P(D \,|\, DP) &= \frac{P(D \cap DP)}{P(DP)}\\
&= \frac{P(D)\cdot P(DP \,|\, D)}{P(D)\cdot P(DP \,|\, D) + P(D')\cdot P(DP \,|\, D')}\\
&= \frac{(0.005)(0.90)}{(0.005)(0.90)+(0.995)(0.02)}\\
&= \frac{0.0045}{0.0045+0.199} = \frac{0.0045}{0.0244} = 0.1844.
\end{aligned}$$

1.5-14 Let $D = \{\text{defective roll}\}$. Then

$$\begin{aligned}
P(I \,|\, D) &= \frac{P(I \cap D)}{P(D)}\\
&= \frac{P(I)\cdot P(D \,|\, I)}{P(I)\cdot P(D \,|\, I) + P(II)\cdot P(D \,|\, II)}\\
&= \frac{(0.60)(0.03)}{(0.60)(0.03)+(0.40)(0.01)}\\
&= \frac{0.018}{0.018+0.004} = \frac{0.018}{0.022} = 0.818.
\end{aligned}$$



Hence $P(W|B_1) = \frac{2}{2} = 1$, $P(W|B_3) = \frac{2}{4} = \frac{1}{2}$.

$P(W|B_2) = \frac{0}{2} = 0$, $P(W|B_4) = \frac{3}{4}$

$$P(W) = P(W \cap B_1) + P(W \cap B_2) + P(W \cap B_3) + P(W \cap B_4)$$

$$= P(B_1) \cdot P(W|B_1) + P(B_2) \cdot P(W|B_2) + P(B_3) \cdot P(W|B_3) + P(B_4) \cdot P(W|B_4)$$

$$= \frac{1}{2} \times 1 + \frac{1}{4} \times (0) + \frac{1}{8} \times \frac{1}{2} + \frac{1}{8} \times \frac{3}{4} = \frac{1}{2} + \frac{1}{16} + \frac{3}{32}$$

$$P(W) = \frac{21}{32}$$

(b) $$P(B_1|W) = \frac{P(B_1) \cdot P(W|B_1)}{P(W)} = \frac{\frac{1}{2} \times 1}{\frac{21}{32}} = \frac{16}{21}$$

P(A) → Prob. that the selected seed is from supplier A

P(B) → Prob. that the selected seed is from supplier B

$P(A) = 40\% = 0.4$

$P(B) = 60\% = 0.6$

$P(G/A) = 85\% = 0.85$

$P(G/B) = 75\% = 0.75$

(a) $P(G) = P(G \cap A) + P(G \cap B)$

$P(G) = P(A) \cdot P(G/A) + P(B) \cdot P(G/B)$

$P(G) = 0.4 \times 0.85 + 0.6 \times 0.75$

$\boxed{P(G) = 0.79}$

(b) $P(A/G) = \dfrac{P(A) \cdot P(G/A)}{P(G)}$

$= \dfrac{0.4 \times 0.85}{0.79}$

$\boxed{P(A/G) = 0.43}$

3

	Blood Pressure			
	high	Normal	Low	Total
regular	0.1005	0.5785	0.151	1−0.17=0.83
irregular	0.35(0.17)=0.0595	0.0715	0.039	0.17
Total	0.16	0.65	0.19	1

(Blue pen → solved part.) ← for table
(Black pen → given in question)

P (Reg. Heatbeat and low bloodpressure) = 0.151
so 15.1 have Regular Heart beat and low B.P.

4 event A: company's insured drivers has an accident
event B: driver is in the 16-25 age group

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.05\times 0.10}{\left(\begin{array}{l}0.05\times0.10 + 0.02\times0.55 + \\ 0.03\times0.20 + 0.04\times0.15\end{array}\right)}$$

$$= \frac{5\times10^{-3}}{5\times10^{-3} + \cancel{57\times10^{-2}}\ 11\times10^{-2} + 6\times10^{-3} + 6\times10^{-3}}$$

$$= \frac{5\times10^{-3}}{12.7\times10^{-2}} = \frac{5}{127} = 0.03937$$

let $C \to$ Critical class

$S \to$ serious class

$T \to$ stable class

$D \to$ Dies

$P(C) = 20\% = 0.2$, $P(S) = 30\% = 0.3$

$P(T) = 50\% = 0.5$

Also given $P(D/C) = 30\% = 0.30$

$P(D|S) = 10\% = 0.10$

$P(D/T) = 1\% = 0.01$

we need to find $P(C|D)$

$$P(D) = P(D \cap C) + P(D \cap S) + P(D \cap T)$$
$$= P(C) \cdot P(D|C) + P(S) \cdot P(D|S) + P(T) \cdot P(D|T)$$
$$= 0.2 \times 0.3 + 0.3 \times 0.1 + 0.5 \times 0.01$$
$$= 0.06 + 0.03 + 0.005$$
$$P(D) = 0.095$$

$$P(C|D) = \frac{P(C) \cdot P(D|C)}{P(D)} = \frac{0.2 \times 0.3}{0.095}$$

$$P(C|D) = 0.632$$

Let $S \rightarrow$ Standard Policy
$T \rightarrow$ Preferred Policy
$U \rightarrow$ Ultrapreferred Policy
$D \rightarrow$ Dies next year

Given $P(S) = 60\% = 0.6$, $P(D|S) = 0.01$
$P(T) = 30\% = 0.3$, $P(D|T) = 0.008$
$P(U) = 10\% = 0.1$, $P(D|U) = 0.007$

$$P(D) = P(D \cap S) + P(D \cap T) + P(D \cap U)$$
$$= P(S) \cdot P(D|S) + P(T) \cdot P(D|T) + P(U) \cdot P(D|U)$$
$$= 0.6 \times 0.01 + 0.3 \times 0.008 + 0.1 \times 0.007$$
$$= 0.006 + 0.0024 + 0.0007$$
$$P(D) = 0.0091$$

$$P(S|D) = \frac{P(S)\,P(D|S)}{P(D)} = \frac{0.6 \times 0.01}{0.0091} = 0.659$$

$$P(T|D) = \frac{P(T) \cdot P(D|T)}{P(D)} = \frac{0.3 \times 0.008}{0.0091} = 0.264$$

$$P(U|D) = \frac{P(U) \cdot P(D|U)}{P(D)} = \frac{0.1 \times 0.007}{0.0091} = 0.0769$$

Let P(A) → Prob that the compound has impurities

P(B) → compound does not have impurities

P(I) → Prob. that the test detects impurity

$P(A) = 20\% = 0.20$

$P(B) = 80\% = 0.80$

$P(I/A) = 0.90$

$P(I|B) = 5\% = 0.05$

$P(I) = P(I \cap A) + P(I \cap B)$

$= P(A) \cdot P(I|A) + P(B) \cdot P(I|B)$

$= 0.20 \times 0.90 + 0.80 \times 0.05$

$P(I) = 0.22$

$$P(A|I) = \frac{P(A) \cdot P(I/A)}{P(I)}$$

$$= \frac{0.20 \times 0.90}{0.22}$$

$$\boxed{P(A|I) = 0.818}$$

Let $P(R)$ be the probability that the tablet needs repair

Given that

$P(B_1) = 40\% = 0.4$

$P(B_2) = 30\% = 0.3$

$P(B_3) = 20\% = 0.2$

$P(B_4) = 10\% = 0.1$

and

$P(R/B_1) = 0.10$

$P(R|B_2) = 0.05$

$P(R|B_3) = 0.03$

$P(R|B_4) = 0.02$

Now

$$P(R) = P(R \cap B_1) + P(R \cap B_2) + P(R \cap B_3) + P(R \cap B_4)$$

$$P(R) = P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) + P(B_3) \cdot P(R|B_3) + P(B_4) \cdot P(R|B_4)$$

$$P(R) = 0.4 \times 0.1 + 0.3 \times 0.05 + 0.2 \times 0.03 + 0.1 \times 0.02$$

$$P(R) = 0.063$$

Hence

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)} = \frac{0.4 \times 0.1}{0.063}$$

$$\boxed{P(B_1|R) = 0.635}$$

$$P(B_2|R) = \frac{P(B_2) \cdot P(R|B_2)}{P(R)} = \frac{0.3 \times 0.05}{0.063}$$

$$\boxed{P(B_2|R) = 0.238}$$

$$P(B_3|R) = \frac{P(B_3) \cdot P(R|B_3)}{P(R)} = \frac{0.2 \times 0.03}{0.063}$$

$$\boxed{P(B_3|R) = 0.095}$$

$$P(B_4|R) = \frac{P(B_4) \cdot P(R|B_4)}{P(R)} = \frac{0.1 \times 0.02}{0.063}$$

$$\boxed{P(B_4|R) = 0.032}$$

9. event A : Person have disease

event B : test indicate, the person has disease

$P(A) = 5 \times 10^{-4}$

$P(B|A) = 0.99$

$P(B|A') = 0.03$

a) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B|A) \times P(A)}{P(B)}$

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= P(B|A) \cdot P(A) + P(B|A') P(A') \\ &= 0.99 \times 5 \times 10^{-4} + 0.03 \times (1 - 5 \times 10^{-4}) \\ &= 0.03 + 0.96 \times 5 \times 10^{-4} \\ &= 3 \times 10^{-2} + 0.048 \times 10^{-2} = 3.048 \times 10^{-2} \end{aligned}$$

$P(A|B) = \dfrac{0.99 \times 5 \times 10^{-4}}{3.048 \times 10^{-2}} = 0.0162$

b) $P(A'|B) = 1 - P(A|B) = 0.9838 \approx 0.984$

10. A^+ : Abused Child D^+ : doctor say abused child

A^- : Non abused child D^- : doctor say non abused child

$P(D^- | A^+) = 0.08 \Rightarrow P(D^+ | A^+) = 0.92$

$P(D^+ | A^-) = 0.05 \Rightarrow P(D^- | A^-) = 0.95$

if $P(A^+) = 0.02$, $P(A^-) = 0.98$

a) $P(D^+) = P(D^+ \cap A^+) + P(D^+ \cap A^-)$

$= P(A^+)\,P(D^+|A^+) + P(A^-)\,P(D^+|A^-)$

$= 0.02 \times 0.92 + 0.98 \times 0.05$

$= 0.0674$

b) $P(A^- | D^+) = \dfrac{P(D^+|A^-)\,P(A^-)}{P(D^+)} = \dfrac{0.05 \times 0.98}{0.0674}$

$= 0.727$

$P(A^+ | D^+) = \dfrac{P(D^+|A^+) \cdot P(A^+)}{P(D^+)} = \dfrac{0.92 \times 0.02}{0.0674}$

$= 0.273$

c) $P(A^- | D^-) = \dfrac{P(D^-|A^-)\,P(A^-)}{P(D^-)} = \dfrac{0.95 \times 0.98}{(1 - 0.0674)}$

$= 0.9983$

$P(A^+ | D^-) = 1 - P(A^- | D^-) = 0.0017$

d) Yes, it is because probability of $P(A^- | D^+)$ is very high (as it tell doctor result is child is abused but he/she actually not)

But in (c) part, it is not alarming

Let $P(H) \rightarrow$ Prob. that a heavy smoker is selected

$P(L) \rightarrow$ Prob. that a light smoker is selected

$P(N) \rightarrow$ Prob. that a non-smoker is selected

Hence $P(H) = 0.15$

$P(L) = 0.30$

$P(N) = 0.55$

As the death rates of heavy and light smokers were five and and three times that of the nonsmokers.

The prob of death of heavy smoker $\underset{P(D|H)}{} = 5p$

and that of light smoker $P(D|L) = 3p$

where $p = P(D|N) \rightarrow$ Prob of death of a nonsmoker

Prob. of death of a participant

$P(D) = P(D \cap H) + P(D \cap L) + P(D \cap N)$

$P(D) = P(H)P(D|H) + P(L) \cdot P(D|L) + P(N)P(D|N)$

$= 0.15 \times 5p + 0.3 \times 3p + 0.55 \times p$

$P(D) = 2.2 \times p$

Prob that the dead participant was a nonsmoker is

$$P(N|D) = \frac{P(N) \cdot P(D|N)}{P(D)} = \frac{0.55 \times \cancel{p}}{2.2 \times \cancel{p}}$$

$\boxed{P(N|D) = 0.25}$

As 0.5% of the population has the disease,

Prob. that the selected person has the disease: ~~P(U) = 2%~~

$$P(U) = 0.5\% = 0.005$$

Prob. that the selected person does not have the disease

$$P(H) = 1 - P(U)$$

$$= 1 - 0.005$$

$$P(H) = 0.995$$

Let $P(I)$ be the prob. that the test indicates the disease

So

$$P(I/U) = 90\% = 0.90$$

$$P(I/H) = 2\% = 0.02$$

$$P(I) = P(I \cap U) + P(I \cap H)$$

$$P(I) = P(U) \cdot P(I/U) + P(H) \cdot P(I/H)$$

$$= 0.005 \times 0.9 + 0.995 \times 0.02$$

$$P(I) = 0.0244$$

$$P(U/I) = \frac{P(U) \cdot P(I/U)}{P(I)} = \frac{0.005 \times 0.90}{0.0244}$$

$$\boxed{P(U/I) = 0.1844}$$

13 flu vaccine $= \frac{2}{5}A + \frac{3}{5}B$

$P(A) \times P(2 \text{ ineffective} | A) = 0.4 \times {}^{25}C_2 \times 0.03 \times 0.03 \times (0.97)^{23}$

$= X$

$\Rightarrow X = 0.0536$

$P(B) \times P(2 \text{ ineffective} | B) = 0.6 \times {}^{25}C_2 \times (0.02)^2 (0.98)^{23}$

$= 0.0452 = Y$

$$P(A | 2 \text{ ineffective}) = \frac{P(2 \text{ ineffective} | A) \times P(A)}{P(2 \text{ ineffective})}$$

$$= \frac{X}{X+Y} = \frac{0.0536}{0.0536 + 0.0452} = 0.54229$$

Prob. that the selected roll is from process I.

$$P(I) = 60\% = 0.60$$

and that from process II

$$P(II) = 40\% = 0.40$$

Let $P(D) \rightarrow$ Prob. that the roll is deffective

$$P(D/I) = 3\% = 0.03$$

$$P(D/II) = 1\% = 0.01$$

$$P(D) = P(D \cap I) + P(D \cap II)$$

$$= P(I) \cdot P(D/I) + P(II) \cdot P(D/II)$$

$$= 0.60 \times 0.03 + 0.40 \times 0.01$$

$$P(D) = 0.022$$

$$P(I|D) = \frac{P(I) \cdot P(D/I)}{P(D)}$$

$$P(I|D) = \frac{0.60 \times 0.03}{0.022}$$

$$\boxed{P(I|D) = 0.818}$$

Chapter 1

1.1-1 0.68.

1.1-3 **(a)** 12/52; **(b)** 2/52; **(c)** 16/52; **(d)** 1; **(e)** 0.

1.1-5 **(a)** 1/6; **(b)** 5/6; **(c)** 1.

1.1-7 0.63.

1.1-9 **(a)** $3(1/3) - 3(1/3)^2 + (1/3)^3$;
(b) $P(A_1 \cup A_2 \cup A_3) = 1 - [1 - 3(1/3) + 3(1/3)^2 - (1/3)^3] = 1 - (1 - 1/3)^3$.

1.1-11 **(a)** $S = \{00, 0, 1, 2, 3, \ldots, 36\}$;
(b) $P(A) = 2/38$;
(c) $P(B) = 4/38$;
(d) $P(D) = 18/38$.

1.1-13 2/3.

1.2-1 4096.

1.2-3 **(a)** 6,760,000; **(b)** 17,576,000.

1.2-5 **(a)** 24; **(b)** 256.

1.2-7 **(a)** 0.0024; **(b)** 0.0012; **(c)** 0.0006; **(d)** 0.0004.

1.2-9 **(a)** 2; **(b)** 8; **(c)** 20; **(d)** 40.

1.2-11 **(a)** 362,880; **(b)** 84; **(c)** 512.

1.2-13 **(a)** 0.00539; **(b)** 0.00882; **(c)** 0.00539; **(d)** Yes.

1.2-17 **(a)** 0.00024; **(b)** 0.00144; **(c)** 0.02113; **(d)** 0.04754; **(e)** 0.42257.

1.3-1 **(a)** 5000/1,000,000; **(b)** 78,515/1,000,000; **(c)** 73,630/995,000; **(d)** 4,885/78,515.

1.3-3 **(a)** 5/35; **(b)** 26/35; **(c)** 5/19; **(d)** 9/23; **(e)** Left.

1.3-5 **(a)** $S = \{(R, R), (R, W), (W, R), (W, W)\}$; **(b)** 1/3.

1.3-7 1/5.

1.3-9 **(f)** $1 - 1/e$.

1.3-11 **(a)** 365^r; **(b)** ${}_{365}P_r$; **(c)** $1 - {}_{365}P_r/365^r$; **(d)** 23.

1.3-13 **(b)** 8/36; **(c)** 5/11; **(e)** $8/36 + 2[(5/36)(5/11) + (4/36)(4/10) + (3/36)(3/9)] = 0.49293$.

1.3-15 11.

1.4-1 **(a)** 0.14; **(b)** 0.76; **(c)** 0.86.

1.4-3 **(a)** 1/6; **(b)** 1/12; **(c)** 1/4; **(d)** 1/4; **(e)** 1/2.

1.4-5 Yes; $0.9 = 0.8 + 0.5 - (0.8)(0.5)$.

1.4-7 **(a)** 0.29; **(b)** 0.44.

1.4-9 **(a)** 0.36; **(b)** 0.49; **(c)** 0.01.

1.4-11 **(a)** No, unless $P(A) = 0$ or $P(B) = 0$;
(b) Only if $P(A) = 0$ or $P(B) = 1$.

1.4-13 $(2/3)^3(1/3)^2$; $(2/3)^3(1/3)^2$.

1.4-15 **(a)** $1/16, 1/8, 5/32, 5/32$;
(b) $14/323, 35/323, 105/646, 60/323$;
(c) Neither model is very good.

1.4-17 **(a)** $1 - (11/12)^{12}$; **(b)** $1 - (11/12)^{11}$.

1.4-19 **(b)** $1 - 1/e$.

1.5-1 **(a)** 21/32; **(b)** 16/21.

1.5-3 15.1%.

1.5-5 $60/95 = 0.632$.

1.5-7 0.8182.

1.5-9 **(a)** $495/30{,}480 = 0.016$; **(b)** $29{,}985/30{,}480 = 0.984$.

1.5-11 1/4.

1.5-13 0.54229.

Chapter 2 Discrete Distributions

2.1 Random Variables of the Discrete Type

Exercises

2.1-1. Let the pmf of X be defined by $f(x) = x/9$, $x = 2, 3, 4$.

(a) Draw a line graph for this pmf.

(b) Draw a probability histogram for this pmf.

2.1-2. Let a chip be taken at random from a bowl that contains six white chips, three red chips, and one blue chip. Let the random variable $X = 1$ if the outcome is a white chip, let $X = 5$ if the outcome is a red chip, and let $X = 10$ if the outcome is a blue chip.

(a) Find the pmf of X.

(b) Graph the pmf as a line graph.

2.1-3. For each of the following, determine the constant c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a line graph:

(a) $f(x) = x/c, \quad x = 1, 2, 3, 4.$

(b) $f(x) = cx, \quad x = 1, 2, 3, \ldots, 10.$

(c) $f(x) = c(1/4)^x, \quad x = 1, 2, 3, \ldots.$

(d) $f(x) = c(x + 1)^2, \quad x = 0, 1, 2, 3.$

(e) $f(x) = x/c, \quad x = 1, 2, 3, \ldots, n.$

(f) $f(x) = \dfrac{c}{(x+1)(x+2)}, \quad x = 0, 1, 2, 3, \ldots.$

Hint: In part (f), write $f(x) = 1/(x + 1) - 1/(x + 2)$.

2.1-4. The state of Michigan generates a three-digit number at random twice a day, seven days a week for its Daily 3 game. The numbers are generated one digit at a time. Consider the following set of 50 three-digit numbers as 150 one-digit integers that were generated at random:

169	938	506	757	594	656	444	809	321	545
732	146	713	448	861	612	881	782	209	752
571	701	852	924	766	633	696	023	601	789
137	098	534	826	642	750	827	689	979	000
933	451	945	464	876	866	236	617	418	988

Let X denote the outcome when a single digit is generated.

(a) With true random numbers, what is the pmf of X? Draw the probability histogram.

(b) For the 150 observations, determine the relative frequencies of 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, respectively.

(c) Draw the relative frequency histogram of the observations on the same graph paper as that of the probability histogram. Use a colored or dashed line for the relative frequency histogram.

2.1-5. The pmf of X is $f(x) = (5 - x)/10$, $x = 1, 2, 3, 4$.

(a) Graph the pmf as a line graph.

(b) Use the following independent observations of X, simulated on a computer, to construct a table like Table 2.1-1:

3	1	2	2	3	2	2	2	1	3	3	2	3	2	4	4	2	1	1	3
3	1	2	2	1	1	4	2	3	1	1	1	2	1	3	1	1	3	3	1
1	1	1	1	1	4	1	3	1	2	4	1	1	2	3	4	3	1	4	2
2	1	3	2	1	4	1	1	1	2	1	3	4	3	2	1	4	4	1	3
2	2	2	1	2	3	1	1	4	2	1	4	2	1	2	3	1	4	2	3

(c) Construct a probability histogram and a relative frequency histogram like Figure 2.1-3.

2.1-6. Let a random experiment be the casting of a pair of fair dice, each having six faces, and let the random variable X denote the sum of the dice.

(a) With reasonable assumptions, determine the pmf $f(x)$ of X. Hint: Picture the sample space consisting of the 36 points (result on first die, result on second die), and assume that each has probability 1/36. Find the probability of each possible outcome of X, namely, $x = 2, 3, 4, \ldots, 12$.

(b) Draw a probability histogram for $f(x)$.

2.1-7. Let a random experiment be the casting of a pair of fair six-sided dice and let X equal the minimum of the two outcomes.

(a) With reasonable assumptions, find the pmf of X.

(b) Draw a probability histogram of the pmf of X.

(c) Let Y equal the range of the two outcomes (i.e., the absolute value of the difference of the largest and the smallest outcomes). Determine the pmf $g(y)$ of Y for $y = 0, 1, 2, 3, 4, 5$.

(d) Draw a probability histogram for $g(y)$.

2.1-8. A fair four-sided die has two faces numbered 0 and two faces numbered 2. Another fair four-sided die has its faces numbered 0, 1, 4, and 5. The two dice are rolled. Let X and Y be the respective outcomes of the roll. Let $W = X + Y$.

(a) Determine the pmf of W.

(b) Draw a probability histogram of the pmf of W.

2.1-9. The pmf of X is $f(x) = (1 + |x - 3|)/11$, for $x = 1, 2, 3, 4, 5$. Graph this pmf as a line graph.

2.1-10. Suppose there are 3 defective items in a lot (collection) of 50 items. A sample of size 10 is taken at random and without replacement. Let X denote the number of defective items in the sample. Find the probability that the sample contains

(a) Exactly one defective item.

(b) At most one defective item.

2.1-11. In a lot (collection) of 100 light bulbs, there are 5 defective bulbs. An inspector inspects 10 bulbs selected at random. Find the probability of finding at least one defective bulb. HINT: First compute the probability of finding no defectives in the sample.

2.1-12. Let X be the number of accidents per week in a factory. Let the pmf of X be

$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}, \qquad x = 0, 1, 2, \ldots.$$

Find the conditional probability of $X \geq 4$, given that $X \geq 1$.

2.1-13. A professor gave her students six essay questions from which she will select three for a test. A student has time to study for only three of these questions. What is the probability that, of the questions studied,

(a) at least one is selected for the test?

(b) all three are selected?

(c) exactly two are selected?

2.1-14. Often in buying a product at a supermarket, there is a concern about the item being underweight. Suppose there are 20 "one-pound" packages of frozen ground turkey on display and 3 of them are underweight. A consumer group buys 5 of the 20 packages at random. What is the probability of at least one of the five being underweight?

2.1-15. Five cards are selected at random without replacement from a standard, thoroughly shuffled 52-card deck of playing cards. Let X equal the number of face cards (kings, queens, jacks) in the hand. Forty observations of X yielded the following data:

2 1 2 1 0 0 1 0 1 1 0 2 0 2 3 0 1 1 0 3

1 2 0 2 0 2 0 1 0 1 1 2 1 0 1 1 2 1 1 0

(a) Argue that the pmf of X is

$$f(x) = \frac{\binom{12}{x}\binom{40}{5-x}}{\binom{52}{5}}, \qquad x = 0, 1, 2, 3, 4, 5,$$

and thus, that $f(0) = 2109/8330$, $f(1) = 703/1666$, $f(2) = 209/833$, $f(3) = 55/833$, $f(4) = 165/21{,}658$, and $f(5) = 33/108{,}290$.

(b) Draw a probability histogram for this distribution.

(c) Determine the relative frequencies of $0, 1, 2, 3$, and superimpose the relative frequency histogram on your probability histogram.

2.1-16. (Michigan Mathematics Prize Competition, 1992, Part II) From the set $\{1, 2, 3, \ldots, n\}$, k distinct integers are selected at random and arranged in numerical order (from lowest to highest). Let $P(i, r, k, n)$ denote the probability that integer i is in position r. For example, observe that $P(1, 2, k, n) = 0$, as it is impossible for the number 1 to be in the second position after ordering.

(a) Compute $P(2, 1, 6, 10)$.

(b) Find a general formula for $P(i, r, k, n)$.

2.1-17. A bag contains 144 ping-pong balls. More than half of the balls are painted orange and the rest are painted blue. Two balls are drawn at random without replacement. The probability of drawing two balls of the same color is the same as the probability of drawing two balls of different colors. How many orange balls are in the bag?

2.1 Random Variables of the Discrete Type

2.1-2 **(a)**

$$f(x) = \begin{cases} 8/13, & x = 1, \\ 4/13, & x = 4, \\ 1/13, & x = 8, \end{cases}$$

(b)

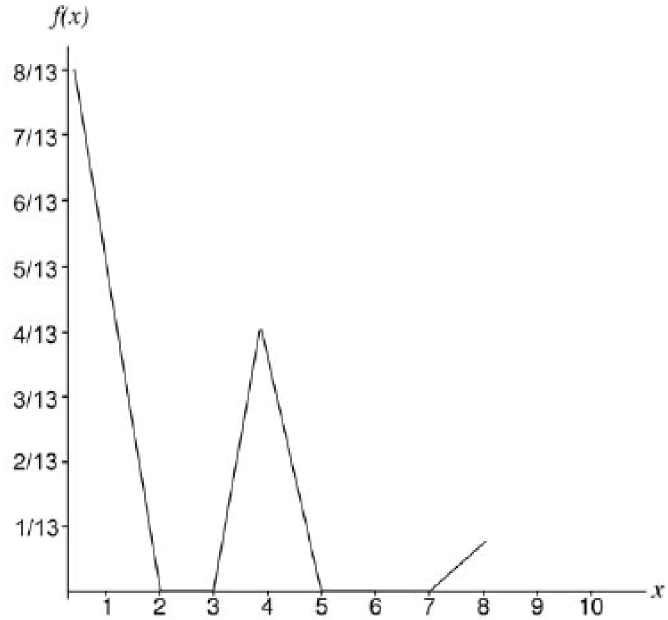


Figure 2.1–2: Line graph.

2.1-4 **(a)** $f(x) = \dfrac{1}{10}, \qquad x = 0, 1, 2, \cdots, 9;$

(b)
$\mathcal{N}(\{0\})/150 = 11/150 = 0.073;$ $\mathcal{N}(\{5\})/150 = 13/150 = 0.087;$
$\mathcal{N}(\{1\})/150 = 14/150 = 0.093;$ $\mathcal{N}(\{6\})/150 = 22/150 = 0.147;$
$\mathcal{N}(\{2\})/150 = 13/150 = 0.087;$ $\mathcal{N}(\{7\})/150 = 16/150 = 0.107;$
$\mathcal{N}(\{3\})/150 = 12/150 = 0.080;$ $\mathcal{N}(\{8\})/150 = 18/150 = 0.120;$
$\mathcal{N}(\{4\})/150 = 16/150 = 0.107;$ $\mathcal{N}(\{9\})/150 = 15/150 = 0.100.$

(c)

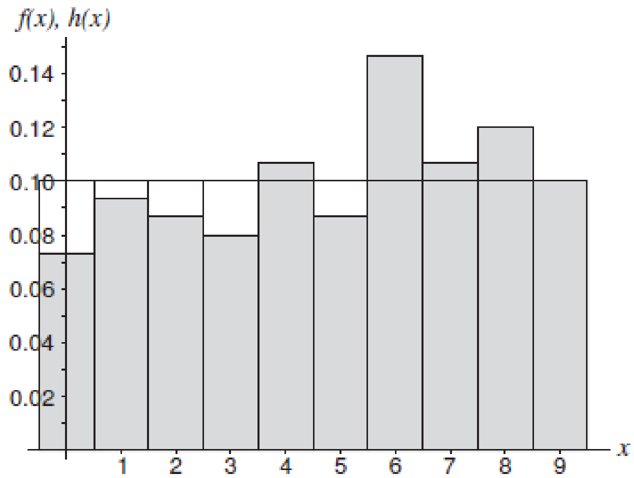


Figure 2.1–4: Michigan daily lottery digits

2.1-6 **(a)** $f(x) = \dfrac{6 - |\,7 - x\,|}{36}$, $x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$

(b)

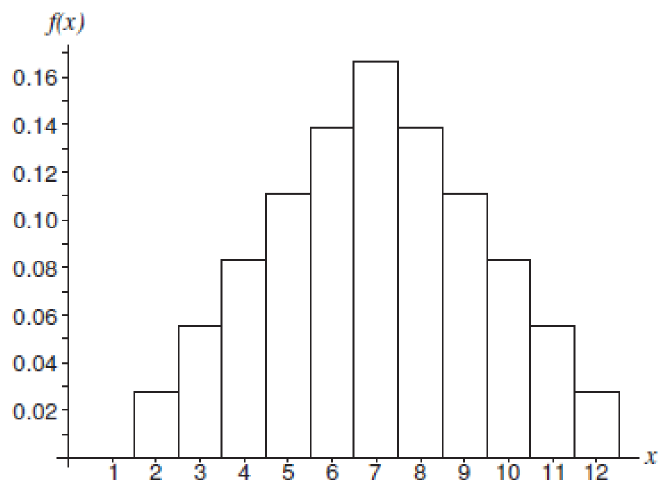


Figure 2.1–6: Probability histogram for the sum of a pair of dice

2.1-8 **(a)** The space of W is $S = \{1, 3, 5, 7, 9, 11, 13\}$.

$$f(w) = \begin{cases} 1/12, & w = 1 \\ 1/6, & w = 3 \\ 1/6, & w = 5 \\ 1/6, & w = 7 \\ 1/6, & w = 9 \\ 1/6, & w = 11 \\ 1/12, & w = 13 \end{cases}$$

(b)

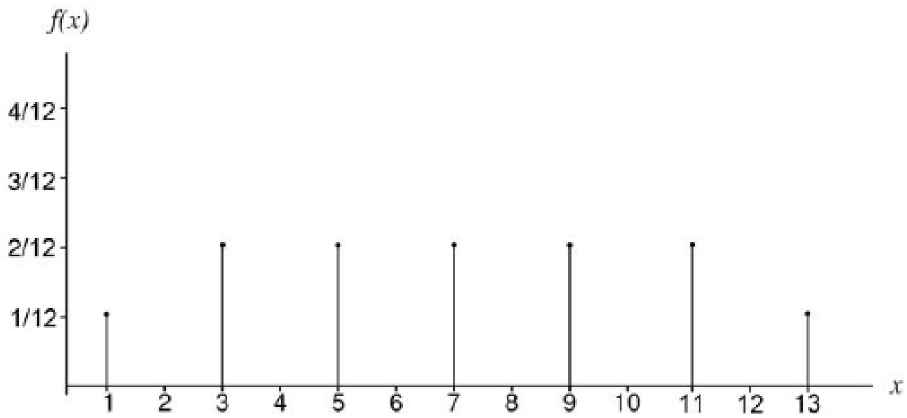


Figure 2.1–8: Probability histogram of sum of two special dice

2.1-10 **(a)** Probability of the sample containing exactly one defective item

$$= \frac{{}^{4}C_1 \times {}^{96}C_{19}}{{}^{100}C_{20}} = 0.42$$

(b) Probability that the sample contains at most one defective item

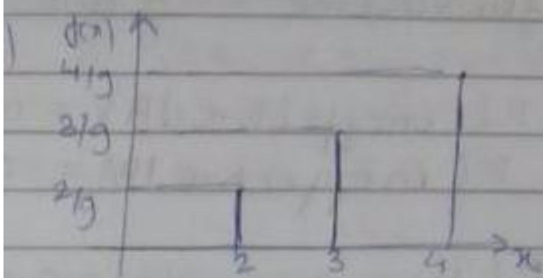
$$= \frac{{}^{4}C_0 \times {}^{96}C_{20}}{{}^{100}C_{20}} + \frac{{}^{4}C_1 \times {}^{96}C_{19}}{{}^{100}C_{20}} = 0.4$$

2.1-12
$$\begin{aligned} P(X \geq 4 \,|\, X \geq 1) &= \frac{P(X \geq 4)}{P(X \geq 1)} = \frac{1 - P(X \leq 3)}{1 - P(X = 0)} \\ &= \frac{1 - [1 - 1/2 + 1/2 - 1/3 + 1/3 - 1/4 + 1/4 - 1/5]}{1 - [1 - 1/2]} = \frac{2}{5}. \end{aligned}$$

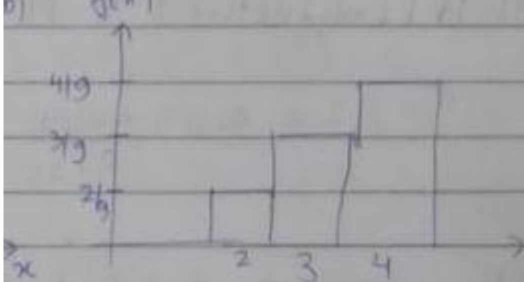
2.1-14 $P(X \geq 1) = 1 - P(X = 0) = 1 - \dfrac{\dbinom{3}{0}\dbinom{17}{5}}{\dbinom{20}{5}} = 1 - \dfrac{91}{228} = \dfrac{137}{228} = 0.60.$

2.1-16 **(a)** $P(2, 1, 6, 10)$ means that 2 is in position 1 so 1 cannot be selected. Thus
$$P(2, 1, 6, 10) = \frac{\dbinom{1}{0}\dbinom{1}{1}\dbinom{8}{5}}{\dbinom{10}{6}} = \frac{56}{210} = \frac{4}{15};$$

(b) $$P(i, r, k, n) = \frac{\dbinom{i-1}{r-1}\dbinom{1}{1}\dbinom{n-i}{k-r}}{\dbinom{n}{k}}.$$



b)

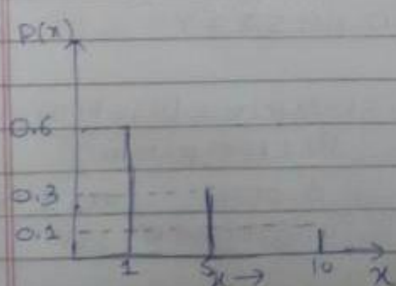


2 Bowl = 6W + 3R + 1B

a) pmf of X, when $X = 1$, $P(X=1) = \frac{6}{10}$

$P(X=5) = \frac{3}{10}$ $\qquad$ $P(X=10) = \frac{1}{10}$

b)

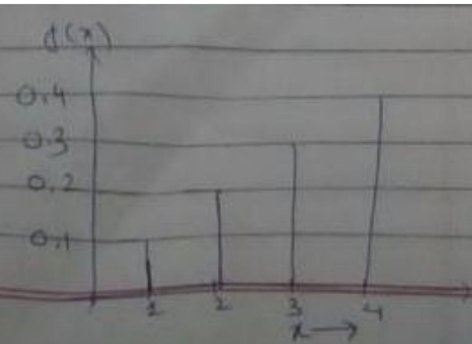


f(x)

3. $\sum_{x} f(x) = 1$

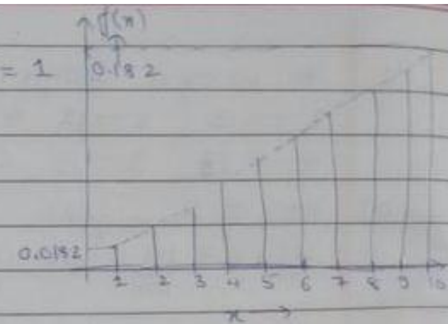
a) $f(x) = \frac{x}{c} \Rightarrow \frac{1+2+3+4}{c} = 1$

$c = 10$

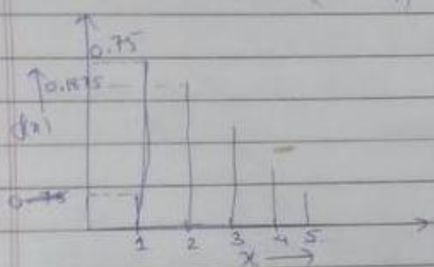


b) $f(x) = cx \Rightarrow c(1+2+\dots 10) = 1$

$c = \frac{1}{55}$

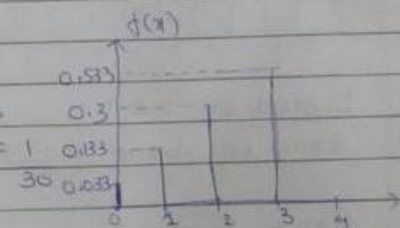


(c) $f(x) = c\left(\frac{1}{4}\right)^x \Rightarrow c\left(\frac{1/4}{1-1/4}\right) = 1 \Rightarrow c = 3$



d) $f(x) = c(x+1)^2 \quad x = 0, 1, 2, 3$

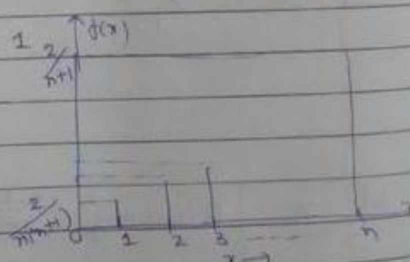
$\Rightarrow c\,[1+4+9+16] = 1 \Rightarrow c = \frac{1}{30}$



e) $f(x) = \frac{x}{c} \Rightarrow \frac{1+2+3+\dots n}{c} = 1$

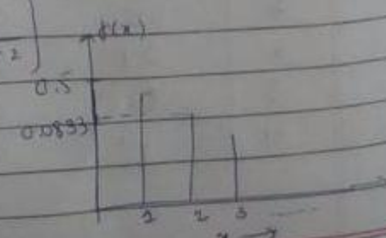
$\Rightarrow c = \left(\frac{2}{n(n+1)}\right)^{-1}$

$c = \frac{n(n+1)}{2}$



(f) $f(x) = \frac{c}{(x+1)(x+2)} = c\left[\frac{1}{x+1} - \frac{1}{x+2}\right]$

$c\,[1 - 0] = 1 \Rightarrow c = 1$

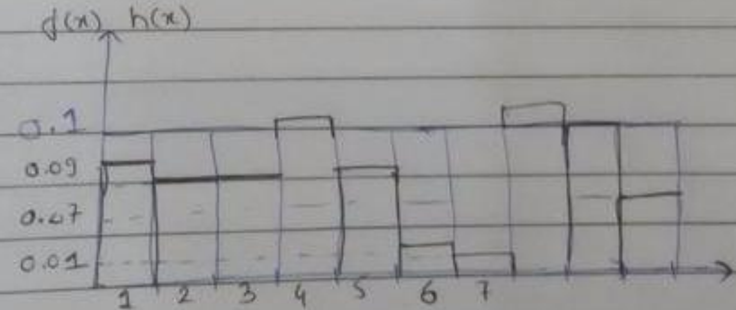


4. $1 \to 14$ time $2 \to 13$ $3 \to 12$ $4 \to 16$ $5 \to 13$ $6 \to 22$
$7 \to 16$ $8 \to 18$ $9 \to 15$ $0 \to 11$ time

a) pmf of x is $f(x)$
$f(x = x_i) = \frac{1}{10}$, where x_i 1 to 9 or 0

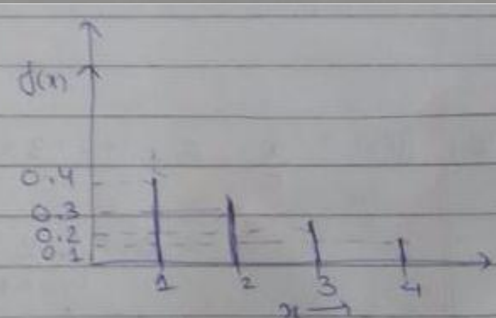
b) or (c)

x_i	1	2	3	4	5	6	7	8	9	0
relative freq	0.093	0.0867	0.08	0.1067	0.0867	0.1467	0.1067	0.12	0.1	0.073



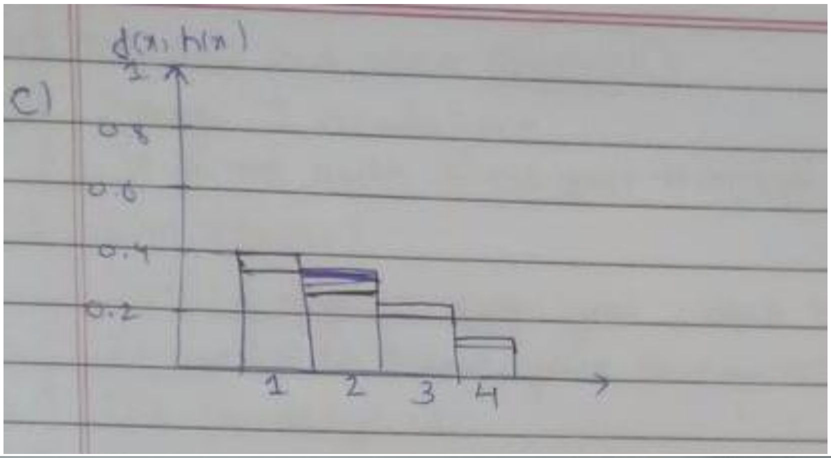
c) In above graph

5. a) $f(x) = \frac{5-x}{10}$



b)

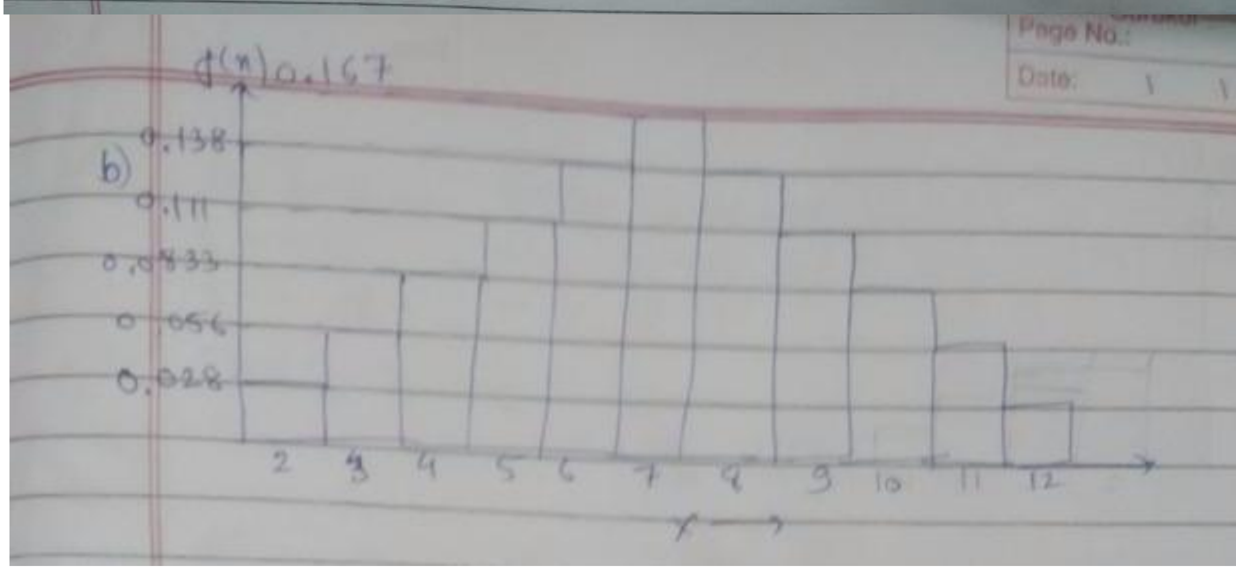
x	Frequency	Relative frequency	$f(x)$
1	38	0.38	0.40
2	27	0.27	0.30
3	21	0.21	0.20
4	14	0.14	0.10



6. a) $P(X=2) = \frac{1}{36}$ $P(X=3) = \frac{2}{36}$ $P(X=4) = \frac{3}{36}$

$P(X=5) = \frac{4}{36}$ $P(X=6) = \frac{5}{36}$ $P(X=7) = \frac{6}{36}$ $P(X=8) = \frac{5}{36}$

$P(X=9) = \frac{4}{36}$ $P(X=10) = \frac{3}{36}$ $P(X=11) = \frac{2}{36}$ $P(X=12) = \frac{1}{36}$



7 a) $P(X = x_i) = \frac{(6-x_i)\times 2 + 1}{36} = \frac{13-2x_i}{36}$

where $x_i = 1, 2, 3, 4, 5, 6$

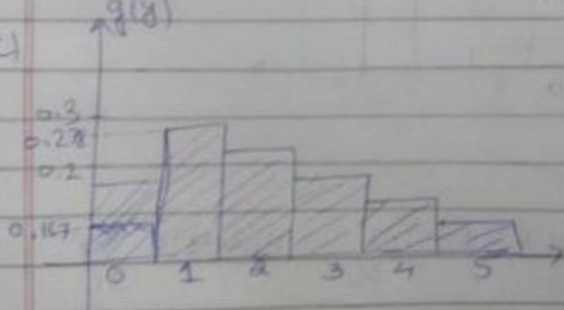
Therefore $f(x) = \frac{13-2x}{36}$, $x = 1, 2, 3, 4, 5, 6$

b) $P(Y = y_i) = ?$

$g(0) = \frac{6}{36}$ $g(1) = \frac{2\times(5)}{36} = \frac{10}{36}$ & similarly

$g(y) = \frac{2\times(6-y)}{36} = \frac{12-2y}{36}$, where $y = 1, 2, 3, 4, 5$

c)



$g(0) = 0.167$ $g(1) = 0.278$ $g(2) = 0.22$ $g(3) = 0.167$

$g(4) = 0.111$ $g(5) = 0.056$

8. 1st Dice → 2 face of "0" and 2 face of "2"
2nd Dice → faces have number 0, 1, 4, 5
$W = X + Y$
Possible value of $W = 0, 1, 2, 3, 4, 5, 6, 7$

a) $f(W=0) = \frac{2}{4} \times \frac{1}{4} = \frac{2}{16} = 0.125$

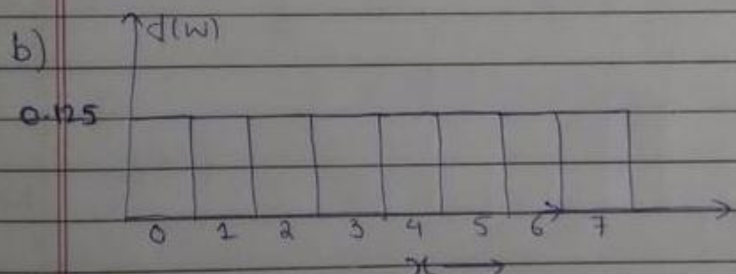
$f(W=1) = \frac{2}{4} \times \frac{1}{4} = 0.125$ $\quad f(W=2) = \frac{2}{4} \times \frac{1}{4} = 0.125$

$f(W=3) = \frac{2}{4} \times \frac{1}{4} = 0.125$ $\quad f(W=4) = \frac{2}{4} \times \frac{1}{4} = 0.125$

$f(W=5) = \frac{2}{4} \times \frac{1}{4} = 0.125$ $\quad f(W=6) = \frac{2}{4} \times \frac{1}{4} = 0.125$

$f(W=7) = \frac{2}{4} \times \frac{1}{4} = 0.125$

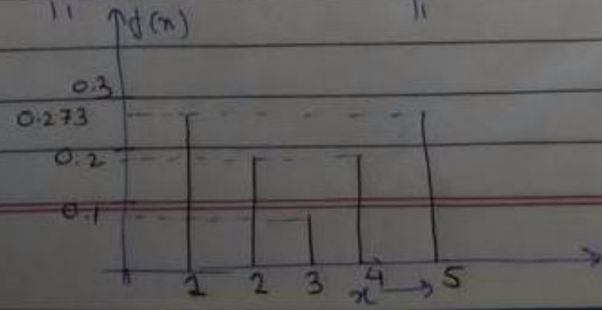
$\therefore f(W) = 0.125$, for $W = 0, 1, 2, 3, 4, 5, 6, 7$

b)


9 $f(x) = \frac{(1 + |x-3|)}{11}$, $x = 1, 2, 3, 4, 5$

$f(1) = \frac{3}{11} = 0.273$; $f(2) = \frac{2}{11} = 0.182$; $f(3) = \frac{1}{11} = 0.091$

$f(4) = \frac{2}{11} = 0.182$; $f(5) = \frac{3}{11} = 0.273$



10. 50 item = 3 defective + 47 not defective

a) $P(X=1) = \frac{{}^{3}C_{1} \times {}^{47}C_{9}}{{}^{50}C_{10}} = \frac{39}{98} = 0.397959$

b) $P(X \le 1) = P(X=0) + P(X=1)$

$= \frac{{}^{3}C_{0} \times {}^{47}C_{10}}{{}^{50}C_{10}} + \frac{{}^{3}C_{1} \times {}^{47}C_{9}}{{}^{50}C_{10}} = \frac{247}{490} + \frac{39}{98} = \frac{221}{245}$

$= 0.902041$

1 100 buls = 5 defective + 95 don't

X: no. of defective bulb in 10 bulbs selected

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - \frac{{}^{5}C_{0} \times {}^{95}C_{10}}{{}^{100}C_{10}} = 1 - 0.583752$

$= 0.416248$

12 $f(x) = \frac{1}{x+1} - \frac{1}{x+2}$, $x = 0, 1, 2, \ldots$

$$P(X \geq 4 \mid x \geq 1) = \frac{P((X \geq 4) \cap (X \geq 1))}{P(X \geq 1)}$$

$$= \frac{P(X \geq 4)}{P(X \geq 1)} = \frac{\sum_{4}^{\infty} f(x)}{\sum_{1}^{\infty} f(x)} = \frac{1/1+4}{1/1+1} = \frac{2}{5}$$

$$= 0.4$$

13 a) $P\left(\frac{\text{at least one}}{\text{selected}}\right) = 1 - P\left(\frac{\text{none of them}}{\text{selected}}\right)$

Let $\frac{\text{6 essay}}{\text{question}} = \frac{\text{3 are}}{\text{readed}} + \frac{\text{3 aren't}}{\text{readed}}$

$$P\left(\frac{\text{at leas one}}{\text{selected}}\right) = 1 - \frac{^3C_3}{^6C_3} = 1 - \frac{1}{20} = \frac{19}{20}$$

$$= 0.95$$

b) $P\left(\frac{\text{all three are}}{\text{selected}}\right) = \frac{^3C_3}{^6C_3} = \frac{1}{20} = 0.05$

c) $P\left(\frac{\text{exactly 2 are}}{\text{seleted}}\right) = \frac{^3C_2 \times {}^3C_1}{^6C_3} = \frac{9}{20} = 0.45$

4. 20 packages = 3 underweight + 17 normal

$P\left(\frac{\text{at least one of}}{\text{five being underweight}}\right) = 1 - P\left(\frac{\text{non of them is}}{\text{underweight}}\right)$

$$= 1 - \frac{^{17}C_5}{^{20}C_5} = 1 - \frac{6188}{15504} \approx 0.6009$$

15 a) X: no of face cards.

12 = total no of face cards   40 = total no. of unface cards

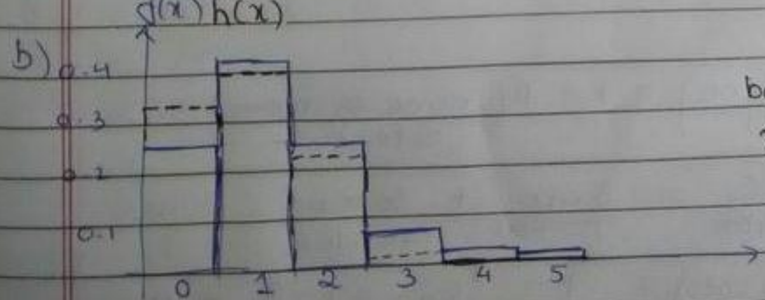
$$f(x) = \frac{{}^{12}C_x * {}^{40}C_{5-x}}{{}^{52}C_x}, \quad x = 0,1,2,3,4,5$$

that's because ${}^{12}C_x$ tells in how many ways we can choose x card from 12.

${}^{40}C_{5-x}$ tells in how many ways (5-x) cards can be choosen

$f(0) = 0.2532$ $f(1) = 0.42197$ $f(2) = 0.251$

$f(3) = 0.066$ $f(4) = 0.00762$ $f(5) = 0.00030$

b) 

both probility and (—)
relative freq graph (---)

c) In given table

x_i	times	h(x), relative freq
0	13	13/40 = 0.325
1	16	16/40 = 0.4
2	9	9/40 = 0.225
3	2	2/40 = 0.05
4	0	0
5	0	0

16 a) $P(2,1,6,10) = \frac{{}^{8}C_5}{{}^{10}C_6} = \frac{4}{15}$

b) $P(i,r,k,n) = \begin{cases} \frac{{}^{i-1}C_{r-1} \times {}^{n-i}C_{k-r}}{{}^{n}C_k} & \text{if } i \geq r \\ 0 & \text{if } i < r \end{cases}$

7. 144 balls = x orange + $(144-x)$ blue

$P\left(\substack{\text{drawing two ball}\\\text{of same color}}\right) = P\left(\substack{\text{drawing two ball of}\\\text{diff color}}\right)$ ...... i)

$P\left(\substack{\text{drawing two ball}\\\text{of same color}}\right) = P(OO) + P(BB)$

$= \frac{{}^{x}C_2}{{}^{144}C_2} + \frac{{}^{144-x}C_2}{{}^{144}C_2}$

$P\left(\substack{\text{drawing two ball}\\\text{of different color}}\right) = P(OB) = \frac{{}^{x}C_1 \, {}^{144-x}C_1}{{}^{144}C_2}$

from i) eqn. we get $\Rightarrow {}^{x}C_2 + {}^{144-x}C_2 = {}^{x}C_1 \, {}^{144-x}C_1$

$\Rightarrow \frac{x(x-1)}{2} + \frac{(144-x)(143-x)}{2} = x(144-x)$

$x(x-1) + (144-x)(143-x) = 2x(144-x)$

$x(x-1) + 143(144-x) = 3x(144-x)$

$x^2 - x + 20592 - 143x = 3 \times 144x - 3x^2$

$4x^2 - 4 \times 144x + 20592 = 0$

$x^2 - 144x + 5148 = 0$

$x = \frac{144 \pm \sqrt{(144)^2 - 4(5148)}}{2} = 78, 66$

Since $x > \frac{144}{2} \Rightarrow x > 72$

Therefore $x = 78$, 78 orange balls are there

2.2 MATHEMATICAL EXPECTATION

Exercises

2.2-1. Find $E(X)$ for each of the distributions given in Exercise 2.1-3.

2.2-2. Let the random variable X have the pmf

$$f(x) = \frac{(|x|+1)^2}{9}, \qquad x = -1, 0, 1.$$

Compute $E(X)$, $E(X^2)$, and $E(3X^2 - 2X + 4)$.

2.2-3. Let the random variable X be the number of days that a certain patient needs to be in the hospital. Suppose X has the pmf

$$f(x) = \frac{5-x}{10}, \qquad x = 1, 2, 3, 4.$$

If the patient is to receive \$200 from an insurance company for each of the first two days in the hospital and \$100 for each day after the first two days, what is the expected payment for the hospitalization?

2.2-4. An insurance company sells an automobile policy with a deductible of one unit. Let X be the amount of the loss having pmf

$$f(x) = \begin{cases} 0.9, & x = 0, \\ \dfrac{c}{x}, & x = 1, 2, 3, 4, 5, 6, \end{cases}$$

where c is a constant. Determine c and the expected value of the amount the insurance company must pay.

2.2-5. In Example 2.2-1 let $Z = u(X) = X^3$.

(a) Find the pmf of Z, say $h(z)$.

(b) Find $E(Z)$.

(c) How much, on average, can the young man expect to win on each play if he charges \$10 per play?

2.2-6. Let the pmf of X be defined by $f(x) = 6/(\pi^2 x^2)$, $x = 1, 2, 3, \ldots$. Show that $E(X) = +\infty$ and thus, does not exist.

2.2-7. In the gambling game chuck-a-luck, for a \$1 bet it is possible to win \$1, \$2, or \$3 with respective probabilities 75/216, 15/216, and 1/216. One dollar is lost with probability 125/216. Let X equal the payoff for this game and find $E(X)$. Note that when a bet is won, the \$1 that was bet, in addition to the \$1, \$2, or \$3 that is won, is returned to the bettor.

2.2-8. Let X be a random variable with support $\{1, 2, 3, 5, 15, 25, 50\}$, each point of which has the same probability 1/7. Argue that $c = 5$ is the value that minimizes $h(c) = E(\,|X - c|\,)$. Compare c with the value of b that minimizes $g(b) = E[(X - b)^2]$.

2.2-9. A roulette wheel used in a U.S. casino has 38 slots, of which 18 are red, 18 are black, and 2 are green. A roulette wheel used in a French casino has 37 slots, of which 18 are red, 18 are black, and 1 is green. A ball is rolled around the wheel and ends up in one of the slots with equal probability. Suppose that a player bets on red. If a \$1 bet is placed, the player wins \$1 if the ball ends up in a red slot. (The player's \$1 bet is returned.) If the ball ends up in a black or green slot, the player loses \$1. Find the expected value of this game to the player in

(a) The United States.

(b) France.

2.2-10. In the casino game called **high–low**, there are three possible bets. Assume that \$1 is the size of the bet. A pair of fair six-sided dice is rolled and their sum is calculated. If you bet **low**, you win \$1 if the sum of the dice is $\{2, 3, 4, 5, 6\}$. If you bet **high**, you win \$1 if the sum of the dice is $\{8, 9, 10, 11, 12\}$. If you bet on $\{7\}$, you win \$4 if a sum of 7 is rolled. Otherwise, you lose on each of the three bets. In all three cases, your original dollar is returned if you win. Find the expected value of the game to the bettor for each of these three bets.

2.2-11. In the gambling game craps (see Exercise 1.3-13), the player wins \$1 with probability 0.49293 and loses \$1 with probability 0.50707 for each \$1 bet. What is the expected value of the game to the player?

2.2-12. Suppose that a school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students, for a total of 1000 students.

(a) What is the average class size?

(b) Select a student randomly out of the 1000 students. Let the random variable X equal the size of the class to which this student belongs, and define the pmf of X.

(c) Find $E(X)$, the expected value of X. Does this answer surprise you?

2.2 Mathematical Expectation

2.2-2 $E(X) = (-2)\left(\frac{9}{27}\right) + (-1)\left(\frac{4}{27}\right) + (0)\left(\frac{1}{27}\right) + (1)\left(\frac{4}{27}\right) + (2)\left(\frac{9}{27}\right) = 0$

$$E(X^2) = (-2)^2\left(\frac{9}{27}\right) + (-1)^2\left(\frac{4}{27}\right) + (0)^2\left(\frac{1}{27}\right) + (1)^2\left(\frac{4}{27}\right) + (2)^2\left(\frac{9}{27}\right) = \frac{80}{27}$$

$$E(X^2 - 3X + 9) = E(X^2) - 3E(X) + 9 = \frac{80}{27} - 0 + 9 = \frac{323}{27}$$

2.2-4

$$\begin{aligned} 1 &= \sum_{x=0}^{6} f(x) = \frac{9}{10} + c\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) \\ c &= \frac{2}{49}; \\ E(\text{Payment}) &= \frac{2}{49}\left(1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{5} + 5 \cdot \frac{1}{6}\right) = \frac{71}{490} \text{ units.} \end{aligned}$$

2.2-6 Note that $\displaystyle\sum_{x=1}^{\infty} \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2}\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2}\frac{\pi^2}{6} = 1$, so this is a pdf

$$E(X) = \sum_{x=1}^{\infty} x\,\frac{6}{\pi^2 x^2} = \frac{6}{\pi^2}\sum_{x=1}^{\infty} \frac{1}{x} = +\infty$$

and it is well known that the sum of this harmonic series is not finite.

2.2-8 $E(|X-c|) = \dfrac{1}{7}\displaystyle\sum_{x \in S} |x-c|$, where $S = \{1, 2, 3, 5, 15, 25, 50\}$.

When $c = 5$,

$$E(|X-5|) = \frac{1}{7}\left[(5-1) + (5-2) + (5-3) + (5-5) + (15-5) + (25-5) + (50-5)\right].$$

If c is either increased or decreased by 1, this expectation is increased by 1/7. Thus $c = 5$, the median, minimizes this expectation while $b = E(X) = \mu$, the mean, minimizes $E[(X-b)^2]$. You could also let $h(c) = E(\,|\,X - c\,|\,)$ and show that $h'(c) = 0$ when $c = 5$.

2.2-10 $(15)\left(\frac{15}{36}\right) + (-15)\left(\frac{21}{36}\right) = \left(-\frac{5}{2}\right)$

$(15)\left(\frac{15}{36}\right) + (-15)\left(\frac{21}{36}\right) = \left(-\frac{5}{2}\right)$

$(20)\left(\frac{6}{36}\right) + (-15)\left(\frac{30}{36}\right) = \left(-\frac{55}{6}\right)$

2.2-12 **(a)** The average class size is $\dfrac{(16)(25) + (3)(100) + (1)(300)}{20} = 50$;

(b)

$$f(x) = \begin{cases} 0.4, & x = 25, \\ 0.3, & x = 100, \\ 0.3, & x = 300, \end{cases}$$

(c) $E(X) = 25(0.4) + 100(0.3) + 300(0.3) = 130.$

1. a) $f(x) = \frac{x}{c}$, $x = 1, 2, 3, 4 \Rightarrow \sum_{x} f(x) = 1$

$c = 1+2+3+4 = 10$

$E(x) = \sum x f(x) = 1\times 0.1 + 2\times 0.2 + 3\times 0.3 + 4\times 0.4$

$= 3$

b) $f(x) = cx$, $x = 1, 2, 3, \ldots 10$

$c = \frac{1}{55}$

$E(x) = \sum x f(x) = \frac{1}{55}\left[1^2 + 2^2 + 3^2 + \ldots + 10^2\right]$

$= \frac{1}{55} \times \frac{10(10+1)(20+1)}{6} = 7$

c) $f(x) = c\left(\frac{1}{4}\right)^x$, $x = 1, 2, 3, \ldots$

$c = 3 \Rightarrow E(x) = \sum_x x f(x) = 3\left[1\times\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right)^3 + \ldots\right]$

$E(x) - E(x)\left(\frac{1}{4}\right) = 3\left[1\times\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right)^3 + \ldots\right]$

$- 3\left[1\times\left(\frac{1}{4}\right)^2 + 2\times\left(\frac{1}{4}\right)^3 + \ldots\right]$

$$\frac{3}{4}E(x) = 3\left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\right]$$

$$E(x) = 4 \times \frac{1/4}{1-1/4} = \frac{4}{3}$$

d) $f(x) = c(x+1)^2 \qquad x = 0,1,2,3 \Rightarrow c = \frac{1}{30}$

$$E(x) = \sum x f(x)$$
$$= \frac{1}{30}\left[1\times4 + 2\times9 + 3\times16\right] = \frac{7}{3} = 2.33$$

e) $f(x) = \frac{x}{c} \qquad x = 1,2,3,\ldots n \Rightarrow c = \frac{n(n+1)}{2}$

$$E(x) = \sum x f(x) = \frac{2}{n(n+1)}\sum_{x=1}^{n} x^2 = \frac{2}{n(n+1)}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2n+1}{3}$$

f) $f(x) = c\left[\frac{1}{x+1} - \frac{1}{x+2}\right] \qquad x = 0,1,2,3,\ldots \Rightarrow c = 1$

$$E(x) = \sum_{x=0}^{\infty} x f(x) = \sum\left[\frac{x}{x+1} - \frac{x}{x+2}\right]$$

as $x \to \infty$ $E(x) \to \infty$ $\therefore$ doesn't exist

2 $f(x) = \frac{(|x|+1)^2}{9} \qquad x = -1, 0, 1$

$$E(x) = \sum x f(x) = -1\times\frac{4}{9} + 0 + \frac{4}{9} = 0$$

$$E(x^2) = \sum x^2 f(x) = \frac{4}{9} + 0 + \frac{4}{9} = \frac{8}{9}$$

$$E(3x^2-2x+4) = \sum(3x^2-2x+4)f(x)$$
$$= 3\sum x^2 f(x) - 2\sum x f(x) + 4\sum f(x)$$
$$= 3E(x^2) - 2E(x) + 4$$
$$= 3\times\frac{8}{9} - 0 + 4 = \frac{8}{3} + 4 = \frac{20}{3}$$

3.

x	1	2	3	4
Payment receive	200	400	500	600

$$f(x) = \frac{5-x}{10} \qquad x = 1, 2, 3, 4$$

$$\text{Expected Payment} = 200 \times \frac{4}{10} + 400 \times \frac{3}{10} + 500 \times \frac{2}{10} + 600 \times \frac{1}{10}$$

$$= 80 + 120 + 100 + 60$$

$$= \$360$$

4.

$$f(x) = \begin{cases} 0.9 & x = 0 \\ c/x & x = 1, 2, 3, 4, 5, 6 \end{cases}$$

$$\sum_{x=0}^{6} f(x) = 1 \Rightarrow \sum_{x=1}^{6} f(x) = 0.1$$

$$\Rightarrow c\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right] = 0.1$$

$$c\left[1 + 0.5 + 0.33 + 0.25 + 0.2 + 0.167\right] = 0.1$$

$$c \times 2.447 = 0.1$$

$$c = 0.0409$$

$$E(x) = \sum_{x=0}^{6} x f(x) = \sum_{x=1}^{6} x f(x) = c \sum_{x=1}^{6} 1$$

$$= c \times \frac{6 \times 7}{2} = 0.8582$$

5- a) $f(x) = \frac{(4-x)}{6}$, $x = 1,2,3$

$h(z) = f(x = z^{1/3}) = \frac{(4 - z^{1/3})}{6}$, $z = 1, 8, 27$

b) $E(z) = \sum_{z=1,8,27} z f(z) = 1 \times \frac{1}{2} + 8 \times \frac{1}{3} + \frac{1}{6} \times 27 = \frac{3 + 16 + 27}{6}$

$= \frac{23}{3} = 7.67$

c) So, if each game take \$10 to play

Expected to win = $10 - \frac{23}{3} = \frac{7}{3}$ of a dollar.

$$f(x) = \frac{6}{\pi^2 x^2} \; ; \; x = 1, 2, 3, \ldots$$

$$E(x) = \sum_{x=1}^{\infty} x f(x)$$

$$= \sum_{x=1}^{\infty} x \times \frac{6}{\pi^2 x^2}$$

$$= \frac{6}{\pi^2} \sum_{x=1}^{\infty} \left(\frac{1}{x}\right)$$

$$= \frac{6}{\pi^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \infty\right)$$

$$= \frac{6}{\pi^2} (+\infty)$$

$= +\infty$ Hence not defined

7. For Bet we have to pay \$1,

if \$1 win, $X =$ payoff for this game, $X = \$1$ (P_1)

if \$2 win $X = \$2$ (P_2)

if \$3 win $X = \$3$ (P_3)

if loss $X = -\$1$ (P_4)

It is given, $P_1 = \frac{75}{216}$ $\quad P_2 = \frac{15}{216}$

$P_3 = \frac{1}{216}$ $\quad P_4 = \frac{125}{216}$

$$E(X) = \sum x_i P(x_i)$$

$$= 1 \times P_1 + 2 \times P_2 + 3 \times P_3 + (-1) P_4$$

$$= \frac{75}{216} + \frac{30}{216} + \frac{3}{216} - \frac{125}{216}$$

$$= \frac{-17}{216} = -0.0787$$

8 X : random variable $\rightarrow \{1, 2, 3, 5, 15, 25, 50\}$

$P(x_i) = \frac{1}{7}$, for each point $i = 1$ to 7

$$h(x) = E(|X-c|) = \sum P_i(|x_i - c|)$$
$$= \frac{1}{7} \sum_i |x_i - c| \leq \frac{1}{7}\left[\sum(|x_i| + |c|)\right]$$

i.e $\left|\sum_i |x_i| - \sum_i |c|\right| \leq \sum |x_i - c| \leq \sum(|x_i| + |c|)$

When we uses $c = 5$, this triangular inequality holds true for equal on both sides. therefore $c = 5$ as it is the number in variable that come between

Now,

$$(x-b)^2 = x^2 + b^2 - 2xb$$
$$E[(x-b)^2] = E[x^2 + b^2 - 2xb]$$
$$= E[x^2] + E(b^2) - 2bE(x)$$
$$= E[x^2] + b^2 - 2bE(x)$$

It minimizes when, $b = E(x) = \frac{1}{7}[1+2+3+5+15+25+50]$
(using quadratic principle)

$$b = 14.4286$$

If we have to choose b from random variable given then $b = 15$ minimize $E[(x-b)^2]$.

9 U.S. Casino: 38 slots $\rightarrow$ 18 red + 18 black + 2 green

French casino: 37 slots $\rightarrow$ 18 red + 18 black + 1 green

a) $E(X) = \sum x_i P(x_i)$ [Player win if red comes, otherwise loss]

$$= \$1 \times P(\text{red comes}) - \$1 \cdot P(\text{red doesn't come})$$
$$= \$1 \times \frac{{}^{18}C_1}{{}^{38}C_1} - \$1 \times \frac{{}^{20}C_1}{{}^{38}C_1}$$
$$= \$\left(-\frac{2}{38}\right) = -\$\frac{1}{19}$$

$E(X) = \sum x_i P(x_i)$ (similarly above)

$= \$1 \times \frac{^{18}C_1}{^{37}C_1} - \$1 \frac{^{19}C_1}{^{37}C_1}$

$= -\$\frac{1}{37}$

10 For bet "low"

$P(\text{win}) = ?$ Total outcome = 36

Fav outcome are, sum {2, 3, 4, 5, 6}

Fav outcome = $1+2+3+4+5 = 15$

$P(\text{win}) = \frac{15}{36}$ $P(\text{loss}) = 1 - \frac{15}{36} = \frac{21}{36}$

$E(X) = \$1 \times \frac{15}{36} - \$1 \times \frac{21}{36} = -\frac{\$6}{36}$

$= -\$\frac{1}{6} = -\$0.167$

For bet "high"

Fav outcome {8, 9, 10, 11, 12} → sum of two dice

$P(\text{Fav. outcome}) = \frac{5+4+3+2+1}{36} = \frac{15}{36}$

similarly above, $E(X) = -\frac{\$1}{6} = -\$0.167$

For bet {7}

Fav outcome = 6

$P(\text{win}) = \frac{6}{36}$ $P(\text{loss}) = 1 - \frac{6}{36} = \frac{30}{36}$

$E(X) = \$4 \times \frac{6}{36} - \$1 \times \frac{30}{36}$

$= -\$\frac{1}{6} = -\$0.167$

11. $P(\$1 \text{ win}) = 0.49293$ $P(\$1 \text{ Loss}) = 0.50707$

Expected value $= \$1 \times P(\text{win}) - \$1 \times P(\text{loss})$

$= \$0.49293 - \$0.50707$

$= -\$0.01414$

12. Total 25 classes = 16 Classes (25 student) + 3 Classes (100 student) + 1 Class (300 student)

a) Let X: no. of student in class (or class size)

$$\text{Avg class size } \cancel{E(X)} = \frac{16\times 25 + 3\times 100 + 1\times 300}{25}$$

$$= \frac{1000}{25} = 40$$

b) X can be 25, 100, 300

$$P(X=25) = \frac{16\times 25}{1000} = \frac{4}{10} = \frac{2}{5} = 0.4$$

$$P(X=100) = \frac{3\times 100}{1000} = 0.3 = \frac{3}{10}$$

$$P(X=300) = \frac{1\times 300}{1000} = \frac{3}{10} = 0.3$$

c) $E(X) = \sum_i x_i P(x_i)$

$$= 25\times\frac{4}{10} + 100\times\frac{3}{10} + 300\times\frac{3}{10}$$

$$= 10 + 120 = 130$$

2.3 SPECIAL MATHEMATICAL EXPECTATIONS

Exercises

2.3-1. Find the mean and variance for the following discrete distributions:

(a) $f(x) = \dfrac{1}{5}$, $\quad x = 5, 10, 15, 20, 25$.

(b) $f(x) = 1$, $\quad x = 5$.

(c) $f(x) = \dfrac{4-x}{6}$, $\quad x = 1, 2, 3$.

2.3-2. For each of the following distributions, find $\mu = E(X)$, $E[X(X-1)]$, and $\sigma^2 = E[X(X-1)] + E(X) - \mu^2$:

(a) $f(x) = \dfrac{3!}{x!(3-x)!}\left(\dfrac{1}{4}\right)^x\left(\dfrac{3}{4}\right)^{3-x}$, $\quad x = 0, 1, 2, 3$.

(b) $f(x) = \dfrac{4!}{x!(4-x)!}\left(\dfrac{1}{2}\right)^4$, $\quad x = 0, 1, 2, 3, 4$.

2.3-3. Given $E(X+4) = 10$ and $E[(X+4)^2] = 116$, determine **(a)** $\text{Var}(X+4)$, **(b)** $\mu = E(X)$, and **(c)** $\sigma^2 = \text{Var}(X)$.

2.3-4. Let μ and σ^2 denote the mean and variance of the random variable X. Determine $E[(X-\mu)/\sigma]$ and $E\{[(X-\mu)/\sigma]^2\}$.

2.3-5. Consider an experiment that consists of selecting a card at random from an ordinary deck of cards. Let the random variable X equal the value of the selected card, where Ace = 1, Jack = 11, Queen = 12, and King = 13. Thus, the space of X is $S = \{1, 2, 3, \ldots, 13\}$. If the experiment is performed in an unbiased manner, assign probabilities to these 13 outcomes and compute the mean μ of this probability distribution.

2.3-6. Place eight chips in a bowl: Three have the number 1 on them, two have the number 2, and three have the number 3. Say each chip has a probability of 1/8 of being drawn at random. Let the random variable X equal the number on the chip that is selected, so that the space of X is $S = \{1, 2, 3\}$. Make reasonable probability assignments to each of these three outcomes, and compute the mean μ and the variance σ^2 of this probability distribution.

2.3-7. Let X equal an integer selected at random from the first m positive integers, $\{1, 2, \ldots, m\}$. Find the value of m for which $E(X) = \text{Var}(X)$. (See Zerger in the references.)

2.3-8. Let X equal the larger outcome when a pair of fair four-sided dice is rolled. The pmf of X is

$$f(x) = \frac{2x-1}{16}, \qquad x = 1, 2, 3, 4.$$

Find the mean, variance, and standard deviation of X.

2.3-9. A warranty is written on a product worth \$10,000 so that the buyer is given \$8000 if it fails in the first year, \$6000 if it fails in the second, \$4000 if it fails in the third, \$2000 if it fails in the fourth, and zero after that. The probability that the product fails in the first year is 0.1, and the probability that it fails in any subsequent year, provided that it did not fail prior to that year, is 0.1. What is the expected value of the warranty?

2.3-10. To find the variance of a hypergeometric random variable in Example 2.3-4 we used the fact that

$$E[X(X-1)] = \frac{N_1(N_1-1)(n)(n-1)}{N(N-1)}.$$

Prove this result by making the change of variables $k = x - 2$ and noting that

$$\binom{N}{n} = \frac{N(N-1)}{n(n-1)}\binom{N-2}{n-2}.$$

2.3-11. If the moment-generating function of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t},$$

find the mean, variance, and pmf of X.

2.3-12. Let X equal the number of people selected at random that you must ask in order to find someone with the same birthday as yours. Assume that each day of the year is equally likely, and ignore February 29.

(a) What is the pmf of X?

(b) Give the values of the mean, variance, and standard deviation of X.

(c) Find $P(X > 400)$ and $P(X < 300)$.

2.3-13. For each question on a multiple-choice test, there are five possible answers, of which exactly one is correct. If a student selects answers at random, give the probability that the first question answered correctly is question 4.

2.3-14. The probability that a machine produces a defective item is 0.01. Each item is checked as it is produced. Assume that these are independent trials, and compute the probability that at least 100 items must be checked to find one that is defective.

2.3-15. Apples are packaged automatically in 3-pound bags. Suppose that 4% of the time the bag of apples weighs less than 3 pounds. If you select bags randomly and weigh them in order to discover one underweight bag of apples, find the probability that the number of bags that must be selected is

(a) At least 20.

(b) At most 20.

(c) Exactly 20.

2.3-16. Let X equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.

(a) Find the pmf of X. HINT: Draw a tree diagram.

(b) Find the moment-generating function of X.

(c) Use the mgf to find the values of **(i)** the mean and **(ii)** the variance of X.

(d) Find the values of **(i)** $P(X \le 3)$, **(ii)** $P(X \ge 5)$, and **(iii)** $P(X = 3)$.

2.3-17. Let X equal the number of flips of a fair coin that are required to observe heads–tails on consecutive flips.

(a) Find the pmf of X. HINT: Draw a tree diagram.

(b) Show that the mgf of X is $M(t) = e^{2t}/(e^t - 2)^2$.

(c) Use the mgf to find the values of **(i)** the mean and **(ii)** the variance of X.

(d) Find the values of **(i)** $P(X \le 3)$, **(ii)** $P(X \ge 5)$, and **(iii)** $P(X = 3)$.

2.3-18. Let X have a geometric distribution. Show that

$$P(X > k + j \mid X > k) = P(X > j),$$

where k and j are nonnegative integers. NOTE: We sometimes say that in this situation there has been loss of memory.

2.3-19. Given a random permutation of the integers in the set $\{1, 2, 3, 4, 5\}$, let X equal the number of integers that are in their natural position. The moment-generating function of X is

$$M(t) = \frac{44}{120} + \frac{45}{120}e^t + \frac{20}{120}e^{2t} + \frac{10}{120}e^{3t} + \frac{1}{120}e^{5t}.$$

(a) Find the mean and variance of X.

(b) Find the probability that at least one integer is in its natural position.

(c) Draw a graph of the probability histogram of the pmf of X.

2.3 Special Mathematical Expectations

2.3-2 (a)

$$\begin{aligned}
\mu &= E(X) \\
&= \sum_{x=1}^{3} x \frac{3!}{x!\,(3-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \\
&= 3\left(\frac{1}{4}\right)\sum_{k=0}^{2}\frac{2!}{k!\,(2-k)!}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{2-k} \\
&= 3\left(\frac{1}{4}\right)\left(\frac{1}{4}+\frac{3}{4}\right)^2 = \frac{3}{4}; \\
E[X(X-1)] &= \sum_{x=2}^{3} x(x-1)\frac{3!}{x!\,(3-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \\
&= 2(3)\left(\frac{1}{4}\right)^2\frac{3}{4} + 6\left(\frac{1}{4}\right)^3 \\
&= 6\left(\frac{1}{4}\right)^2 = 2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right); \\
\sigma^2 &= E[X(X-1)] + E(X) - \mu^2 \\
&= (2)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 \\
&= (2)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right);
\end{aligned}$$

$$\begin{aligned}
\textbf{(b)}\quad \mu &= E(X) \\
&= \sum_{x=1}^{4} x\,\frac{4!}{x!\,(4-x)!}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{4-x} \\
&= 4\left(\frac{1}{2}\right)\sum_{k=0}^{3}\frac{3!}{k!\,(3-k)!}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{3-k} \\
&= 4\left(\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}\right)^{3} = 2;
\end{aligned}$$

$$\begin{aligned}
E[X(X-1)] &= \sum_{x=2}^{4} x(x-1)\,\frac{4!}{x!\,(4-x)!}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{4-x} \\
&= 2(6)\left(\frac{1}{2}\right)^{4} + (6)(4)\left(\frac{1}{2}\right)^{4} + (12)\left(\frac{1}{2}\right)^{4} \\
&= 48\left(\frac{1}{2}\right)^{4} = 12\left(\frac{1}{2}\right)^{2}; \\
\sigma^2 &= (12)\left(\frac{1}{2}\right)^{2} + \frac{4}{2} - \left(\frac{4}{2}\right)^{2} = 1.
\end{aligned}$$

2.3-4 $E[(X-\mu)/\sigma] = (1/\sigma)[E(X)-\mu] = (1/\sigma)(\mu-\mu) = 0;$

$E\{[(X-\mu)/\sigma]^2\} = (1/\sigma^2)E[(X-\mu)^2] = (1/\sigma^2)(\sigma^2) = 1.$

2.3-6 $f(1) = \dfrac{3}{8},\ f(2) = \dfrac{2}{8},\ f(3) = \dfrac{3}{8}$

$\mu = 1\cdot\dfrac{3}{8} + 2\cdot\dfrac{2}{8} + 3\cdot\dfrac{3}{8} = 2,$

$\sigma^2 = 1^2\cdot\dfrac{3}{8} + 2^2\cdot\dfrac{2}{8} + 3^2\cdot\dfrac{3}{8} - 2^2 = \dfrac{3}{4}.$

2.3-8

$$\begin{aligned}
E(X) &= \sum_{x=1}^{4} x\cdot\frac{2x-1}{16} \\
&= \frac{50}{16} = 3.125; \\
E(X^2) &= \sum_{x=1}^{4} x^2\cdot\frac{2x-1}{16} \\
&= \frac{85}{8}; \\
\mathrm{Var}(X) &= \frac{85}{8} - \left(\frac{25}{8}\right)^{2} = \frac{55}{64} = 0.8594; \\
\sigma &= \frac{\sqrt{55}}{8} = 0.9270.
\end{aligned}$$

2.3-10 We have $N = N_1 + N_2$. Thus

$$\begin{aligned}
E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)f(x) \\
&= \frac{\displaystyle\sum_{x=2}^{n} x(x-1)\,\frac{N_1!}{x!\,(N_1-x)!}\cdot\frac{N_2!}{(n-x)!\,(N_2-n+x)!}}{\dbinom{N}{n}} \\
&= N_1(N_1-1)\,\frac{\displaystyle\sum_{x=2}^{n} \frac{(N_1-2)!}{(x-2)!\,(N_1-x)!}\cdot\frac{N_2!}{(n-x)!\,(N_2-n+x)!}}{\dbinom{N}{n}}.
\end{aligned}$$

In the summation, let $k = x - 2$, and in the denominator, note that

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{N(N-1)}{n(n-1)}\binom{N-2}{n-2}.$$

Thus

$$\begin{aligned}
E[X(X-1)] &= \frac{N_1(N_1-1)}{\dfrac{N(N-1)}{n(n-1)}}\sum_{k=0}^{n-2}\frac{\dbinom{N_1-2}{k}\dbinom{N_2}{n-2-k}}{\dbinom{N-2}{n-2}} \\
&= \frac{N_1(N_1-1)(n)(n-1)}{N(N-1)}.
\end{aligned}$$

2.3-12 **(a)** $f(x) = \left(\dfrac{364}{365}\right)^{x-1}\left(\dfrac{1}{365}\right), \qquad x = 1, 2, 3, \ldots,$

(b)
$$\begin{aligned}
\mu &= \frac{1}{\dfrac{1}{365}} = 365, \\
\sigma^2 &= \frac{\dfrac{364}{365}}{\left(\dfrac{1}{365}\right)^2} = 132{,}860, \\
\sigma &= 364.500;
\end{aligned}$$

(c)
$$\begin{aligned}
P(X > 400) &= \left(\frac{364}{365}\right)^{400} = 0.3337, \\
P(X < 300) &= 1 - \left(\frac{364}{365}\right)^{299} = 0.5597.
\end{aligned}$$

2.3-14 $P(X \geq 100) = P(X > 99) = (0.99)^{99} = 0.3697.$

2.3-16 **(a)** $f(x) = (1/2)^{x-1}, \quad x = 2, 3, 4, \ldots;$

$$
\begin{aligned}
\textbf{(b)} \quad M(t) &= E[e^{tx}] \;=\; \sum_{x=2}^{\infty} e^{tx}(1/2)^{x-1} \\
&= 2\sum_{x=2}^{\infty}(e^t/2)^x \\
&= \frac{2(e^t/2)^2}{1-e^t/2} \;=\; \frac{e^{2t}}{2-e^t}, \qquad t < \ln 2;
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(c)} \quad M'(t) &= \frac{4e^{2t}-e^{3t}}{(2-e^t)^2} \\
\mu &= M'(0) \;=\; 3; \\
M''(t) &= \frac{(2-e^t)^2(8e^{2t}-3e^{3t})-(4e^{2t}-e^{3t})2*(2-e^t)(-e^t)}{(2-e^t)^4} \\
\sigma^2 &= M''(0)-\mu^2 \;=\; 11-9 \;=\; 2;
\end{aligned}
$$

(d) **(i)** $P(X \le 3) = 3/4$; **(ii)** $P(X \ge 5) = 1/8$; **(iii)** $P(X = 3) = 1/4$.

2.3-18

$$
\begin{aligned}
P(X > k+j \,|\, X > k) &= \frac{P(X > k+j)}{P(X > k)} \\
&= \frac{q^{k+j}}{q^k} \;=\; q^j = P(X > j).
\end{aligned}
$$

(a) $f(x) = \frac{1}{5}$; $x = 5, 10, 15, 20, 25$

$$E(x) = \sum_{x \in S} x f(x)$$

$$= 5 \times \frac{1}{5} + 10 \times \frac{1}{5} + 15 \times \frac{1}{5} + 20 \times \frac{1}{5} + 25 \times \frac{1}{5}$$

$$= 1 + 2 + 3 + 4 + 5$$

$$E(x) = 15$$

~~Mean~~

$$E(x^2) = \sum_{x \in S} x^2 f(x)$$

$$= (5)^2 \times \frac{1}{5} + (10)^2 \times \frac{1}{5} + (15)^2 \times \frac{1}{5} + (20)^2 \times \frac{1}{5} + (25)^2 \times \frac{1}{5}$$

$$E(x)^2 = 275$$

~~Mean~~

Hence: mean $= E(x) = 15$

$$\text{Variance} = E(x^2) - \{E(x)\}^2$$

$$= 275 - (15)^2$$

$$= 50$$

(b) $f(x) = 1 \quad ; \; x = 5$

$$E(x) = \sum_{x \in S} x f(x)$$

$$E(x) = 5 \times 1 = 5$$

$$E(x^2) = \sum_{x \in S} x^2 f(x)$$

$$= (5)^2 \times 1$$

$$= 25$$

Mean $= E(x) = 5$

Variance $= E(x^2) - [E(x)]^2$

$$= 25 - (5)^2$$

$$= 0$$

(c) $f(x) = \frac{4-x}{6}$; $x = 1, 2, 3$

$$E(x) = \sum_{x \in S} x f(x)$$

$$= 1 \times \left(\frac{4-1}{6}\right) + 2 \times \left(\frac{4-2}{6}\right) + 3 \times \left(\frac{4-3}{6}\right)$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{1}{2}$$

$$E(x) = \frac{5}{3}$$

$$E(x^2) = (1)^2 \times \left(\frac{4-1^2}{6}\right) + 2^2 \times \left(\frac{4-2^2}{6}\right) + 3^2 \times \left(\frac{4-3^2}{6}\right)$$

$$= \frac{1}{2} + 0 - \frac{15}{2}$$

$$E(x^2) = -7$$

$$\text{Mean} = E(x) = \frac{5}{3}$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$= -7 - \left(\frac{5}{3}\right)^2$$

$$= -7 - \frac{25}{9}$$

$$= -\frac{88}{9}$$

2. a) $f(x) = \frac{3!}{x!(3-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}$, $x = 0, 1, 2, 3$

$f(0) = \left(\frac{3}{4}\right)^3$; $f(1) = \left(\frac{3}{4}\right)^3$;

$f(2) = \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$; $f(3) = \left(\frac{1}{4}\right)^3$

$E(X) = \sum_{i=1}^{4} x_i P(x_i) = 0\times\left(\frac{3}{4}\right)^3 + 1\times\left(\frac{3}{4}\right)^3 + \frac{2}{4}\cdot\left(\frac{3}{4}\right)^2 + 3\times\left(\frac{1}{4}\right)^3$

$E(X) = \frac{3}{3}\cdot\left(\frac{3}{4}\right)^3 + \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{3}{4^3} = \frac{3}{4^3}\left[9+6+1\right] = \frac{3}{4}$

$E[X(X-1)] = 0(0-1)\times\left(\frac{3}{4}\right)^3 + 1(1-1)\cdot\left(\frac{3}{4}\right)^3$

$+ 2(2-1)\times\frac{1}{4}\left(\frac{3}{4}\right)^2 + 3(3-1)\times\left(\frac{1}{4}\right)^3$

$= \frac{1}{2}\times\left(\frac{3}{4}\right)^2 + \frac{3\times}{2}\cdot\left(\frac{1}{4}\right)^2 = \frac{12}{32} = \frac{3}{8}$

$\sigma^2 = E[X(X-1)] + E(X) - \mu^2$

$= \frac{3}{8} + \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{8} + \frac{3}{4}\cdot\frac{1}{4} = \frac{9}{16}$

$\sigma^2 = \frac{9}{16}$

b) $f(x) = {}^4C_x\left(\frac{1}{2}\right)^4$, $x = 0, 1, 2, 3, 4$

$\mu = E(X) = \sum_{i=1}^{5} x_i f(x_i)$

$= {}^4C_0\left(\frac{1}{2}\right)^4\times 0 + {}^4C_1\left(\frac{1}{2}\right)^4\cdot 1 + {}^4C_2\left(\frac{1}{2}\right)^4\times 2$

$+ {}^4C_3\left(\frac{1}{2}\right)^4\times 3 + {}^4C_4\left(\frac{1}{2}\right)^4\times 4$

$= \frac{1}{2^4}\left[4 + 12 + 12 + 4\right] = \frac{32}{16} = 2$

$$E(X(X-1)) = \left[0\times(0-1)\,{}^4C_0 + 1(1-1)\,{}^4C_1 + 2(2-1)\,{}^4C_2 + 3(3-1)\,{}^4C_3 + 4(4-1)\,{}^4C_4\right]\cdot\left(\frac{1}{2}\right)^4$$

$$= \frac{[12 + 24 + 12]}{2^4} = \frac{48}{16} = 3$$

$$\sigma^2 = E[X(X-1)] + E(X) - \mu^2$$

$$= 3 + 2 - (2)^2 = 1$$

$$E(X+4)=10$$

$$\Rightarrow E(X)+4=10$$

$$\Rightarrow E(X)=6$$

and $$E[(X+4)^2]=116$$

$$\Rightarrow E(X^2+8X+16)=116$$

$$\Rightarrow E(X^2)+8E(X)+16=116$$

$$E(X^2)+8\times 6+16=116$$

$$E(X^2)=52$$

(a) $$Var(X+4) = E[(X+4)^2] - [E(X+4)]^2$$
$$= 116 - 10^2 = 16$$

(b) $$\mu = E(X) = 6$$

(c) $$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = 52 - 6^2$$
$$= 16$$

4 $E(X) = \mu \quad E[(X-\mu)^2] = \sigma^2 = E(X^2) - \mu^2$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)\right] = \frac{1}{\sigma} E[X-\mu] = \frac{1}{\sigma}\left[E(X) - E(\mu)\right]$$

(Since, σ is constant taken outside)

$$E\left[\left(\frac{X-\mu}{\sigma}\right)\right] = \frac{1}{\sigma}\{\mu - \mu\}$$
$$= 0$$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = E\left[\frac{(X-\mu)^2}{\sigma^2}\right]$$

{since σ^2 is not variable taken outside}

$$= \frac{1}{\sigma^2} E[(X-\mu)^2]$$

$$= \frac{\sigma^2}{\sigma^2} = 1$$

$S = \{1, 2, 3, \ldots 13\}$

There are 4 Aces in a deck of 52 cards

Hence $P(X=1) = \frac{^4C_1}{^{52}C_1} = \frac{4}{52} = \frac{1}{13}$

Similarly $P(X=2) = \frac{4}{52} = \frac{1}{13}$

Similarly we can find that

$P(X) = \frac{1}{13}$ for $X = \{1, 2, 3, \ldots 13\}$

Hence pmf is

$f(x) = \frac{1}{13}$; $x = \{1, 2, \ldots 13\}$

mean $\mu = E(x)$

$= \sum_{x \in S} x f(x)$

$= \sum_{x=1}^{13} x f(x)$

$= 1 \times \frac{1}{13} + 2 \times \frac{1}{13} + \cdots + 13 \times \frac{1}{13}$

$= \frac{1}{13} [1 + 2 + 3 + \cdots + 13]$

$= \frac{1}{\cancel{13}} \left[\frac{\cancel{13}}{2} (1 + 13) \right]$

$= 7$

$S = \{1, 2, 3\}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{2}{8} = \frac{1}{4}$

$P(X=3) = \frac{3}{8}$

Hence pmf is

$$f(x) = \left.\begin{array}{ll} \frac{3}{8}, & x = 1, 3 \\ = \frac{1}{4}, & x = 2 \end{array}\right\}$$

mean $\mu = \sum_{x \in S} x f(x)$

$\mu = \sum_{x=1}^{3} x f(x)$

$\mu = 1 \times \frac{3}{8} + 2 \times \frac{1}{4} + 3 \times \frac{3}{8}$

$\boxed{\mu = 2}$

$E(x^2) = (1)^2 \times \frac{3}{8} + (2)^2 \times \frac{1}{4} + (3)^2 \times \frac{3}{8}$

$E(x^2) = \frac{19}{4}$

$Var(x) = \sigma^2 = E(x^2) - \mu^2$

$= \frac{19}{4} - (2)^2$

$\sigma^2 = \frac{3}{4}$

$S = \{1, 2, 3, \ldots, m\}$

$P(X) = \frac{1}{m}$ for $x = 1, 2, 3, \ldots m$

Hence pmf is

$f(x) = \frac{1}{m}$, $x = 1, 2, 3, \ldots m$

$E(X) = \sum_{x=1}^{m} x f(x)$

$= 1 \times \frac{1}{m} + 2 \times \frac{1}{m} + \cdots + m \times \frac{1}{m}$

$= \frac{1}{m}(1 + 2 + 3 + \cdots + m)$

$= \frac{1}{m}\left[\frac{m}{2}(1+m)\right]$

$E(X) = \frac{m+1}{2}$

$E(X^2) = \sum_{x=1}^{m} x^2 f(x)$

$= (1)^2 \times \frac{1}{m} + (2)^2 \times \frac{1}{m} + \cdots + (m)^2 \times \frac{1}{m}$

$= \frac{1}{m}\left[(1)^2 + (2)^2 + \cdots + (m)^2\right]$

$= \frac{1}{m}\left[\frac{1}{6} m(m+1)(2m+1)\right]$

$E(X^2) = \frac{(m+1)(2m+1)}{6}$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{(m+1)(2m+1)}{6} - \left(\frac{m+1}{2}\right)^2$

$= (m+1)\left[\frac{2m+1}{6} - \frac{m+1}{4}\right]$

$Var(X) = \frac{(m+1)(m-1)}{12}$

Now $E(X) = Var(X)$

$\frac{m+1}{2} = \frac{(m+1)(m-1)}{12}$

$m - 1 = 6$

$\boxed{m = 7}$

$$f(x) = \frac{2x-1}{16} \; ; x = 1, 2, 3, 4$$

$$E(x) = \sum_{x \in S} x f(x)$$

$$= 1 \times \frac{2(1)-1}{16} + 2 \times \frac{2(2)-1}{16} + 3 \times \frac{2(3)-1}{16} + 4 \times \frac{2(4)-1}{16}$$

$$= \frac{1}{16} + \frac{6}{16} + \frac{15}{16} + \frac{28}{16}$$

$$E(x) = \frac{25}{8}$$

$$E(x^2) = \sum_{x \in S} x^2 f(x)$$

$$= (1^2) \times f(1) + 2^2 f(2) + 3^2 \times f(3) + 4^2 f(4)$$

$$= \frac{1}{16} + \frac{12}{16} + \frac{45}{16} + \frac{112}{16}$$

$$E(x^2) = \frac{85}{8}$$

Mean $\mu = E(x) = \frac{25}{8}$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{85}{8} - \left(\frac{25}{8}\right)^2$$

$$= \frac{55}{64}$$

Standard deviation

$$\sigma = \sqrt{\text{Variance}} = \sqrt{\frac{55}{64}} = 0.927$$

9. P(pdt fail in 1st Yr) = 0.1

P(pdt fail in any subsequent Yr) = 0.1

Expected value = E(X) , where X: warrenty return of warrent

X_1 = \$8000 $\quad P(x_1) = 0.1$

X_2 = \$6000 $\quad P(x_2) = (1-0.1)\times 0.1 = 0.09$

X_3 = \$4000 $\quad P(x_3) = (1-0.1)^2\times 0.1 = 0.081$

X_4 = \$2000 $\quad P(x_4) = (1-0.1)^3\times 0.1 = 0.0729$

E(X) = \$8000×0.1 + \$6000×0.09 + \$4000×0.081

+ \$2000×0.0729

= \$1809.8

10. X: hypergeometric distribution

n selected (obj) from $N = N_1 + N_2$

$n = x + (n-x)$, where x: selected from N_1

$n-x$: selected from N_2

$$\therefore P(x) = \frac{{}^{N_1}C_x \; {}^{N_2}C_{n-x}}{{}^{N}C_n}$$

$$E(X) = \mu = \sum_{x\in S} x \frac{{}^{N_1}C_x \; {}^{N_2}C_{n-x}}{{}^{N}C_n} = n\left(\frac{N_1}{N}\right)$$

$$E[X(X-1)] = \sum_{x\in S} x(x-1) \frac{{}^{N_1}C_x \; {}^{N_2}C_{n-x}}{{}^{N}C_n}$$

$$= \sum_{x\in S} x(x-1) \frac{N_1!}{x!(N_1-x)!} \cdot \frac{N_2!}{(n-x)!(N_2-n+x)!} \times \frac{n!(N-n)!}{N!}$$

$$= \sum_{x\in S} \left\{ \frac{x(x-1) \frac{N_1(N_1-1)}{x(x-1)} {}^{N_1-2}C_{x-2} \cdot \frac{N_2(N_2-1)}{(n-x)(n-x-1)} {}^{N_2-2}C_{n-x-2}}{\times {}^{N}C_n} \right\}$$

$$= N_1(N_1-1) \sum_{x\in S} \left\{ \frac{{}^{N_1-2}C_{x-2} \cdot {}^{N_2}C_{n-x}}{{}^{N}C_n} \right\}$$

$$\Rightarrow \frac{{}^{N_2}C_{n-x}}{{}^{N}C_n} = \frac{N_2(N_2-1)}{(n-x)(n-x-1)} \cdot \frac{n(n-1)}{(N)(N-1)} \times \frac{{}^{N_2-2}C_{(n-2)-x}}{{}^{N-2}C_{n-2}}$$

$$= \frac{{}^{N_2}C_{n-x}}{{}^{N-2}C_{n-2}} \times \frac{n(n-1)}{N(N-1)}$$

$$E[X(X-1)] = \frac{N_1(N_1-1)\,n(n-1)}{N(N-1)} \sum \frac{{}^{N_1-2}C_{x-2} \; {}^{N_2}C_{n-x}}{{}^{N-2}C_{n-2}}$$

Let $k = x-2$, then $\sum \frac{{}^{N_1-2}C_{x-2} \times {}^{N_2}C_{(n-2)-(x-2)}}{{}^{N-2}C_{n-2}} = \sum P(k_i)$

$= 1$

$$\therefore E[X(X-1)] = \frac{N_1(N_1-1)\,n(n-1)}{N(N-1)}$$

11 $M(t) = \frac{2}{5}e^{t} + \frac{e^{2t}}{5} + \frac{2}{5}e^{3t}$

$M'(t) = \frac{2}{5}e^{t} + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t}$

$M''(t) = \frac{2}{5}e^{t} + \frac{4}{5}e^{2t} + \frac{18}{5}e^{3t}$

$\mu = E(X) = M'(0) = 2$

$\sigma^2 = E(X^2) - [E(X)]^2$

$= M''(0) - [M'(0)]^2$

$= \frac{24}{5} - 4 = \frac{4}{5}$

pmf

$$f(x) = \begin{cases} 2/5 & x = 1 \\ 1/5 & x = 2 \\ 2/5 & x = 3 \end{cases}$$

as $M(t) = p(x)e^{xt}$

12 If we ignore Feb 29 then,

No of Days in 1 year = 365

a) $f(X=x) = \frac{1}{365} \cdot \left(\frac{364}{365}\right)^{x-1}$ where ~~$0 \le x \le 365$~~

as $p = \frac{1}{365}$; P : Probability that this person have same birthday

pmf of X

$$f(X=x) = p(1-p)^{x-1}$$

$$= \frac{1}{365} \left(\frac{364}{365}\right)^{x-1}, \quad x = 1, 2, 3, \ldots \infty$$

b) $\mu = \sum_{i=1}^{\infty} x_i f(x_i) = \left(\frac{364}{365}\right)^0 \cdot \frac{1}{365} + 2 \cdot \frac{1}{365} \cdot \left(\frac{364}{365}\right)^1 + \ldots$

$\Rightarrow \frac{364}{365}\mu = \left(\frac{364}{365}\right)^1 \cdot \frac{1}{365} + 2 \cdot \frac{1}{365} \cdot \left(\frac{364}{365}\right)^2 + \ldots$

$\mu - \frac{364}{365}\mu = \frac{1}{365}\left[\left(\frac{364}{365}\right)^0 + \left(\frac{364}{365}\right)^1 + \left(\frac{364}{365}\right)^2 + \ldots\right]$

$$\mu = \frac{1}{1-\frac{364}{365}} = 365$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$E(X^2) = 1^2 \times \frac{1}{365} \times \left(\frac{364}{365}\right)^0 + 2^2 \times \frac{1}{365} \times \left(\frac{364}{365}\right)^1 + 3^2 \times \frac{1}{365} \times \left(\frac{364}{365}\right)^2 + \cdots$$

$$E(X^2) = \frac{1}{365}\left[1 \cdot \left(\frac{364}{365}\right)^0 + 4 \cdot \left(\frac{364}{365}\right)^1 + 9 \cdot \left(\frac{364}{365}\right)^2 + \cdots\right]$$

$$\frac{364}{365} E(X^2) = \frac{1}{365}\left[1 \times \left(\frac{364}{365}\right)^1 + 4 \times \left(\frac{364}{365}\right)^2 + 9 \times \left(\frac{364}{365}\right)^3 + \cdots\right]$$

$$E(X^2)\left[1 - \frac{364}{365}\right] = \frac{1}{365}\left[1 \times \left(\frac{364}{365}\right)^0 + 3 \times \left(\frac{364}{365}\right)^1 + 5\left(\frac{364}{365}\right)^2 + 7\left(\frac{364}{365}\right)^3 + \cdots\right]$$

$$E(X^2) = 1\left(\frac{364}{365}\right)^0 + 3 \times \left(\frac{364}{365}\right)^1 + 5 \times \left(\frac{364}{365}\right)^2 + \cdots$$

$$E(X^2) - \frac{364}{365}E(X^2) = \left(\frac{364}{365}\right)^0 + 2\left(\frac{364}{365}\right)^1 + 2\left(\frac{364}{365}\right)^2 + \cdots$$

$$E(X^2) \times \frac{1}{365} = 1 + 2\left(\frac{\frac{364}{365}}{1-\frac{364}{365}}\right) = 1 + 728$$

$$E(X^2) = 365 \times 729 = 266085$$

$$\sigma^2 = E(X^2) - \mu^2$$
$$= 266085 - (365)^2 = 365(729 - 365)$$
$$= 365 \times 364 = 132{,}860$$

$$\sigma = \text{S.D.} = \sqrt{132{,}860} = 364.4996$$

13. $P(\text{question is correct}) = \frac{1}{5}$

$$P\left(\begin{array}{l}\text{question 4 is} \\ \text{correct}\end{array}\right) = (1-p)^3 \, p$$

$$= \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right) = 0.1024$$

14. $P(\text{defective item}) = 0.01$

$$P\left(\begin{array}{l}\text{at least 100 item must} \\ \text{checked to fine one defective}\end{array}\right) = \sum_{i=299}^{\infty} p(1-p)^i$$

$$= p \sum_{i=299}^{\infty} (0.99)^i$$

$$P = 0.01 \times \frac{(0.99)^{299}}{1-0.99} = (0.99)^{299} = 0.04954$$

15. $P\left(\begin{array}{c}\text{bag of apple less than} \\ \text{3 pounds}\end{array}\right) = 0.04 = P$

a) $P\left(\begin{array}{c}\text{at least 20 bags} \\ \text{to find one underweight}\end{array}\right) = \sum_{i=19}^{\infty} P(1-P)^i$

$$= P \times \frac{(1-P)^{19}}{1-(1-P)} \quad (\text{G.P.})$$

$$= (0.96)^{19} = 0.46042$$

b) $P\left(\begin{array}{c}\text{at most} \\ \text{20 bags}\end{array}\right) = 1 - P\left(\begin{array}{c}\text{at least 21 bags} \\ \text{to find one underweight}\end{array}\right)$

$$= 1 - \sum_{i=20}^{\infty} P(1-P)^i$$

$$= 1 - (0.96)^{20} \quad [\text{similarly as above}]$$

$$= 0.557998 \approx 0.558$$

c) $P\left(\begin{array}{c}\text{exactly 20 bags} \\ \text{to find one underweight}\end{array}\right) = P(1-P)^{19}$

$$= 0.04 \times (0.96)^{19}$$

$$= 0.0184$$

16. a) $f(x) = \left(\frac{1}{2}\right)^{x-1}, \quad x = 2, 3, 4, \ldots$

possible case HTHT.... or THTH....

So $f(x) = 2 \cdot \left(\frac{1}{2}\right)^{x-1} \cdot \frac{1}{2}$ [as last two should be same]

$$= \left(\frac{1}{2}\right)^{x-1}$$

b) $M(t) = E(e^{tx}) = \sum_{x \in S} e^{tx} f(x)$

$M'(t) = \sum_{x \in S} x e^{tx} f(x)$; $M''(t) = \sum_{x \in S} x^2 e^{tx} f(x)$

$M'(0) = \sum_{x \in S} x f(x) = E(x)$; $M''(0) = \sum_{x \in S} x^2 f(x) = E(x^2)$

$$M(t) = \sum_{x \in S} e^{tx} f(x) = e^{2t}\left(\frac{1}{2}\right) + e^{3t}\left(\frac{1}{2}\right)^2 + e^{4t}\left(\frac{1}{2}\right)^3 + \ldots$$

$$= \frac{e^{2t}(1/2)}{1 - e^t(1/2)} \quad ; \text{ provided } \frac{e^t}{2} \leq 1$$

$$= \frac{e^{2t}}{2 - e^t}$$

(c) $M'(t) = \dfrac{2e^{2t}(2-e^t) + e^{3t}}{(2-e^t)^2} = \dfrac{4e^{2t} - e^{3t}}{(2-e^t)^2}$

$E(x) = u = M'(0) = 3 = \text{Mean}$

$$M''(t) = \frac{(2-e^t)^2\left[8e^{2t} - 3e^{3t}\right] + (4e^{2t} - e^{3t}) \times \left[2(2-e^t)e^t\right]}{(2-e^t)^4}$$

$M''(0) = 5 + 3 \times 2 = 11$

$\therefore E(X^2) = 11$

Now $\sigma^2 = E(x^2) - (E(x))^2$

$\sigma^2 = 11 - 9$

$\sigma^2 = 2 = \text{variance}$

(d) i) $P(X \leq 3) = f(x=2) + f(x=3)$

$$= \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$$

ii) $P(X \geq 5) = f(x=5) + f(x=6) + \ldots$

$$= \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \ldots$$

$$= \frac{(1/2)^4}{1 - 1/2} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

iii) $P(X=3) = f(x=3) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

17 a). Let $x = 2$, possible case $\{HT\}$

$x = 3$ $\{HHT, THT\}$

$x = 4$ $\{HHHT$, ~~HTHT~~, $TTHT$, TH

$x = 5$ $\{HHHHT$, ~~HHTHT~~, ~~HTHHT~~, $THHHT$

$TTTHT\}$

So, $f(X=x) = (x-1) \times \left(\frac{1}{2}\right)^x$, $x \geq 2$;

b) $M(t) = E(e^{tx}) = \sum_{x \in S} e^{tx} f(x)$

$$M(t) = e^{2t} \times \left(\frac{1}{2}\right)^2 + e^{3t} \times \frac{2}{2^3} + e^{4t} \times \frac{3}{2^4} + \cdots$$

$$M(t) - \frac{M(t)e^t}{2} = e^{2t} \times \left(\frac{1}{2}\right)^2 + e^{3t} \times \frac{1}{2^3} + e^{4t} \times \frac{1}{2^4} + \cdots$$

$$\frac{(2-e^t)}{2} M(t) = \frac{e^{2t} \times (1/2)^2}{1 - e^t/2}, \text{ provided } \frac{e^t}{2} < 1$$

$$M(t) = \frac{e^{2t}}{(2-e^t)^2}$$

c) $M'(t) = \dfrac{2e^{2t}(2-e^t)^2 + 2(2-e^t)e^{3t}}{(2-e^t)^4}$

$M'(t) = \dfrac{2e^{2t}\left[\,2\,\right]}{(2-e^t)^3} = \dfrac{4e^{2t}}{(2-e^t)^3}$

$M''(t) = \dfrac{4\left[2e^{2t}(2-e^t)^3 + e^{2t}\times 3(2-e^t)^2 e^t\right]}{(2-e^t)^6}$

$M'(0) = 4 \qquad M''(0) = 20$

$E(x) = M'(0) \qquad E(x^2) = M''(0)$

Mean, $\mu = 4$

variance, $\sigma^2 = E(x^2) - (E(x))^2 = 4$

d) $P(X \le 3) = f(x=2) + f(x=3) = \dfrac{1}{4} + \dfrac{2}{8} = \dfrac{1}{2}$

$P(X \ge 5) = f(x=5) + f(x=6) + \dots$

$= 1 - \left[f(x=2) + f(x=3) + f(x=4)\right]$

$= 1 - \left[\dfrac{1}{4} + \dfrac{2}{8} + \dfrac{3}{16}\right] = 5/16$

$P(X=3) = f(x=3) = \dfrac{2}{8} = \dfrac{1}{4}$

18 X have a geometric distribution

So, let say $f(X=x) = A\,r^{x-1}$; $x = 1,2,3$

Now, $P(X > k+j \mid X > k) = \dfrac{P(X > k+j)}{P(X > k)}$

$$= \frac{\sum_{k+j+1}^{\infty} A\,r^{x-1}}{\sum_{k+1}^{\infty} A\,r^{x-1}}$$

$$= \frac{(r^{k+j}/1-r)}{(r^{k}/1-r)}$$

$$= r^{j}$$

Now, since $\sum_{x=1}^{\infty} P(x) = 1$

i.e $\dfrac{A}{1-r} = 1$

So, $P(X > k+j \mid X > k) = \dfrac{A\,r^{j+1-1}}{1-r}$

$$= P(X > j)$$

(Hence proved)

19 $M(t) = \frac{44}{120}e^{0} + \frac{45}{120}e^{t} + \frac{20}{120}e^{2t} + \frac{10}{120}e^{3t}$

$+ 0\,e^{4t} + \frac{1}{120}e^{5t}$

a) $M'(t) = \frac{45}{120}e^{t} + \frac{40}{120}e^{2t} + \frac{30}{120}e^{3t}$

$+ \frac{5}{120}e^{5t}$

$M''(t) = \frac{45}{120}e^{t} + \frac{80}{120}e^{2t} + \frac{90}{120}e^{3t}$

$+ \frac{25}{120}e^{5t}$

$E(X) = \mu = M'(0) = \frac{45+40+30+5}{120}$

$= 1$

$E(X^2) = M''(0) = \frac{45+80+90+25}{120} = 2$

Mean $= 1$

Variance, $\sigma^2 = E(X^2) - (E(X))^2 = 1$

b) $P(X \geq 1) = 1 - P(X=0)$

$= 1 - \frac{44}{120}$ (from m.g.f given)

$= \frac{76}{120} = \frac{19}{30} = 0.633$

c) $f(x=0) = \frac{44}{120} = 0.367$ (from m.g.f)

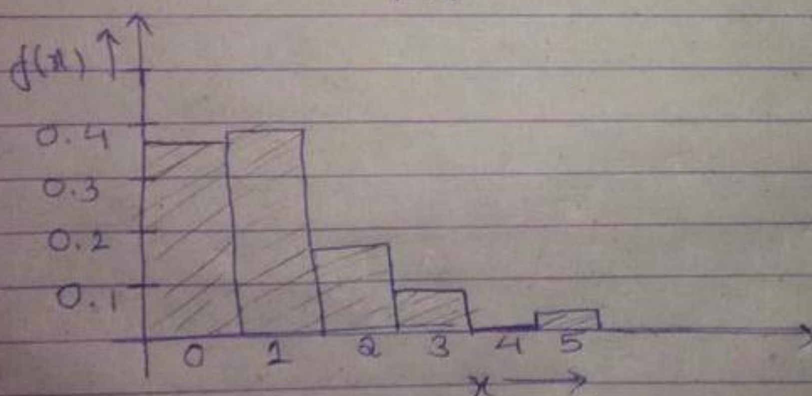
$f(x=1) = \frac{45}{120} = 0.375$

$f(x=2) = \frac{20}{120} = \frac{1}{6} = 0.167$

$f(x=3) = \frac{10}{120} = \frac{1}{12} = 0.083$

$f(x=4) = 0$

$f(x=5) = \frac{1}{120} = 0.0083$



2.4 THE BINOMIAL DISTRIBUTION

Exercises

2.4-1. An urn contains 7 red and 11 white balls. Draw one ball at random from the urn. Let $X = 1$ if a red ball is drawn, and let $X = 0$ if a white ball is drawn. Give the pmf, mean, and variance of X.

2.4-2. Suppose that in Exercise 2.4-1, $X = 1$ if a red ball is drawn and $X = -1$ if a white ball is drawn. Give the pmf, mean, and variance of X.

2.4-3. On a six-question multiple-choice test there are five possible answers for each question, of which one is correct (C) and four are incorrect (I). If a student guesses randomly and independently, find the probability of

(a) Being correct only on questions 1 and 4 (i.e., scoring C, I, I, C, I, I).

(b) Being correct on two questions.

2.4-4. It is claimed that 15% of the ducks in a particular region have patent schistosome infection. Suppose that seven ducks are selected at random. Let X equal the number of ducks that are infected.

(a) Assuming independence, how is X distributed?

(b) Find **(i)** $P(X \geq 2)$, **(ii)** $P(X = 1)$, and **(iii)** $P(X \leq 3)$.

2.4-5. In a lab experiment involving inorganic syntheses of molecular precursors to organometallic ceramics, the final step of a five-step reaction involves the formation of a metal–metal bond. The probability of such a bond forming is $p = 0.20$. Let X equal the number of successful reactions out of $n = 25$ such experiments.

(a) Find the probability that X is at most 4.

(b) Find the probability that X is at least 5.

(c) Find the probability that X is equal to 6.

(d) Give the mean, variance, and standard deviation of X.

2.4-6. It is believed that approximately 75% of American youth now have insurance due to the health care law. Suppose this is true, and let X equal the number of American youth in a random sample of $n = 15$ with private health insurance.

(a) How is X distributed?

(b) Find the probability that X is at least 10.

(c) Find the probability that X is at most 10.

(d) Find the probability that X is equal to 10.

(e) Give the mean, variance, and standard deviation of X.

2.4-7. Suppose that 2000 points are selected independently and at random from the unit square $\{(x, y) : 0 \leq x < 1, 0 \leq y < 1\}$. Let W equal the number of points that fall into $A = \{(x, y) : x^2 + y^2 < 1\}$.

(a) How is W distributed?

(b) Give the mean, variance, and standard deviation of W.

(c) What is the expected value of $W/500$?

(d) Use the computer to select 2000 pairs of random numbers. Determine the value of W and use that value to find an estimate for π. (Of course, we know the real value of π, and more will be said about estimation later in this text.)

(e) How could you extend part (d) to estimate the volume $V = (4/3)\pi$ of a ball of radius 1 in 3-space?

(f) How could you extend these techniques to estimate the "volume" of a ball of radius 1 in n-space?

2.4-8. A boiler has four relief valves. The probability that each opens properly is 0.99.

(a) Find the probability that at least one opens properly.

(b) Find the probability that all four open properly.

2.4-9. Suppose that the percentage of American drivers who are multitaskers (e.g., talk on cell phones, eat a snack, or text message at the same time they are driving) is approximately 80%. In a random sample of $n = 20$ drivers, let X equal the number of multitaskers.

(a) How is X distributed?

(b) Give the values of the mean, variance, and standard deviation of X.

(c) Find the following probabilities: **(i)** $P(X = 15)$, **(ii)** $P(X > 15)$, and **(iii)** $P(X \le 15)$.

2.4-10. A certain type of mint has a label weight of 20.4 grams. Suppose that the probability is 0.90 that a mint weighs more than 20.7 grams. Let X equal the number of mints that weigh more than 20.7 grams in a sample of eight mints selected at random.

(a) How is X distributed if we assume independence?

(b) Find **(i)** $P(X = 8)$, **(ii)** $P(X \le 6)$, and **(iii)** $P(X \ge 6)$.

2.4-11. A random variable X has a binomial distribution with mean 6 and variance 3.6. Find $P(X = 4)$.

2.4-12. In the casino game chuck-a-luck, three fair six-sided dice are rolled. One possible bet is \$1 on fives, and the payoff is equal to \$1 for each five on that roll. In addition, the dollar bet is returned if at least one five is rolled. The dollar that was bet is lost only if no fives are rolled. Let X denote the payoff for this game. Then X can equal -1, 1, 2, or 3.

(a) Determine the pmf $f(x)$.

(b) Calculate μ, σ^2, and σ.

(c) Depict the pmf as a probability histogram.

2.4-13. It is claimed that for a particular lottery, 1/10 of the 50 million tickets will win a prize. What is the probability of winning at least one prize if you purchase **(a)** 10 tickets or **(b)** 15 tickets?

2.4-14. For the lottery described in Exercise 2.4-13, find the smallest number of tickets that must be purchased so that the probability of winning at least one prize is greater than **(a)** 0.50; **(b)** 0.95.

2.4-15. A hospital obtains 40% of its flu vaccine from Company A, 50% from Company B, and 10% from Company C. From past experience, it is known that 3% of the vials from A are ineffective, 2% from B are ineffective, and 5% from C are ineffective. The hospital tests five vials from each shipment. If at least one of the five is ineffective, find the conditional probability of that shipment's having come from C.

2.4-16. A company starts a fund of M dollars from which it pays \$1000 to each employee who achieves high performance during the year. The probability of each employee achieving this goal is 0.10 and is independent of the probabilities of the other employees doing so. If there are $n = 10$ employees, how much should M equal so that the fund has a probability of at least 99% of covering those payments?

2.4-17. Your stockbroker is free to take your calls about 60% of the time; otherwise, he is talking to another client or is out of the office. You call him at five random times during a given month. (Assume independence.)

(a) What is the probability that he will take every one of the five calls?

(b) What is the probability that he will accept exactly three of your five calls?

(c) What is the probability that he will accept at least one of the calls?

2.4-18. In group testing for a certain disease, a blood sample was taken from each of n individuals and part of each sample was placed in a common pool. The latter was then tested. If the result was negative, there was no more testing and all n individuals were declared negative with one test. If, however, the combined result was found positive, all individuals were tested, requiring $n+1$ tests. If $p = 0.05$ is the probability of a person's having the disease and $n = 5$, compute the expected number of tests needed, assuming independence.

2.4-19. Define the pmf and give the values of μ, σ^2, and σ when the moment-generating function of X is defined by

(a) $M(t) = 1/3 + (2/3)e^t$.

(b) $M(t) = (0.25 + 0.75e^t)^{12}$.

2.4-20. **(i)** Give the name of the distribution of X (if it has a name), **(ii)** find the values of μ and σ^2, and **(iii)** calculate $P(1 \le X \le 2)$ when the moment-generating function of X is given by

(a) $M(t) = (0.3 + 0.7e^t)^5$.

(b) $M(t) = \dfrac{0.3e^t}{1 - 0.7e^t}, \qquad t < -\ln(0.7)$.

(c) $M(t) = 0.45 + 0.55e^t$.

(d) $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$.

(e) $M(t) = \sum_{x=1}^{10} (0.1)e^{tx}$.

2.4 The Binomial Distribution

2.4-2 $f(-1) = \dfrac{11}{18}, \qquad f(1) = \dfrac{7}{18};$

$$\mu = (-1)\frac{11}{18} + (1)\frac{7}{18} = -\frac{4}{18};$$

$$\sigma^2 = \left(-1 + \frac{4}{18}\right)^2\left(\frac{11}{18}\right) + \left(1 + \frac{4}{18}\right)^2\left(\frac{7}{18}\right) = \frac{77}{81}.$$

2.4-4 **(a)** X is $b(7,\ 0.15)$;

(b) (i) $P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.7166 = 0.2834;$

(ii) $P(X = 1) = P(X \leq 1) - P(X \leq 0) = 0.7166 - 0.3206 = 0.3960;$

(iii) $P(X \leq 3) = 0.9879.$

2.4-6 **(a)** X is $b(15, 0.75)$; $\qquad 15 - X$ is $b(15, 0.25)$;

(b) $P(X \geq 10) = P(15 - X \leq 5) = 0.8516;$

(c) $P(X \leq 10) = P(15 - X \geq 5) = 1 - P(15 - X \leq 4) = 1 - 0.6865 = 0.3135;$

(d) $P(X = 10) = P(X \geq 10) - P(X \geq 11)$

$= P(15 - X \leq 5) - P(15 - X \leq 4) = 0.8516 - 0.6865 = 0.1651;$

(e) $\mu = (15)(0.75) = 11.25, \quad \sigma^2 = (15)(0.75)(0.25) = 2.8125; \quad \sigma = \sqrt{2.8125} = 1.67705.$

2.4-8 **(a)** $1 - 0.01^4 = 0.99999999;$ **(b)** $0.99^4 = 0.960596.$

2.4-10 **(a)** X is $b(8,\ 0.90)$;

(b) (i) $P(X = 8) = P(8 - X = 0) = 0.4305;$

(ii) $P(X \leq 6) = P(8 - X \geq 2)$

$= 1 - P(8 - X \leq 1) = 1 - 0.8131 = 0.1869;$

(iii) $P(X \geq 6) = P(8 - X \leq 2) = 0.9619.$

2.4-16 It is given that X is $b(10,\ 0.10)$. We are to find M so that $P(1000X \leq M) \geq 0.99)$ or $P(X \leq M/1000) \geq 0.99$. From Appendix Table II, $P(X \leq 4) = 0.9984 > 0.99$. Thus $M/1000 = 4$ or $M = 4000$ dollars.

2.4-18 X is $b(5,\ 0.05)$. The expected number of tests is

$1\,P(X = 0) + 6\,P(X > 0) = 1\,(0.7738) + 6\,(1 - 0.7738) = 2.131.$

2.4-20 **(a)** **(i)** $b(5,\ 0.7)$; **(ii)** $\mu = 3.5, \sigma^2 = 1.05$; **(iii)** 0.1607;

(b) **(i)** geometric, $p = 0.3$; **(ii)** $\mu = 10/3, \sigma^2 = 70/9$; **(iii)** 0.51;

(c) **(i)** Bernoulli, $p = 0.55$; **(ii)** $\mu = 0.55, \sigma^2 = 0.2475$; **(iii)** 0.55;

(d) **(ii)** $\mu = 2.1, \sigma^2 = 0.89$; **(iii)** 0.7;

(e) **(i)** discrete uniform on $1, 2, \ldots, 10$; **(ii)** 5.5, 8.25; **(iii)** 0.2.

1 Urn = 7 red + 11 white

$X = 1$, $P(x=1) = \frac{7}{18}$ = Prob that red ball drawn

$X = 0$; $P(x=0) = \frac{11}{18}$ = Prob that white ball drawn

So, $X : Ber\left(\frac{7}{18}\right)$

$$f(x) = \left(\frac{7}{18}\right)^x \left(\frac{11}{18}\right)^{1-x} \quad x = 0, 1.$$

$$\text{Mean} = \sum x_i f(x_i)$$
$$= 0 \times f(0) + 1 \times f(1)$$
$$= \frac{7}{18}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$
$$= \left[0^2 \times f(0) + 1^2 \times f(1)\right] - \left(\frac{7}{18}\right)^2$$
$$= \frac{7}{18} \times \frac{11}{18} = \frac{77}{324}$$

2. $P(x=1) = \frac{7}{18}$ and $P(x=-1) = \frac{11}{18}$

$$f(x) = \begin{cases} 7/18 & x = 1 \\ 11/18 & x = -1 \end{cases}$$

$$\text{Mean, } E(X) = \frac{1 \times 7}{18} + \frac{(-1) \times 11}{18} = \frac{-4}{18} = \frac{-2}{9}$$

$$E(x^2) = 1^2 \times \frac{7}{18} + (-1)^2 \times \frac{11}{18} = 1$$

Variance, $\sigma^2 = E(x^2) - (E(x))^2$

$$= 1 - \left(\frac{-2}{9}\right)^2$$

$$= \frac{77}{81}$$

Prob of an answer being correct

$$P(C) = \frac{1}{5}$$

Incorrect $P(I) = \frac{4}{5}$

(a) $P(C \cap I \cap I \cap C \cap I \cap I) = P(C).P(I).P(I).P(C)P(I)P(I)$

$$= \frac{1}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{1}{5} \times \frac{4}{5} \times \frac{4}{5}$$

$$= \frac{(4)^4}{(5)^6} = 0.0164$$

(b) Prob. of being correct on two que. $= {}^6C_2 \{P(C)\}^2 \{P(I)\}^4$

$$= {}^6C_2 \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4$$

$$= \frac{\lfloor 6}{\lfloor 2 \lfloor 4} \times \frac{(4)^4}{(5)^6}$$

$$= 0.2458$$

Prob. that a duck is infected $(p) = 15\% = 0.15$

(a) $P(x) = {}^7C_x (p)^x (1-p)^{7-x}$; $x = 0, 1, \dots 7$

(b) (i) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - \sum_{X=0}^{1} P(x)$

$= 1 - \left[{}^7C_0 \, p^0 (1-p)^7 + {}^7C_1 \, p^1 (1-p)^6 \right]$

$= 1 - \left[(0.85)^7 + 7 \times (0.15) \times (0.85)^6 \right]$

$= 0.2834$

(ii) $P(X = 1) = {}^7C_1 \, p^1 (1-p)^6 = 7 \times 0.15 \times (0.85)^6$

$= 0.396$

(iii) $P(X \leq 3) = \sum_{X=0}^{3} P(x)$

$= {}^7C_0 \, p^0 (1-p)^7 + {}^7C_1 \, p^1 (1-p)^6 + {}^7C_2 \, p^2 (1-p)^5$
$+ {}^7C_3 \, p^3 (1-p)^4$

$= (0.85)^7 + 7 \times 0.15 \times (0.85)^6 + 21 \times (0.15)^2 \times (0.85)^5$
$+ 35 \times (0.15)^3 \times (0.85)^4$

$= 0.988$

$p = 0.20$

$n = 25$

$P(x) = {}^{n}C_x \, p^x (1-p)^{n-x} \;; x = 0, 1, \ldots, 25$

$P(x) = {}^{25}C_x \, p^x (1-p)^{25-x}$

(a) $P(x \le 4) = \sum_{x=0}^{4} p(x)$

$= {}^{25}C_0 \, p^0 (1-p)^{25} + {}^{25}C_1 \, p^1 (1-p)^{24} + {}^{25}C_2 \, p^2 (1-p)^{23}$

$+ {}^{25}C_3 \, p^3 (1-p)^{22} + {}^{25}C_4 \, p^4 (1-p)^{21}$

$= (0.80)^{25} + 25 \times 0.20 \times (0.80)^{24} + 300 \times (0.20)^2 \times (0.80)^{23}$

$+ 2300 \times (0.20)^3 \times (0.80)^{22} + 12650 \times (0.2)^4 (0.8)^{21}$

$P(x \le 4) = 0.42$

(b) $P(x \ge 5) = 1 - P(x \le 4)$

$= 1 - 0.42$

$= 0.58$

(c) $P(x=6) = {}^{25}C_6 \, (0.2)^6 (0.8)^{19}$

$= 0.163$

(d) mean of binomial $(\mu) = np = 25 \times 0.2$

~~$\mu = np = 25 \times 0.2$~~ $= 5$

~~$= 5$~~

~~$\mu = np(1-p) = 25 \times 0.2 \times 0.8$~~

~~$= 4$~~

Variance $= np(1-p)$

$= 25 \times 0.2 \times 0.8$

$= 4$

Standard deviation $= \sqrt{\text{Variance}}$

$= \sqrt{4}$

$= 2$

$p = 75\% = 0.75$

$n = 15$

(a) $P(x) = {}^{15}C_x (p)^x (1-p)^{15-x}$; $x = 0,1,2,\ldots 15$

(c) $P(X \le 10) = 1 - P(X > 10)$

Part (c)

$$= 1 - \sum_{x=11}^{15} P(x)$$

$$= 1 - \left[{}^{15}C_{11} p^{11} (1-p)^4 + {}^{15}C_{12} p^{12} (1-p)^3 + {}^{15}C_{13} p^{13} (1-p)^2 + {}^{15}C_{14} p^{14} (1-p) + {}^{15}C_{15} p^{15} \right]$$

$$= 1 - 0.686$$

$$= 0.314$$

(b) $P(X \ge 10) = 1 - P(X \le 10) + P(X = 10)$

Part (b)

$$= 1 - 0.314 + {}^{15}C_{10} (0.75)^{10} \times (0.25)^5$$

$$= 1 - 0.314 + 0.165$$

$$= 0.815$$

(d) $P(X = 10) = {}^{15}C_{10} (0.75)^{10} \times (0.25)^5$

$$= 0.165$$

(e) Mean

$\mu = np$

$= 15 \times 0.75$

$= 11.25$

$Var(X) = np(1-p)$

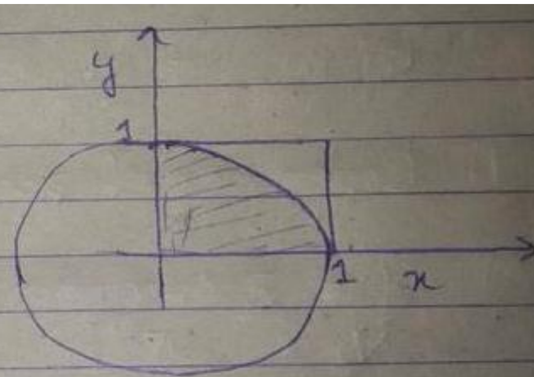
$= 15 \times 0.75 \times 0.25$

$= 2.8125$

$s.d. = \sqrt{Var(X)} = \sqrt{2.8125}$

$= 1.677$

7 shaded region in the figure shown that point which lies in square and satisfy the condition $x^2+y^2<1$



p: that point lie in shaded region

$$p = \frac{\frac{\pi(1)^2}{4}}{1} = \frac{\pi}{4}$$

a) if point lie inside the circle, it is success otherwise failure

so, $X : b(2000, \pi/4)$ $\quad X \sim b(n, p)$

b) for Binomial distribution

$$\text{Mean} = np = \frac{2000 \times \pi}{4} = 500\pi = 1570.796$$

Variance, $\sigma^2 = np(1-p)$

$$= 2000 \times \frac{\pi}{4}\left(1-\frac{\pi}{4}\right)$$

$$= 337.09578$$

$$\approx 337.0956$$

Standard Deviation, $\sigma = \sqrt{337.0956}$

$$= 18.360$$

(c) $E\left(\frac{W}{500}\right) = \frac{1}{500} E(W) = \frac{500\pi}{500} = \pi$

(d) Experimental part

(e) for 3-dimensional space

$V_3 = \frac{\pi^{3/2}}{\Gamma(3/2+1)} = \frac{\pi^{3/2}}{\Gamma(5/2)}$, is the volume of ball of radius 1 in 3D space

where Γ is a gamma function

(f) Similarly for n-D space volume of ball of radius 1, $V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ in n-D space

$n=4,\ p=0.99$

X = No. of valves that open properly

$$p(x) = {}^4C_x\, p^x (1-p)^{4-x} \ ;\ x = 0,1,2,3,4$$

(a) $p(x \geq 1) = 1 - p(x < 1)$

$$= 1 - \sum_{x=0}^{0} {}^4C_x\, p^x (1-p)^x$$

$$= 1 - {}^4C_0\, p^0 (1-p)^4$$

$$= 1 - (0.01)^4$$

$$= 0.99999999$$

(b) $p(x=4) = {}^4C_4\, p^4 (1-p)^0$

$$= (0.99)^4 = 0.9606$$

$n = 20,$

$p = 80\% = 0.8$

(a) $p(x) = {}^{20}C_x \, p^x (1-p)^{20-x}$; $x = 0, 1, 2 \ldots 20$

(b) Mean

$\mu = np = 20 \times 0.8 = 16$

$Var(x) = np(1-p) = 20 \times 0.8 \times 0.2 = 3.2$

$S.d. = \sqrt{Var(x)} = \sqrt{3.2} = 1.79$

(c) (i) $p(x=15) = {}^{20}C_{15} \, p^{15} (1-p)^5$

$= {}^{20}C_{15} (0.8)^{15} (0.2)^5$

$= 0.1746$

(ii) $p(x>15) = \sum_{x=16}^{20} p(x)$

$= {}^{20}C_{16} \, p^{16} (1-p)^4 + {}^{20}C_{17} \, p^{17} (1-p)^3 +$

${}^{20}C_{18} \, p^{18} (1-p)^2 + {}^{20}C_{19} \, p^{19} (1-p) + {}^{20}C_{20} \, p^{20}$

$= 0.6296$

(iii) $p(x \le 15) = 1 - p(x>15)$

$= 1 - 0.6296$

$= 0.3704$

$n = 8$

$p = 0.90$

(a) $P(X) = {}^{8}C_{x}\, p^{x}(1-p)^{8-x}$; $x = 0,1,2,\ldots 8$

(b) (i) $P(X=8) = {}^{8}C_{8}\, p^{8}(1-p)^{0}$

$= (0.9)^{8}$

$= 0.43$

(ii) $P(X \leq 6) = 1 - P(X > 6)$

$= 1 - \sum_{x=7}^{8} {}^{8}C_{x}\, p^{x}(1-p)^{8-x}$

$= 1 - \left[{}^{8}C_{7}\, p^{7}(1-p) + {}^{8}C_{8}\, p^{8}(1-p)^{0} \right]$

$= 0.187$

(iii) $P(X \geq 6) = \sum_{x=6}^{8} {}^{8}C_{x}\, p^{x}(1-p)^{8-x}$

$= {}^{8}C_{6}\, p^{6}(1-p)^{2} + {}^{8}C_{7}\, p^{7}(1-p) + {}^{8}C_{8}\, p^{8}$

$= 0.962$

$\mu = 6$

$np = 6$

$Var(x) = 3.6$

$np(1-p) = 3.6$

$\Rightarrow 6(1-p) = 3.6$

$1-p = 0.6$

$\boxed{p = 0.4}$

$np = 6$

$\boxed{n = 15}$

Hence $p(x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

$p(x) = {}^{6}C_{x} (0.4)^{x} (0.6)^{6-x}$

$p(x=4) = {}^{6}C_{4} (0.4)^{4} (0.6)^{2}$

$= 0.13824$

2.4.12 (a)

$$f(x)=\begin{cases} x=-1 & 125/216 \\ x=1 & 75/216 \\ x=2 & 15/216 \\ x=3 & 1/216 \end{cases}$$

(b) $\mu = -1\times f(-1) + 1\times f(1) + 2\times f(2) + 3\times f(3)$

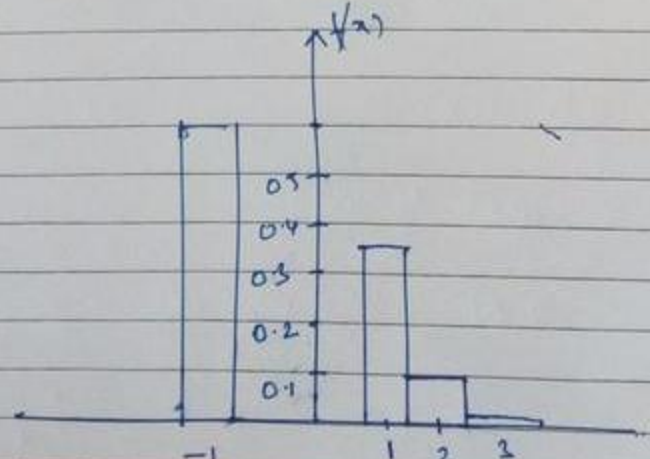
$= -1\times \frac{125}{216} + 1\times \frac{75}{216} + \frac{2\times 15}{216} + \frac{3\times 1}{216}$

$= -\frac{17}{216}$

$\sigma^2 = E(X^2) - \mu^2 = \frac{289}{216} - \left(\frac{-17}{216}\right)^2 = 1.239$

$\sigma = 1.11$

(c)



(a) $n = 10$, $p = \frac{1}{10} = 0.1$

Hence

$$P(x) = {}^{10}C_x\, p^x (1-p)^{10-x} \; ; \; x = 0, 1, \dots 10$$

$$P(x \geq 1) = 1 - P(x < 1)$$

$$= 1 - P(x = 0)$$

$$= 1 - {}^{10}C_0 (0.1)^0 (0.9)^{10}$$

$$= 0.651$$

(b) $n = 15$, $p = 0.1$

$$P(x) = {}^{15}C_x\, p^x (1-p)^{15-x} \; ; \; x = 0, 1, 2 \dots 15$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$= 1 - {}^{15}C_0\, p^0 (1-p)^{15}$$

$$= 0.794$$

$$P(x) = {}^{n}C_x (0.1)^x (0.9)^{n-x}$$

(a) $P(x \geq 1) > 0.50$

$1 - P(x=0) > 0.50$

$1 - {}^{n}C_0 (0.1)^0 (0.9)^n > 0.50$

$1 - (0.9)^n > 0.50$

$0.50 > (0.9)^n$

$\Rightarrow \boxed{n \geq 7}$ Hence minimum 7 tickets have to be purchased

(b) $P(x \geq 1) > 0.95$

$1 - (0.9)^n > 0.95$

$0.05 > (0.9)^n$

$\Rightarrow \boxed{n \geq 29}$ Hence minimum 29 tickets have to be purchased

Pg 23

15) First let us define the events for this question

A → hospital obtains vaccine from company A

B → hospital obtains vaccine from company B

C → hospital obtains vaccine from company ~~B~~ C

E → one of the 5 vials is ineffectiv

$P(E/A) = 0.03 \times 0.97^4$

$P(E/B) = 0.02 \times 0.98^4$

$P(E/C) = 0.05 \times 0.95^4$

Now, according to question, we have to find out

$$P(C/E) = \frac{P(C \cap E)}{P(E)} = \frac{P(C)\,P(E/C)}{P(E)} \quad \text{[Bayes' Theorem]}$$

Now, $P(C) = 0.1 \quad P(B) = 0.5 \quad P(A) = 0.4$

and $P(E) = P(A)P(E/A) + P(B)\,P\left(\frac{E}{B}\right) + P(C)\,P\left(\frac{E}{C}\right)$

$= 0.4 \times 0.03 \times 0.97^4 + 0.5 \times 0.02 \times 0.98^4 + 0.1 \times 0.05 \times 0.95^4$

$P(E) = 0.023$

$$\therefore P(C/E) = \frac{0.1 \times 0.05 \times 0.95^4}{0.023}$$

$P(C/E) = 0.177$

4.16 It is given that X is $b(10, 0.10)$. We are to find M so that

$$P(1000X < M) > 0.99$$

or

$$P(X \leq M/1000) \geq 0.99$$

$$\therefore \; P(X \leq 4) = 0.9984 > 0.99$$

Thus $M/1000 = 4$ or $M = 4000\,\$$

$n = 5$

$p = 60\% = 0.6$

Let x be the number of calls answered

(a) $P(x) = {}^5C_x \, p^x (1-p)^{5-x}$; $x = 0, 1, \ldots 5$

(a) $P(X=5) = {}^5C_5 (0.6)^5 (1-0.6)^0$

$= 0.0778$

(b) $P(X=3) = {}^5C_3 (0.6)^3 (0.4)^2$

$= 0.3456$

(c) $P(X \geq 1) = 1 - P(X=0)$

$= 1 - {}^5C_0 (0.6)^0 (0.4)^5$

$= 0.9898$

Pg 23
(B) Probability of a person having a disease = $p = 0.05$

number of individuals = $n = 5$

number of tests $\to t$	1	2	3	4	5	6
$P(t)$	${}^5C_0 (0.95)^5$	${}^5C_1\, 0.05(0.95)^4$	${}^5C_2\, 0.05^2(0.95)^3$	${}^5C_3\, 0.05^3(0.95)^2$	${}^5C_4 (0.05)^4\, 0.95$	${}^5C_5 (0.05)^5$

$$\therefore E(t) = \sum_{i=p}^{5} t_i P(t_i) = 1.25$$

1

$a)\ M(t) = \frac{1}{3} + \frac{2}{3}e * t = (1 - \frac{2}{3}) + \frac{2}{3}e^t$

Now, $\mu = np = 1 * \frac{2}{3} = \frac{2}{3}$

$\sigma^2 = npq = 1 * \frac{2}{3} * \frac{1}{3} = 2/9\ and\ \sigma = \sqrt{\frac{2}{9}}$

This implies that our distribution is a binomial distribution with number of trials=n=1 and p=2/3

2

$b)\ M(t) = (0.25 + 0.75e^t)^{12} = ((1 - 0.75) + 0.75e^t)^{12}$

Following the same steps as before,

$\mu = np = 12 * 0.75 = 9$

$\sigma^2 = npq = 12 * 0.75 * 0.25 = 2.25\ and\ \sigma = \sqrt{2.25}$

Simlar to the above problem, this again is a binomial distribution with n=12 and p=0.75

RESULT

$$a)\ b(1, \frac{2}{3})$$

$$b)\ b(12, 0.75)$$

2.5 THE NEGATIVE BINOMIAL DISTRIBUTION

Exercises

2.5-1. An excellent free-throw shooter attempts several free throws until she misses.

(a) If $p = 0.9$ is her probability of making a free throw, what is the probability of having the first miss on the 13th attempt or later?

(b) If she continues shooting until she misses three, what is the probability that the third miss occurs on the 30th attempt?

2.5-2. Show that 63/512 is the probability that the fifth head is observed on the tenth independent flip of a fair coin.

2.5-3. Suppose that a basketball player different from the ones in Example 2.5-2 and in Exercise 2.5-1 can make a free throw 60% of the time. Let X equal the minimum number of free throws that this player must attempt to make a total of 10 shots.

(a) Give the mean, variance, and standard deviation of X.

(b) Find $P(X = 16)$.

2.5-4. Suppose an airport metal detector catches a person with metal 99% of the time. That is, it misses detecting a person with metal 1% of the time. Assume independence of people carrying metal. What is the probability that the first metal-carrying person missed (not detected) is among the first 50 metal-carrying persons scanned?

2.5-5. Let the moment-generating function $M(t)$ of X exist for $-h < t < h$. Consider the function $R(t) = \ln M(t)$. The first two derivatives of $R(t)$ are, respectively,

$$R'(t) = \frac{M'(t)}{M(t)} \qquad \text{and} \qquad R''(t) = \frac{M(t)M''(t) - [M'(t)]^2}{[M(t)]^2}.$$

Setting $t = 0$, show that

(a) $\mu = R'(0)$.

(b) $\sigma^2 = R''(0)$.

2.5-6. Use the result of Exercise 2.5-5 to find the mean and variance of the

(a) Bernoulli distribution.

(b) Binomial distribution.

(c) Geometric distribution.

(d) Negative binomial distribution.

2.5-7. If $E(X^r) = 5^r, r = 1, 2, 3, \ldots$, find the moment-generating function $M(t)$ of X and the pmf of X.

2.5-8. The probability that a company's work force has no accidents in a given month is 0.7. The numbers of accidents from month to month are independent. What is the probability that the third month in a year is the first month that at least one accident occurs?

2.5-9. One of four different prizes was randomly put into each box of a cereal. If a family decided to buy this cereal until it obtained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?

2.5-10. In 2012, Red Rose tea randomly began placing 1 of 12 English porcelain miniature figurines in a 100-bag box of the tea, selecting from 12 nautical figurines.

(a) On the average, how many boxes of tea must be purchased by a customer to obtain a complete collection consisting of the 12 nautical figurines?

(b) If the customer uses one tea bag per day, how long can a customer expect to take, on the average, to obtain a complete collection?

.5 The Negative Binomial Distribution

2.5-2 $\dbinom{10-1}{5-1}\left(\dfrac{1}{2}\right)^5\left(\dfrac{1}{2}\right)^5 = \dfrac{126}{1024} = \dfrac{63}{512}.$

2.5-4 Let "being missed" be a success and let X equal the number of trials until the first success. Then $p = 0.01$.

$P(X \le 50) = 1 - 0.99^{50} = 1 - 0.605 = 0.395.$

2.5-6 **(a)**
$$\begin{aligned}
R(t) &= \ln(1-p+pe^t),\\
R'(t) &= \left[\frac{pe^t}{1-p+pe^t}\right]_{t=0} = p,\\
R''(t) &= \left[\frac{(1-p+pe^t)(pe^t)-(pe^t)(pe^t)}{(1-p+pe^t)^2}\right]_{t=0} = p(1-p);
\end{aligned}$$

(b)
$$\begin{aligned}
R(t) &= n\ln(1-p+pe^t),\\
R'(t) &= \left[\frac{npe^t}{1-p+pe^t}\right]_{t=0} = np,\\
R''(t) &= n\left[\frac{(1-p+pe^t)(pe^t)-(pe^t)(pe^t)}{(1-p+pe^t)^2}\right]_{t=0} = np(1-p);
\end{aligned}$$

(c)
$$\begin{aligned}
R(t) &= \ln p + t - \ln[1-(1-p)e^t],\\
R'(t) &= \left[1+\frac{(1-p)e^t}{1-(1-p)e^t}\right]_{t=0} = 1+\frac{1-p}{p} = \frac{1}{p},\\
R''(t) &= \left[(-1)\{1-(1-p)e^t\}^2\{-(1-p)e^t\}\right]_{t=0} = \frac{1-p}{p};
\end{aligned}$$

(d)
$$\begin{aligned}
R(t) &= r\left[\ln p + t - \ln\{1-(1-p)e^t\}\right],\\
R'(t) &= r\left[\frac{1}{1-(1-p)e^t}\right]_{t=0} = \frac{r}{p},\\
R''(t) &= r\left[(-1)\{1-(1-p)e^t\}^{-2}\{-(1-p)e^t\}\right]_{t=0} = \frac{r(1-p)}{p^2}.
\end{aligned}$$

2.5-8 $(0.7)(0.7)(0.3) = 0.147.$

2.5-10 **(a)** Let X equal the number of boxes that must be purchased. Then
$$E(X) = \sum_{x=1}^{12}\frac{1}{(13-x)/12} = \frac{86021}{2310} = 37.2385;$$

(b) $\dfrac{100\cdot E(X)}{365} \approx 10.2.$

Pg 78
2.5-1

$p = 0.9$

a) $E \rightarrow$ first miss on 13^{th} attempt or later

This can be understood as 12 successes in 12 attempts

$$\therefore P(E) = 0.9^{12} = 0.2824$$

b) $E \rightarrow$ Third miss on 30^{th} attempt

i.e. 2 misses and 27 success in 29 attempts and a success in the 30^{th} attempt

$$P(E) = {}^{29}C_2 \times 0.1^2 \times 0.9^{27} \times 0.1$$

$$= 0.0236$$

Pg 78
Q2.

$E \rightarrow$ Fifth head is observed on the tenth flip of an unbiased coin

$E \equiv$ 4 heads in 9 trials and the tenth trial having a head

$$\therefore P(E) = \underbrace{{}^{9}C_4 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^5}_{1} \times \left(\frac{1}{2}\right)$$

1. Probability of 4 heads in 9 trials

$$P(E) = \frac{\cancel{6} \times 7 \times \cancel{8}^{2} \times 9}{\cancel{24}} \times \frac{1}{2^{\cancel{10}\,9}}$$

$$\boxed{P(E) = 63/512}$$

Pg. 28

Q4. This problem can be seen as that of a binomial distribution

$p = 0.99 \rightarrow$ person is detected by metal detector

$q = 0.01 \rightarrow$ person is missed by the metal detector

Our event can be mathematically expressed as

$$(q+p)^{50} = \sum_{r=0}^{50} {}^{n}C_r\, q^{50-r} p^r$$

$E \rightarrow$ first metal carrying preson missed is among fist 50

$$P(E) = \sum_{r=0}^{49} {}^{n}C_r\, q^{50-r} p^r = 1 - {}^{50}C_{50} p^{50}$$

$$\boxed{P(E) = 1 - (0.99)^{50} = 0.39}$$

$$R'(t) = \frac{M'(t)}{M(t)} -$$

a) $$R'(0) = \frac{M'(0)}{M(0)} = \frac{\mu}{1} = \mu$$

$$(\because M(0) = E(e^{0 \cdot x}) = E(1) = 1.)$$

b) $$R''(0) = \frac{M(0)\,M''(0) - (M'(0))^2}{(M(0))^2}$$

$$= \frac{M''(0) - (M'(0))^2}{1} = \frac{E(x^2) - (E(x))^2}{1} = \sigma^2$$

From Exercise 2.5-5

$\mu = R'(0)$ where $R'(t) = \dfrac{M'(t)}{M(t)}$

$\sigma^2 = R''(0)$

$$R''(t) = \frac{M(t)M''(t) - [M'(t)]^2}{[M(t)]^2}$$

$M(t)$ is the moment-generating function

a) Bernoulli distribution

$M(t) = 1-p+pe^t$ $\qquad M(0) = 1-p+pe^0 = 1$

$M'(t) = pe^t$ (differentiating w.r.t. t) $\qquad M'(0) = pe^0 = p$

$M''(t) = pe^t$ $\qquad M''(0) = pe^0 = p$

$$R'(0) = \frac{M'(0)}{M(0)} = \frac{p}{1} = p$$

$$R''(0) = \frac{M(0)M''(0) - [M'(0)]^2}{(M(0))^2} = \frac{p-p^2}{1^2} = p(1-p)$$

$\mu = p$ and $\sigma^2 = p(1-p)$

b) Binomial distribution.

$$M(t) = (1-p+pe^t)^n$$

$$M'(t) = n(1-p+pe^t)^{n-1}[pe^t]$$

$$M''(t) = n(1-p+pe^t)^{n-1}(pe^t) + (pe^t)^2[n(n-1)(1-p+pe^t)^{n-2}]$$

$$= n(pe^t)(1-p+pe^t)^{n-2}[1-p+pe^t + (n-1)pe^t]$$

$$= n(pe^t)(1-p+pe^t)^{n-2}[1-p+n(pe^t)]$$

$M(0) = (1-p+pe^0)^n = 1$

$M'(0) = n(1)pe^0 = np$

$M''(0) = n(pe^0)(1)(1-p+npe^0) = np(1+(n-1)p)$

$R'(0) = \frac{M'(0)}{M(0)} = \frac{np}{1} = np$

$R''(0) = \frac{M(0)M''(0) - (M'(0))^2}{(M(0))^2} = \frac{np(1+(n-1)p) - (np)^2}{1}$

$= np[1+(n-1)p - np]$

$= np(1-p)$

$\mu = np$ and $\sigma^2 = np(1-p)$

c) Geometric distribution:

$M(t) = \frac{pe^t}{1-qe^t}$ $\quad q = 1-p$

$M'(t) = \frac{(1-qe^t)(pe^t) - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t}{(1-(1-p)e^t)^2}$

$M''(t) = \frac{(1-qe^t)^2 pe^t - 2pe^t(1-qe^t)(-qe^t)}{(1-qe^t)^4}$

$= \frac{pe^t(1+(1-p)e^t)}{(1-(1-p)e^t)^3}$

$$M(0) = 1$$

$$M'(0) = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

$$M''(0) = \frac{p(1+(1-p))}{(1-(1-p))^3} = \frac{2-p}{p^2}$$

$$R'(0) = \frac{M'(0)}{M(0)} = \frac{1}{p}$$

$$R''(0) = \frac{M(0)M''(0) - (M'(0))^2}{(M(0))^2} = \frac{\frac{2-p}{p^2} - \frac{1}{p^2}}{1}$$

$$= \frac{1-p}{p^2}$$

$\mu = 1/p$ and $\sigma^2 = \frac{1-p}{p^2}$

d) Negative Binomial Distribution:

$$M(t) = \frac{(pe^t)^r}{(1-(1-p)e^t)^r}$$

$$M'(t) = \frac{r(pe^t)^r}{(1-(1-p)e^t)^r}$$

$$M''(t) = \frac{r(pe^t)^r(r+1)(1-p)e^t}{(1-(1-p)e^t)^{r+2}} + \frac{r^2(pe^t)^r}{(1-(1-p)e^t)^{r+1}}$$

$$M(0) = \frac{p^r}{p^r} = 1$$

$$M'(0) = \frac{r p^r}{p^{r+1}} = \frac{r}{p}$$

$$M''(0) = \frac{r(r+1)p^r(1-p)}{p^{r+2}} + \frac{r^2 p^r}{p^{r+1}}$$

$$= \frac{r(r+1)(1-p)}{p^2} + \frac{r^2}{p}$$

$$= \frac{r}{p^2}(r+1-p)$$

$$R'(0) = \frac{M'(0)}{M(0)} = \frac{r}{p}$$

$$R''(0) = \frac{M(0)\,M''(0) - (M'(0))^2}{(M(0))^2}$$

$$= \frac{r}{p^2}(r+1-p) - \frac{r^2}{p^2}$$

$$= \frac{r(1-p)}{p^2}$$

$$\mu = r/p \quad \text{and} \quad \sigma^2 = \frac{r(1-p)}{p^2}$$

Pg 79.
2.52

$$E(X^r) = 5^r \quad \text{for } r = 1, 2, 3, \ldots$$

$$M(t) = M(0) + M'(0)\frac{t}{1!} + M''(0)\frac{t^2}{2!} + M'''(0)\frac{t^3}{3!} + \cdots$$

$$\Rightarrow M(t) = 1 + \frac{5t}{1!} + \frac{5^2t^2}{2!} + \frac{5^3t^3}{3!} + \cdots = \sum_{r=0}^{\infty} \frac{(5t)^r}{r!}$$

$$\Rightarrow M(t) = e^{5t}$$

Thus,

$$P(X=5) = 1 \longrightarrow \text{proof}$$

.S.8 Given Probability of no accident, $P(A) = 0.7$

$\therefore$ Probability of accident, $P(B) = 1 - 0.7$

$= 0.3$

$\therefore$ Probability that the third month in a year is the first month that atleast one accident

= No accident in first month $\times$ No accident in Second Month $\times$ Accident in third month

$= 0.7 \times 0.7 \times 0.3 = 0.147$

Pg 99
2.5-9

The first box will give us a new prize.

For the consecutive boxes, the probability of observing a new kind of prize becomes $3/4$, therefore expected number of boxes $= \frac{1}{3/4} = 4/3$

Similary, after obtaining 2 prizes, the probability of finding another new prize $= 2/4$ and finally for the last one, will be $1/4$

$\therefore$ Expected number of cereal boxes to be bougth $= 1 + 4/3 + 4/2 + 4/1 = 25/3$

2.5.10 (a) Let X be the equal numbers of boxes that must be purchased. Then boxes that must be purchased

$$E(X) = \sum_{r=1}^{12} \frac{1}{(13-r)/12}$$

$$= 37.23$$

(b) To obtain complete collection

$$= \frac{\text{No. of bag box of tea} \times \text{No. of boxes that must be purchased}}{\text{Total no. of days in a year}}$$

$$= \frac{100 \times 37.23}{365} \simeq 10.2$$

2.6 THE POISSON DISTRIBUTION

Exercises

2.6-1. Let X have a Poisson distribution with a mean of 4. Find

(a) $P(2 \le X \le 5)$.

(b) $P(X \ge 3)$.

(c) $P(X \le 3)$.

2.6-2. Let X have a Poisson distribution with a variance of 3. Find $P(X = 2)$.

2.6-3. Customers arrive at a travel agency at a mean rate of 11 per hour. Assuming that the number of arrivals per hour has a Poisson distribution, give the probability that more than 10 customers arrive in a given hour.

2.6-4. Find $P(X = 4)$ if X has a Poisson distribution such that $3P(X = 1) = P(X = 2)$.

2.6-5. Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume a Poisson distribution, find the probability of at most one flaw appearing in 225 square feet.

2.6-6. A certain type of aluminum screen that is 2 feet wide has, on the average, one flaw in a 100-foot roll. Find the probability that a 50-foot roll has no flaws.

2.6-7. With probability 0.001, a prize of \$499 is won in the Michigan Daily Lottery when a \$1 straight bet is placed. Let Y equal the number of \$499 prizes won by a gambler after placing n straight bets. Note that Y is $b(n, 0.001)$. After placing $n = 2000$ \$1 bets, the gambler is behind or even if $\{Y \le 4\}$. Use the Poisson distribution to approximate $P(Y \le 4)$ when $n = 2000$.

2.6-8. Suppose that the probability of suffering a side effect from a certain flu vaccine is 0.005. If 1000 persons are inoculated, find the approximate probability that

(a) At most 1 person suffers.

(b) 4, 5, or 6 persons suffer.

2.6-9. A store selling newspapers orders only $n = 4$ of a certain newspaper because the manager does not get many calls for that publication. If the number of requests per day follows a Poisson distribution with mean 3,

(a) What is the expected value of the number sold?

(b) What is the minimum number that the manager should order so that the chance of having more requests than available newspapers is less than 0.05?

2.6-10. The mean of a Poisson random variable X is $\mu = 9$. Compute

$$P(\mu - 2\sigma < X < \mu + 2\sigma).$$

2.6-11. An airline always overbooks if possible. A particular plane has 95 seats on a flight in which a ticket sells for \$300. The airline sells 100 such tickets for this flight.

(a) If the probability of an individual not showing up is 0.05, assuming independence, what is the probability that the airline can accommodate all the passengers who do show up?

(b) If the airline must return the \$300 price plus a penalty of \$400 to each passenger that cannot get on the flight, what is the expected payout (penalty plus ticket refund) that the airline will pay?

2.6-12. A baseball team loses \$100,000 for each consecutive day it rains. Say X, the number of consecutive days it rains at the beginning of the season, has a Poisson distribution with mean 0.2. What is the expected loss before the opening game?

2.6-13. Assume that a policyholder is four times more likely to file exactly two claims as to file exactly three claims. Assume also that the number X of claims of this policyholder is Poisson. Determine the expectation $E(X^2)$.

2.6 The Poisson Distribution

2.6-2 $\lambda = \mu = \sigma^2 = 3$ so $P(X = 2) = 0.423 - 0.199 = 0.224$.

2.6-4
$$\begin{aligned} 3\,\frac{\lambda^1 e^{-\lambda}}{1!} &= \frac{\lambda^2 e^{-\lambda}}{2!} \\ e^{-\lambda}\lambda(\lambda - 6) &= 0 \\ \lambda &= 6 \end{aligned}$$
Thus $P(X = 4) = 0.285 - 0.151 = 0.134$.

2.6-6 $\lambda = (1)(50/100) = 0.5$, so $P(X = 0) = e^{-0.5}/0! = 0.607$.

2.6-8 $np = 1000(0.005) = 5$;

(a) $P(X \le 1) \approx 0.040$;

(b) $P(X = 4, 5, 6) = P(X \le 6) - P(X \le 3) \approx 0.762 - 0.265 = 0.497$.

2.6-10 $\sigma = \sqrt{9} = 3$,

$P(3 < X < 15) = P(X \le 14) - P(X \le 3) = 0.959 - 0.021 = 0.938$.

2.6-12 Since $E(X) = 0.2$, the expected loss is $(0.02)(\$10{,}000) = \$2{,}000$.

2.4-12 **(a)**

$$f(x) = \begin{cases} 125/216, & x = -1, \\ 75/216, & x = 1, \\ 15/216, & x = 2, \\ 1/216, & x = 3; \end{cases}$$

(b)
$$\begin{aligned} \mu &= (-1)\cdot\frac{125}{216} + (1)\cdot\frac{75}{216} + (2)\cdot\frac{15}{216} + (3)\cdot\frac{1}{216} = -\frac{17}{216}; \\ \sigma^2 &= E(X^2) - \mu^2 = \frac{269}{216} - \left(-\frac{17}{216}\right)^2 = 1.2392; \\ \sigma &= 1.11; \end{aligned}$$

(c) See Figure 2.4-12.

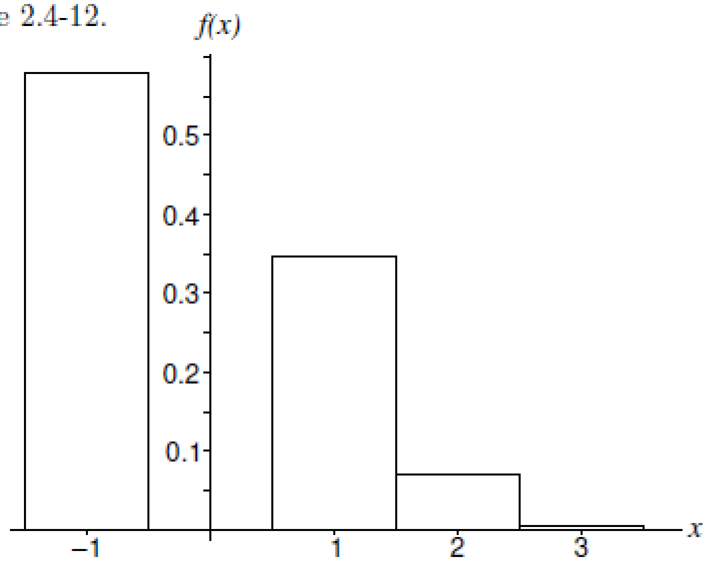


Figure 2.4–12: Losses in chuck-a-luck

2.4-14 Let X equal the number of winning tickets when n tickets are purchased. Then

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \left(\frac{9}{10}\right)^n. \end{aligned}$$

(a)
$$\begin{aligned} 1 - (0.9)^n &= 0.50 \\ (0.9)^n &= 0.50 \\ n \ln 0.9 &= \ln 0.5 \\ n &= \frac{\ln 0.5}{\ln 0.9} = 6.58 \end{aligned}$$

so $n = 7$.

(b)
$$\begin{aligned} 1 - (0.9)^n &= 0.95 \\ (0.9)^n &= 0.05 \\ n &= \frac{\ln 0.05}{\ln 0.09} = 28.43 \end{aligned}$$

$$\mu = 4 = \lambda$$

$$f(x) = \frac{e^{-4}4^x}{x!}$$

(a) $$P(2 \le X \le 5) = \sum_{x=2}^{5} f(x)$$

$$= \sum_{x=0}^{5} \frac{e^{-4}4^x}{x!} - \sum_{x=0}^{1} \frac{e^{-4}4^x}{x!}$$

By table III of appendix B

$$\sum_{x=0}^{5} \frac{e^{-4}4^x}{x!} = 0.785$$

$$\sum_{x=0}^{1} \frac{e^{-4}4^x}{x!} = 0.092$$

Hence $$P(2 \le X \le 5) = 0.785 - 0.092$$
$$= 0.693$$

(b) $$P(X \ge 3) = 1 - P(X < 3)$$

$$= 1 - \sum_{x=0}^{2} f(x)$$

From Table III of appendix B

$$\sum_{x=0}^{2} f(x) = 0.238$$

Hence $$P(X \ge 3) = 1 - 0.238 = 0.762$$

(c) $$P(X \le 3) = \sum_{x=0}^{3} f(x) = 0.433$$

$$\sigma^2 = 3 = \lambda$$

$$f(x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x} \quad ; \quad x = 0, 1, 2 \ldots$$

$$P(x=2) = \frac{e^{-3}3^2}{\lfloor 2} = f(2)$$

$$= 0.224$$

$\mu = 11 = \lambda$

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{\lfloor x} = \frac{e^{-11}\,11^{x}}{\lfloor x} \quad ; \; x = 0, 1, 2 \cdots$$

$$P(X > 10) = 1 - P(X \le 10)$$

$$= 1 - \sum_{x=0}^{10} f(x)$$

$$\cancel{= 1 - e^{-11}\left[\frac{11^{0}}{\lfloor 0} + \frac{11^{1}}{\lfloor 1} + \frac{11^{2}}{\lfloor 2} + \cdots + \frac{11^{10}}{\lfloor 10}\right]}$$

$$\cancel{= 1 - e^{-11}}$$

~~From~~ By Table III in appendix B.

$$\sum_{x=0}^{10} \frac{e^{-11}\,11^{x}}{\lfloor x} = 0.46$$

Hence $P(X > 10) = 1 - 0.46$

$$= 0.54$$

We know, $P(x=n) = \frac{\lambda^n e^{-\lambda}}{n!}$

$\therefore$ Given $3P(x=1) = P(x=2)$

$$\frac{3\lambda^1 e^{-\lambda}}{1!} = \frac{\lambda^2 e^{-\lambda}}{2!}$$

$$6\lambda e^{-\lambda} = \lambda^2 e^{-\lambda}$$

$$\lambda^2 e^{-\lambda} - 6\lambda e^{-\lambda} = 0$$

$$\lambda e^{-\lambda}(\lambda - 6) = 0$$

$$\therefore \lambda = 6$$

Thus $P(x=4) = \frac{6^4 e^{-6}}{4!} = 0.1338$

1 1/150= lamda/225 lamda = 1.5	If one out of every 150 square feet has a flaw, then 1.5 is expected out of 225 to have a flaw. Thus our lamda is 1.5
2 Either look at the table in the back for P(x<_ 1) on page 491 or do the poisson formula for P(x=1)+P(x=0)	

RESULT

0.558

1	We assume the flaws follow a Poisson process	
2	Let X = number of flaws E(X) in a 50 foot section = .5	Expected flaws in a 100 foot section is 1. Therefore the expected flaws in a 50 foot section is half of that, .5
3	f(X) = (.5^x * e^-.5)/ x!	In a Poisson process, E(X) = mean = lambda
4	P(X=0) = (.5^0 * e^-.5)/1 = .606	Plug in 0 for x since we are interested in the probability of 0 flaw

RESULT

P(X=0) = .606

Given data

$n = 2000$

$p = 0.001$

$\lambda = np = 2$

$$f(x) = \frac{2^x e^{-2}}{x!}$$

$P(X \leq 4)$ from the Poisson Distribution Table

is given by

for $\lambda = 2$, $P(X \leq 4) = 0.947$

8. $p = .005$; 1000 people

$$\lambda = .005 \cdot 1000 = 5$$

$$f(x) = \frac{5^x e^{-5}}{x!}$$

a. $f(x \le 1) = \sum_{x=0}^{1} \frac{5^x e^{-5}}{x!} = .0404$

b. $f(4 \le x \le 6) = \sum_{x=4}^{6} \frac{5^x e^{-5}}{x!} = .497$

a) Using $\lambda = \mu = 3$ and X = number of papers requested, let Y = number of papers sold (so $Y \in \{0,1,2,3,4\}$).

Then
$$E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=0) + 2 \cdot P(Y=2) + 3 \cdot P(Y=3) + 4 \cdot P(Y=4)$$
$$= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + 4 \cdot P(X \ge 4)$$

With $P(X=x) = \frac{3^x e^{-x}}{x!}$,

$$E(Y) = 0 \cdot .049787 + 1 \cdot .149361 + 2 \cdot .224042 + 3 \cdot .224042 + 4 \cdot ((1 - .049787 + .149361 + \cdots + .224042)$$
$$= 2.680643 \cong 2.681.$$

(b) When n=6, P(X<=6) = 0.96649 > 0.95, so the probability of having too many requests is 0.03351 which is less than 0.05. (Note n=5 is not sufficient.)

$\mu = 9 = \lambda$

$\sigma^2 = 9$

$\sigma = 3 \quad ; \quad f(x) = \dfrac{e^{-9} 9^x}{x!}$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(3 < X < 15)$

it's (14) $\longrightarrow$ $= \sum_{x=4}^{14} f(x) = \sum_{x=0}^{14} f(x) - \sum_{x=0}^{3} f(x)$

from Table III of appendix B

$\sum_{x=0}^{14} f(x) = 0.959$

$\sum_{x=0}^{3} f(x) = 0.021$

Hence $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.959 - 0.021$

$= 0.938$

I

(100,000)(0.2)

RESULT

$20,000

Given

$P(X=2) = 4P(X=3) \quad -①$

and X is a Poisson Distribution.

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$① \Rightarrow \frac{\lambda^2 . e^{-\lambda}}{2!} = \frac{\lambda^3 e^{-\lambda}}{3!}$$

$$\frac{1}{2} = \frac{\lambda}{6}$$

$$\boxed{\lambda = 3}$$

$$E(x^2) = E(x(x-1) + x)$$

$$= E(x(x-1)) + E(x)$$

$$E(x) = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-3} \sum_{x=0}^{\infty} \frac{3^x}{(x-1)!}$$ [Eliminating $x=0$ because $E(x)=0$ at $x=0$]

$$= e^{-3} . \sum_{x=1}^{\infty} \frac{3^{x-1}}{(x-1)!} . 3$$

$$= 3e^{-3} . e^3 = 3 \quad \left[\sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a \right]$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{3^x e^{-3}}{(x-2)!}$$

$$= e^{-3} \sum_{x=2}^{\infty} \frac{3^{x-2} \cdot 3^2}{(x-2)!}$$

[Eliminating $x=0$ and $x=1$ because $E(X(X-1)) = 0$ at $x=0,1$]

$$= 9e^{-3} \sum_{x=2}^{\infty} \frac{3^{x-2}}{(x-2)!}$$

$$= 9e^{-3} \cdot e^{3}$$

$$= 9.$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$= 12.$$

Chapter 2

2.1-3 **(a)** 10; **(b)** 1/55; **(c)** 3; **(d)** 1/30; **(e)** $n(n+1)/2$; **(f)** 1.

2.1-5 **(b)**

x	Frequency	Relative Frequency	$f(x)$
1	38	0.38	0.40
2	27	0.27	0.30
3	21	0.21	0.20
4	14	0.14	0.10

2.1-7 **(a)** $f(x) = \dfrac{13-2x}{36}$, $x = 1,2,3,4,5,6$;

(b) $g(0) = \dfrac{6}{36}, g(y) = \dfrac{12-2y}{36}$, $y = 1,2,3,4,5$.

2.1-11 0.416.

2.1-13 **(a)** 19/20; **(b)** 1/20; **(c)** 9/20.

2.1-15 **(c)**

x	Frequency	Relative Frequency	$f(x)$
0	13	0.325	0.2532
1	16	0.400	0.4220
2	9	0.225	0.2509
3	2	0.050	0.0660
4	0	0.000	0.0076
5	0	0.000	0.0003

2.1-17 78.

2.2-1 **(a)** 3; **(b)** 7; **(c)** 4/3; **(d)** 7/3; **(e)** $(2n+1)/3$; **(f)** $E(X) = +\infty$, so does not exist.

2.2-3 \$360.

2.2-5 **(a)** $h(z) = (4 - z^{1/3})/6,\ z = 1, 8, 27$; **(b)** 23/3 of a dollar; **(c)** 7/3 of a dollar.

2.2-7 $E(X) = -17/216 = -\$0.0787$.

2.2-9 **(a)** $-\$1/19$; **(b)** $-\$1/37$.

2.2-11 $-\$0.01414$.

2.3-1 **(a)** 15; 50; **(b)** 5; 0; **(c)** 5/3; 5/9.

2.3-3 **(a)** 16; **(b)** 6; **(c)** 16.

2.3-5 $\mu = 7$.

2.3-7 $m = 7$.

2.3-9 \$1809.80.

2.3-11 $\mu = 2, \sigma^2 = 4/5$,

$$f(x) = \begin{cases} 2/5, & x = 1, \\ 1/5, & x = 2, \\ 2/5, & x = 3. \end{cases}$$

2.3-13 $(4/5)^3(1/5)$.

2.3-15 **(a)** 0.4604; **(b)** 0.5580; **(c)** 0.0184.

2.3-17 **(a)** $f(x) = (x-1)/2^x,\ x = 2, 3, \ldots$;
(c) $\mu = 4, \sigma^2 = 4$;
(d) **(i)** 1/2, **(ii)** 5/16, **(iii)** 1/4.

2.3-19 **(a)** $\mu = 1, \sigma^2 = 1$; **(b)** 19/30.

2.4-1 $f(x) = (7/18)^x(11/18)^{1-x},\ x = 0, 1;\ \mu = 7/18;\ \sigma^2 = 77/324$.

2.4-3 **(a)** $(1/5)^2(4/5)^4 = 0.0164$;
(b) $\dfrac{6!}{2!4!}(1/5)^2(4/5)^4 = 0.2458$.

2.4-5 **(a)** 0.4207; **(b)** 0.5793; **(c)** 0.1633; **(d)** $\mu = 5, \sigma^2 = 4, \sigma = 2$.

2.4-7 **(a)** $b(2000, \pi/4)$; **(b)** 1570.796, 337.096, 18.360; **(c)** π;
(f) $V_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of a ball of radius 1 in n-space.

2.4-9 **(a)** $b(20, 0.80)$; **(b)** $\mu = 16, \sigma^2 = 3.2, \sigma = 1.789$;
(c) **(i)** 0.1746, **(ii)** 0.6296, **(iii)** 0.3704.

2.4-11 0.1268.

radius 1 in n-space.

2.4-9 **(a)** $b(20, 0.80)$; **(b)** $\mu = 16, \sigma^2 = 3.2, \sigma = 1.789$; **(c)** **(i)** 0.1746, **(ii)** 0.6296, **(iii)** 0.3704.

2.4-11 0.1268.

2.4-13 **(a)** 0.6513; **(b)** 0.7941.

2.4-15 0.178.

2.4-17 **(a)** 0.0778; **(b)** 0.3456; **(c)** 0.9898.

2.4-19 **(a)** $b(1, 2/3)$; **(b)** $b(12, 0.75)$.

2.5-1 **(a)** $0.9^{12} = 0.2824$; **(b)** 0.0236.

2.5-3 **(a)** $\mu = 10/0.60, \sigma^2 = 4/0.36, \sigma = 2/0.60$; **(b)** 0.1240.

2.5-7 $M(t) = e^{5t}, \ -\infty < t < \infty, \ f(5) = 1$.

2.5-9 25/3.

2.6-1 **(a)** 0.693; **(b)** 0.762; **(c)** 0.433.

2.6-3 0.540.

2.6-5 0.558.

2.6-7 0.947.

2.6-9 **(a)** 2.681; **(b)** $n = 6$.

2.6-11 **(a)** 0.564 using binomial, 0.560 using Poisson approximation.
(b) \$598.56 using binomial, \$613.90 using Poisson approximation.

2.6-13 21/16.

Chapter 3 Continuous Distributions

3.1 Random Variables of the Continuous Type

Exercises

3.1-1. Show that the mean, variance, and mgf of the uniform distribution are as given in this section.

3.1-2. Let $f(x) = 1/2$, $-1 \le x \le 1$, be the pdf of X. Graph the pdf and cdf, and record the mean and variance of X.

3.1-3. Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minutes that the customer arrived. If X is $U(0, 10)$, find

(a) The pdf of X.

(b) $P(X \geq 8)$.

(c) $P(2 \leq X < 8)$.

(d) $E(X)$.

(e) $\text{Var}(X)$.

3.1-4. If the mgf of X is

$$M(t) = \frac{e^{5t} - e^{4t}}{t}, \; t \neq 0, \qquad \text{and} \qquad M(0) = 1,$$

find **(a)** $E(X)$, **(b)** $\text{Var}(X)$, and **(c)** $P(4.2 < X \leq 4.7)$.

3.1-5. Let Y have a uniform distribution $U(0, 1)$, and let

$$W = a + (b - a)Y, \qquad a < b.$$

(a) Find the cdf of W.
HINT: Find $P[a + (b - a)Y \leq w]$.

(b) How is W distributed?

3.1-6. A grocery store has n watermelons to sell and makes \$1.00 on each sale. Say the number of consumers of these watermelons is a random variable with a distribution that can be approximated by

$$f(x) = \frac{1}{200}, \qquad 0 < x < 200,$$

a pdf of the continuous type. If the grocer does not have enough watermelons to sell to all consumers, she figures that she loses \$5.00 in goodwill from each unhappy customer. But if she has surplus watermelons, she loses 50 cents on each extra watermelon. What should n be to maximize profit? HINT: If $X \leq n$, then her profit is $(1.00)X + (-0.50)(n - X)$; but if $X > n$, her profit is $(1.00)n + (-5.00)(X - n)$. Find the expected value of profit as a function of n, and then select n to maximize that function.

3.1-7. For each of the following functions, **(i)** find the constant c so that $f(x)$ is a pdf of a random variable X, **(ii)** find the cdf, $F(x) = P(X \leq x)$, **(iii)** sketch graphs of the pdf $f(x)$ and the cdf $F(x)$, and **(iv)** find μ and σ^2:

(a) $f(x) = 4x^c, \qquad 0 \leq x \leq 1.$

(b) $f(x) = c\sqrt{x}, \qquad 0 \leq x \leq 4.$

(c) $f(x) = c/x^{3/4}, \qquad 0 < x < 1.$

3.1-8. For each of the following functions, **(i)** find the constant c so that $f(x)$ is a pdf of a random variable X, **(ii)** find the cdf, $F(x) = P(X \leq x)$, **(iii)** sketch graphs of the pdf $f(x)$ and the distribution function $F(x)$, and **(iv)** find μ and σ^2:

(a) $f(x) = x^3/4, \qquad 0 < x < c.$

(b) $f(x) = (3/16)x^2, \qquad -c < x < c.$

(c) $f(x) = c/\sqrt{x}, \qquad 0 < x < 1.$ Is this pdf bounded?

3.1-9. Let the random variable X have the pdf $f(x) = 2(1 - x)$, $0 \leq x \leq 1$, zero elsewhere.

(a) Sketch the graph of this pdf.

(b) Determine and sketch the graph of the cdf of X.

(c) Find **(i)** $P(0 \leq X \leq 1/2)$, **(ii)** $P(1/4 \leq X \leq 3/4)$, **(iii)** $P(X = 3/4)$, and **(iv)** $P(X \geq 3/4)$.

3.1-10. The pdf of X is $f(x) = c/x^2$, $1 < x < \infty$.

(a) Calculate the value of c so that $f(x)$ is a pdf.

(b) Show that $E(X)$ is not finite.

3.1-11. The pdf of Y is $g(y) = d/y^3$, $1 < y < \infty$.

(a) Calculate the value of d so that $g(y)$ is a pdf.

(b) Find $E(Y)$.

(c) Show that $\text{Var}(Y)$ is not finite.

3.1-12. Sketch the graphs of the following pdfs and find and sketch the graphs of the cdfs associated with these distributions (note carefully the relationship between the shape of the graph of the pdf and the concavity of the graph of the cdf):

(a) $f(x) = \left(\frac{3}{2}\right)x^2, \qquad -1 < x < 1.$

(b) $f(x) = \frac{1}{2}, \qquad -1 < x < 1.$

(c) $f(x) = \begin{cases} x + 1, & -1 < x < 0, \\ 1 - x, & 0 \leq x < 1. \end{cases}$

3.1-13. The logistic distribution is associated with the cdf $F(x) = (1 + e^{-x})^{-1}$, $-\infty < x < \infty$. Find the pdf of the logistic distribution and show that its graph is symmetric about $x = 0$.

3.1-14. Let $f(x) = 1/2$, $0 < x < 1$ or $2 < x < 3$, zero elsewhere, be the pdf of X.

(a) Sketch the graph of this pdf.

(b) Define the cdf of X and sketch its graph.

(c) Find $q_1 = \pi_{0.25}$.

(d) Find $m = \pi_{0.50}$. Is it unique?

(e) Find $q_3 = \pi_{0.75}$.

3.1-15. The life X (in years) of a voltage regulator of a car has the pdf

$$f(x) = \frac{3x^2}{7^3} e^{-(x/7)^3}, \qquad 0 < x < \infty.$$

(a) What is the probability that this regulator will last at least 7 years?

(b) Given that it has lasted at least 7 years, what is the conditional probability that it will last at least another 3.5 years?

3.1-16. Let $f(x) = (x+1)/2$, $-1 < x < 1$. Find **(a)** $\pi_{0.64}$, **(b)** $q_1 = \pi_{0.25}$, and **(c)** $\pi_{0.81}$.

3.1-17. An insurance agent receives a bonus if the loss ratio L on his business is less than 0.5, where L is the total losses (say, X) divided by the total premiums (say, T). The bonus equals $(0.5 - L)(T/30)$ if $L < 0.5$ and equals zero otherwise. If X (in \$100,000) has the pdf

$$f(x) = \frac{3}{x^4}, \qquad x > 1,$$

and if T (in \$100,000) equals 3, determine the expected value of the bonus.

3.1-18. The weekly demand X for propane gas (in thousands of gallons) has the pdf

$$f(x) = 4x^3 e^{-x^4}, \qquad 0 < x < \infty.$$

If the stockpile consists of two thousand gallons at the beginning of each week (and nothing extra is received during the week), what is the probability of not being able to meet the demand during a given week?

3.1-19. The total amount of medical claims (in \$100,000) of the employees of a company has the pdf that is given by $f(x) = 30x(1-x)^4$, $0 < x < 1$. Find

(a) The mean and the standard deviation of the total in dollars.

(b) The probability that the total exceeds \$20,000.

3.1-20. Nicol (see References) lets the pdf of X be defined by

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ c/x^3, & 1 \le x < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Find

(a) The value of c so that $f(x)$ is a pdf.

(b) The mean of X (if it exists).

(c) The variance of X (if it exists).

(d) $P(1/2 \le X \le 2)$.

3.1-21. Let $X_1, X_2, \ldots, X_k$ be random variables of the continuous type, and let $f_1(x), f_2(x), \ldots, f_k(x)$ be their corresponding pdfs, each with sample space $S = (-\infty, \infty)$. Also, let $c_1, c_2, \ldots, c_k$ be nonnegative constants such that $\sum_{i=1}^{k} c_i = 1$.

(a) Show that $\sum_{i=1}^{k} c_i f_i(x)$ is a pdf of a continuous-type random variable on S.

(b) If X is a continuous-type random variable with pdf $\sum_{i=1}^{k} c_i f_i(x)$ on S, $E(X_i) = \mu_i$, and $\mathrm{Var}(X_i) = \sigma_i^2$ for $i = 1, \ldots, k$, find the mean and the variance of X.

Figure 3.1–8: Melting points of metal alloys

There are 31 observations within one standard deviation of the mean (62%) and 48 observations within two standard deviations of the mean (96%).

Chapter 3

Continuous Distributions

3.1 Random Variables of the Continuous Type

3.1–2 $\mu = 0$, $\sigma^2 = (1+1)^2/12 = 1/3$.

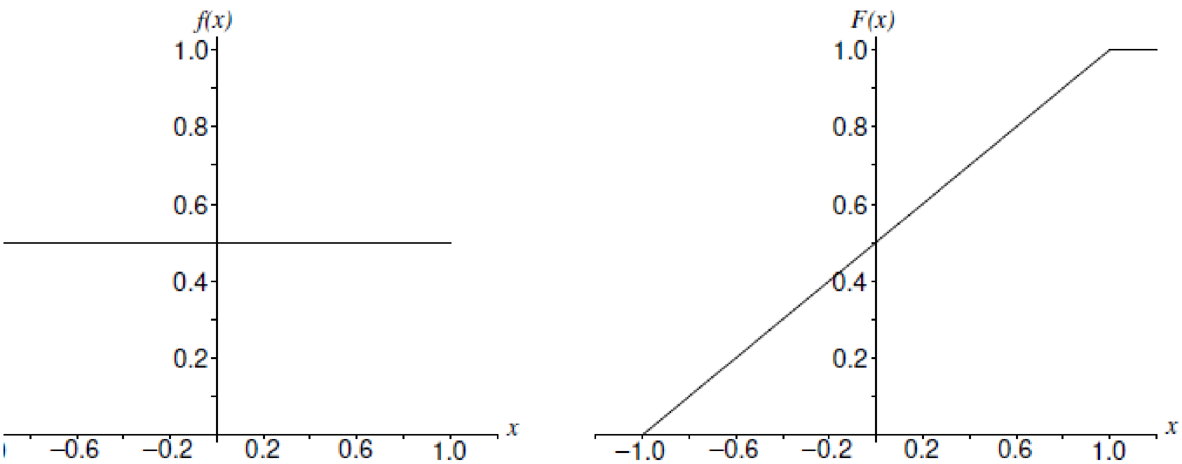


Figure 3.1–2: $f(x) = 1/2$ and $F(x) = (x+1)/2$

3.1–4 X is $U(4,5)$;

(a) $\mu = 9/2$; **(b)** $\sigma^2 = 1/12$; **(c)** 0.5.

3.1–6

$$\begin{aligned}
E(\text{profit}) &= \int_0^n [x - 0.5(n-x)]\frac{1}{200}\,dx + \int_n^{200}[n - 5(x-n)]\frac{1}{200}\,dx \\
&= \frac{1}{200}\left[\frac{x^2}{2} + \frac{(n-x)^2}{4}\right]_0^n + \frac{1}{200}\left[6nx - \frac{5x^2}{2}\right]_n^{200} \\
&= \frac{1}{200}[-3.25n^2 + 1200n - 100000] \\
\text{derivative} &= \frac{1}{200}[-6.5n + 1200] \ = \ 0 \\
n &= \frac{1200}{6.5} \ \approx \ 185.
\end{aligned}$$

3.1–8 **(a)** **(i)**

$$\begin{aligned} \int_0^c x^3/4\,dx &= 1 \\ c^4/16 &= 1 \\ c &= 2; \end{aligned}$$

(ii)

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_0^x t^3/4\,dt \\ &= x^4/16, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ x^4/16, & 0 \le x < 2, \\ 1, & 2 \le x < \infty. \end{cases}$$

(iii)

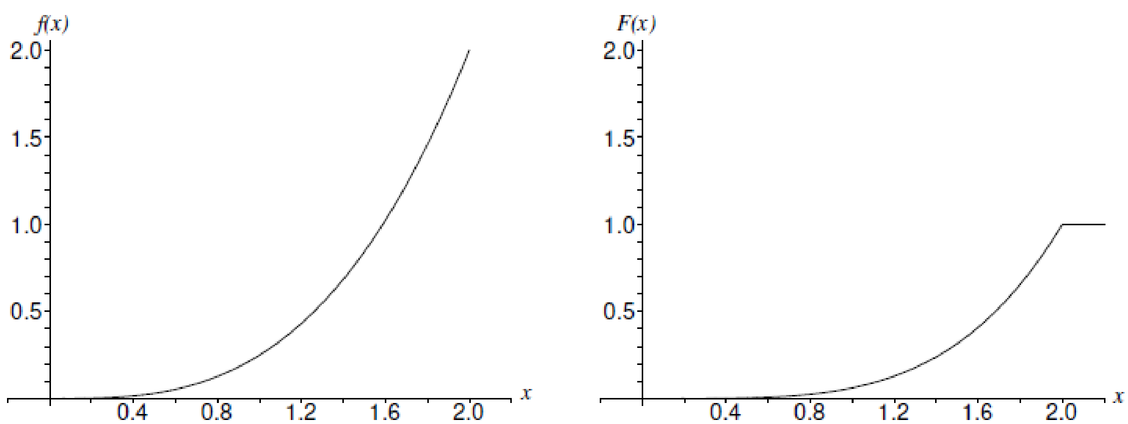


Figure 3.1–8: **(a)** Continuous distribution pdf and cdf

(iv)

$$\begin{aligned} \mu &= \int_0^2 (x)(x^3/4)\,dx \;=\; \frac{8}{5}; \\ E(X^2) &= \int_0^2 (x^2)(x^3/4)\,dx \;=\; \frac{8}{3}; \\ \sigma^2 &= \frac{8}{3} - \left(\frac{8}{5}\right)^2 \;=\; \frac{8}{75}. \end{aligned}$$

(b) (i)
$$\begin{aligned}\int_{-c}^{c}(3/16)x^2\,dx &= 1\\ c^3/8 &= 1\\ c &= 2;\end{aligned}$$

(ii)
$$\begin{aligned}F(x) &= \int_{-\infty}^{x} f(t)\,dt\\ &= \int_{-2}^{x}(3/16)t^2\,dt\\ &= \left[\frac{t^3}{16}\right]_{-2}^{x}\\ &= \frac{x^3}{16}+\frac{1}{2},\end{aligned}$$

$$F(x)=\begin{cases}0, & -\infty < x < -2,\\ \dfrac{x^3}{16}+\dfrac{1}{2}, & -2 \le x < 2,\\ 1, & 2 \le x < \infty.\end{cases}$$

(iii)

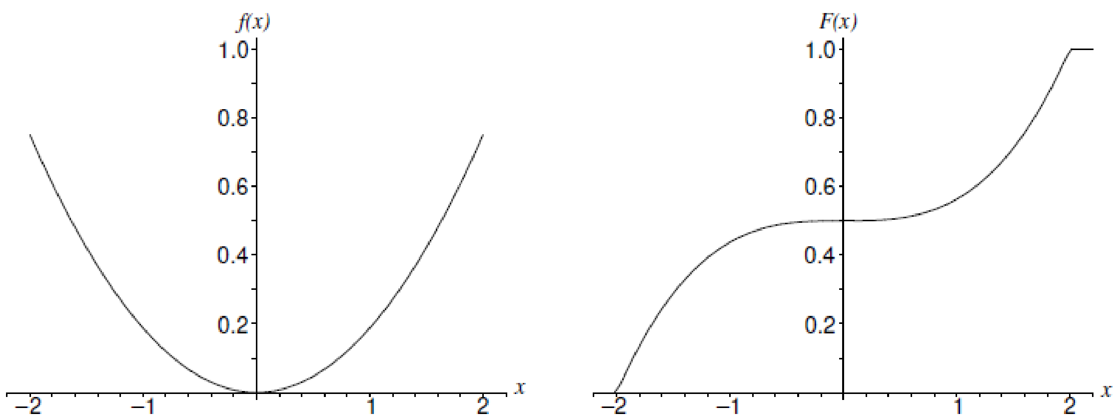


Figure 3.1–8: **(b)** Continuous distribution pdf and cdf

(iv)
$$\begin{aligned}\mu &= \int_{-2}^{2}(x)(3/16)(x^2)\,dx \;=\; 0;\\ \sigma^2 &= \int_{-2}^{2}(x^2)(3/16)(x^2)\,dx \;=\; \frac{12}{5}.\end{aligned}$$

(c) **(i)**

$$\begin{aligned}\int_0^1 \frac{c}{\sqrt{x}}\,dx &= 1\\ 2c &= 1\\ c &= 1/2.\end{aligned}$$

The pdf in part (c) is unbounded.

(ii)

$$\begin{aligned}F(x) &= \int_{-\infty}^{x} f(t)\,dt\\ &= \int_0^x \frac{1}{2\sqrt{t}}\,dt\\ &= \left[\sqrt{t}\right]_0^x \;=\; \sqrt{x},\end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0,\\ \sqrt{x}, & 0 \le x < 1,\\ 1, & 1 \le x < \infty.\end{cases}$$

(iii)

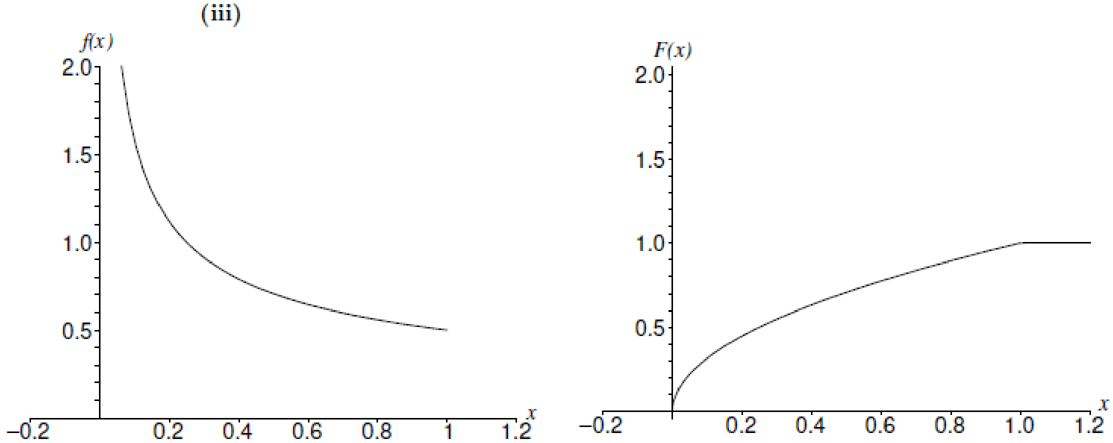


Figure 3.1–8: **(c)** Continuous distribution pdf and cdf

(iv)

$$\begin{aligned}\mu &= \int_0^1 (x)(1/2)/\sqrt{x}\,dx \;=\; \frac{1}{3};\\ E(X^2) &= \int_0^1 (x^2)(1/2)/\sqrt{x}\,dx \;=\; \frac{1}{5};\\ \sigma^2 &= \frac{1}{5} - \left(\frac{1}{3}\right)^2 \;=\; \frac{4}{45}.\end{aligned}$$

3.1–10 **(a)**

$$\begin{aligned}\int_1^\infty \frac{c}{x^2}\,dx &= 1\\ \left[\frac{-c}{x}\right]_1^\infty &= 1\\ c &= 1;\end{aligned}$$

(b) $E(X) = \displaystyle\int_1^\infty \frac{x}{x^2}\,dx = [\ln x]_1^\infty$, which is unbounded.

3.1–12 **(a)**

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x^3+1)/2, & -1 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

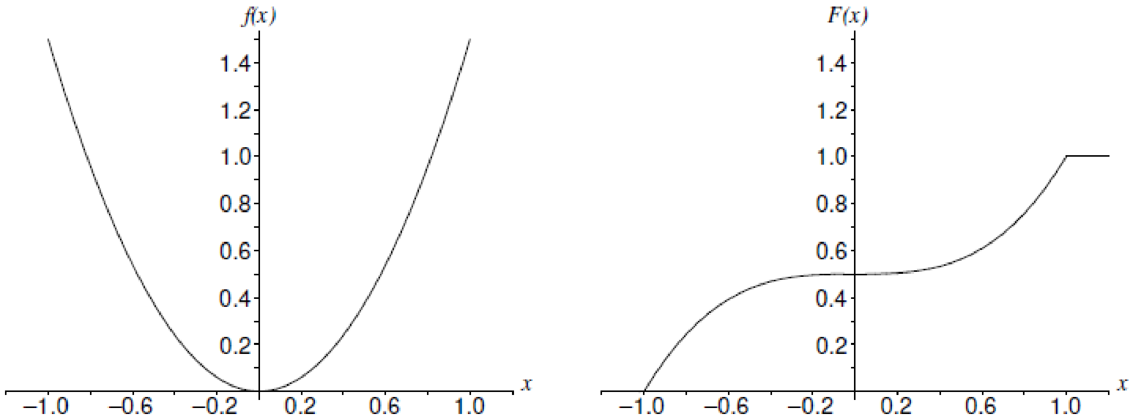


Figure 3.1–12: **(a)** $f(x) = (3/2)x^2$ and $F(x) = (x^3+1)/2$

(b)

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x+1)/2, & -1 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

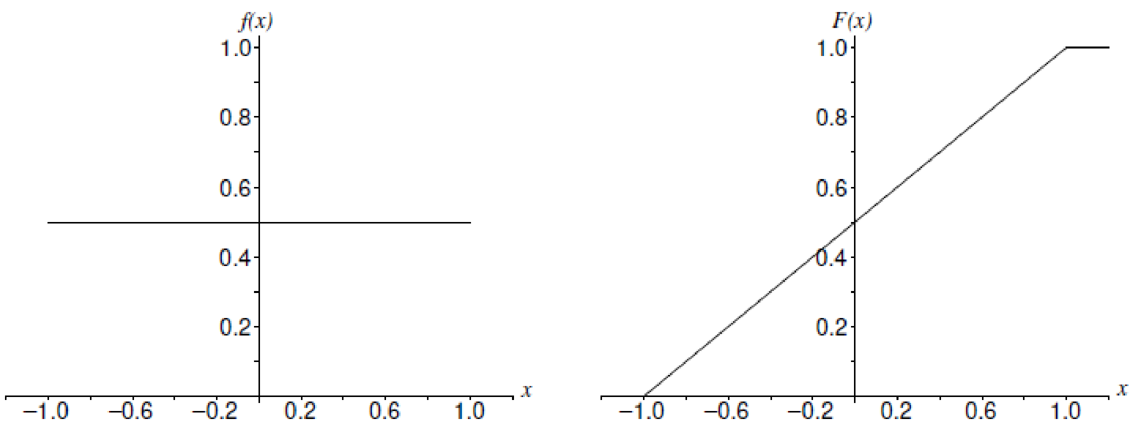


Figure 3.1–12: **(b)** $f(x) = 1/2$ and $F(x) = (x+1)/2$

(c)

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x+1)^2/2, & -1 \le x < 0, \\ 1-(1-x)^2/2, & 0 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

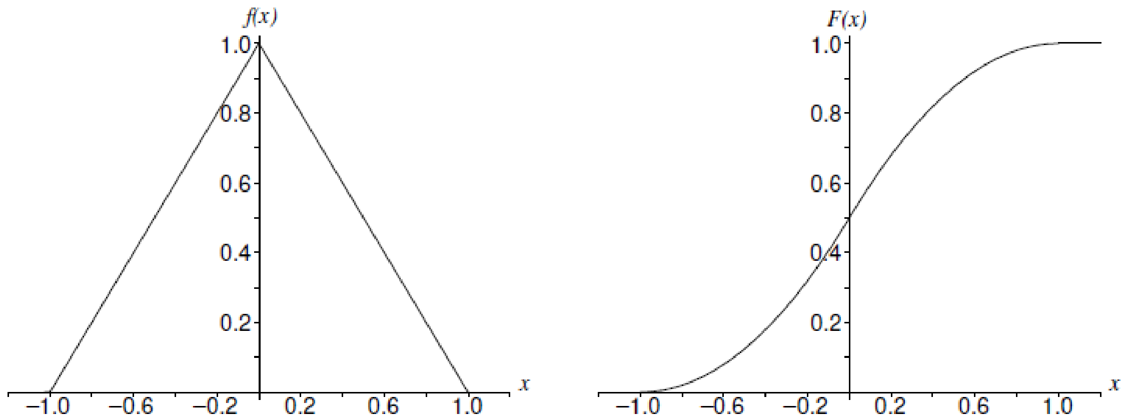


Figure 3.1–12: **(c)** $f(x)$ and $F(x)$ for Exercise **3.1-12(c)**

3.1–14 (b)

$$F(x) = \begin{cases} 0, & -\infty < x \le 0, \\ \dfrac{x}{2}, & 0 < x \le 1, \\ \dfrac{1}{2}, & 1 < x \le 2, \\ \dfrac{x}{2} - \dfrac{1}{2} & 2 \le x < 3, \\ 1, & 3 \le x < \infty; \end{cases}$$

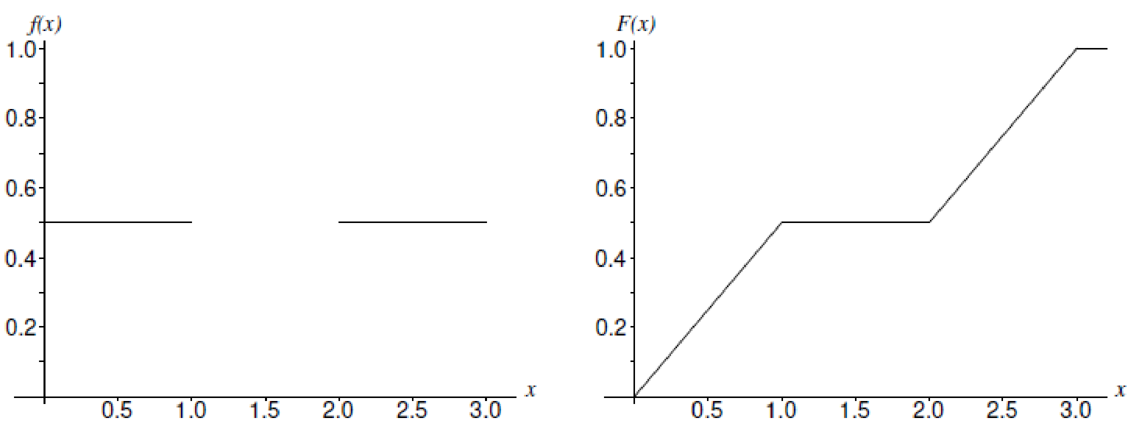


Figure 3.1–14: $f(x)$ and $F(x)$ for Exercise **3.1-14(a)** and **(b)**

3.1 Random Variables of the Continuous Type

3.1–2 $\mu = 0, \quad \sigma^2 = (1+1)^2/12 = 1/3.$

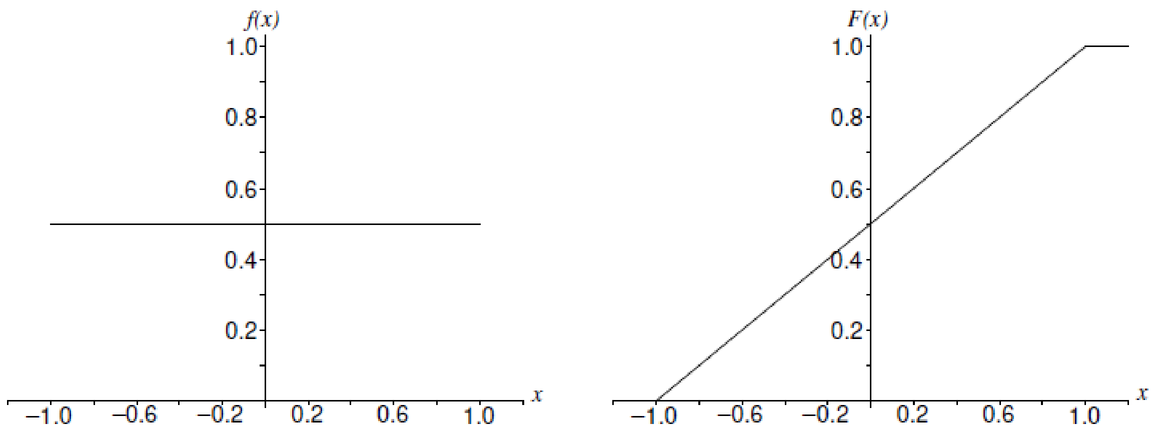


Figure 3.1–2: $f(x) = 1/2$ and $F(x) = (x+1)/2$

3.1–4 X is $U(4,5)$;

(a) $\mu = 9/2$; **(b)** $\sigma^2 = 1/12$; **(c)** 0.5.

3.1–6

$$\begin{aligned}
E(\text{profit}) &= \int_0^n [x - 0.5(n-x)]\frac{1}{200}\,dx + \int_n^{200} [n - 5(x-n)]\frac{1}{200}\,dx \\
&= \frac{1}{200}\left[\frac{x^2}{2} + \frac{(n-x)^2}{4}\right]_0^n + \frac{1}{200}\left[6nx - \frac{5x^2}{2}\right]_n^{200} \\
&= \frac{1}{200}[-3.25n^2 + 1200n - 100000] \\
\text{derivative} &= \frac{1}{200}[-6.5n + 1200] \;=\; 0 \\
n &= \frac{1200}{6.5} \;\approx\; 185.
\end{aligned}$$

3.1–8 **(a)** **(i)**
$$\begin{aligned}\int_0 x^3/4\,dx &= 1\\ c^4/16 &= 1\\ c &= 2;\end{aligned}$$

(ii)
$$\begin{aligned}F(x) &= \int_{-\infty}^{x} f(t)\,dt\\ &= \int_0^x t^3/4\,dt\\ &= x^4/16,\end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0,\\ x^4/16, & 0 \le x < 2,\\ 1, & 2 \le x < \infty.\end{cases}$$

(iii)

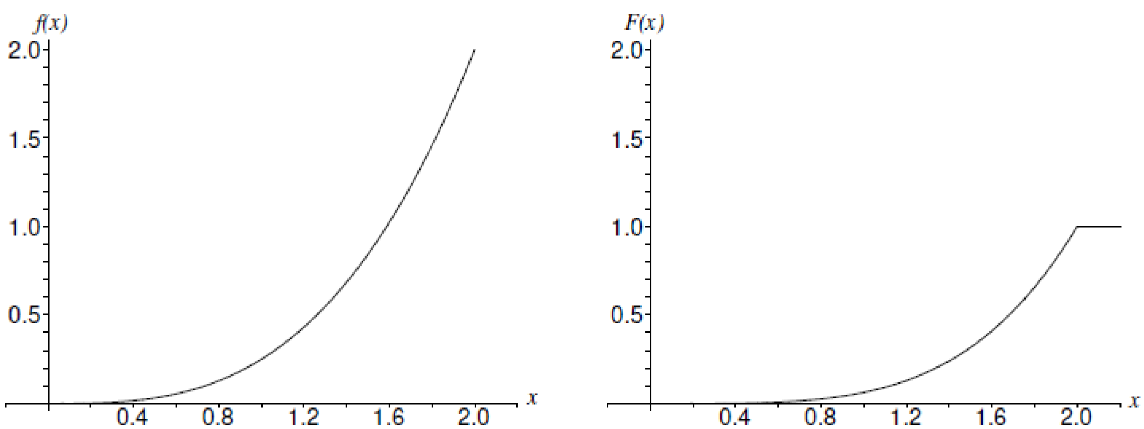


Figure 3.1–8: **(a)** Continuous distribution pdf and cdf

(iv)
$$\begin{aligned}\mu &= \int_0^2 (x)(x^3/4)\,dx \;=\; \frac{8}{5};\\ E(X^2) &= \int_0^2 (x^2)(x^3/4)\,dx \;=\; \frac{8}{3};\\ \sigma^2 &= \frac{8}{3} - \left(\frac{8}{5}\right)^2 \;=\; \frac{8}{75}.\end{aligned}$$

(b) (i)
$$\begin{aligned} \int_{-c}^{c} (3/16)x^2\,dx &= 1 \\ c^3/8 &= 1 \\ c &= 2; \end{aligned}$$

(ii)
$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{-2}^{x} (3/16)t^2\,dt \\ &= \left[\frac{t^3}{16}\right]_{-2}^{x} \\ &= \frac{x^3}{16} + \frac{1}{2}, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < -2, \\ \dfrac{x^3}{16} + \dfrac{1}{2}, & -2 \le x < 2, \\ 1, & 2 \le x < \infty. \end{cases}$$

(iii)

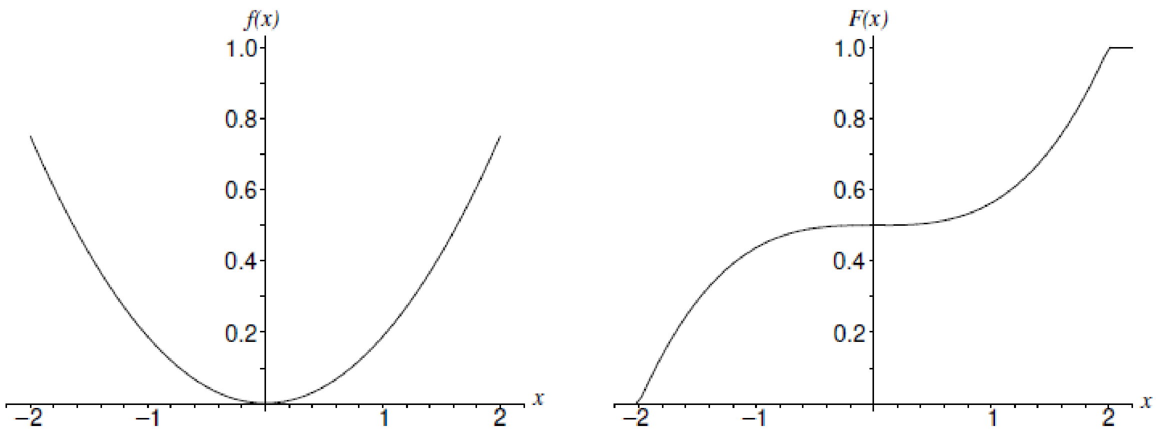


Figure 3.1–8: **(b)** Continuous distribution pdf and cdf

(iv)
$$\begin{aligned} \mu &= \int_{-2}^{2} (x)(3/16)(x^2)\,dx \;=\; 0; \\ \sigma^2 &= \int_{-2}^{2} (x^2)(3/16)(x^2)\,dx \;=\; \frac{12}{5}. \end{aligned}$$

$$\textbf{(c)}\ \textbf{(i)}\quad \begin{aligned} \int_0^1 \frac{c}{\sqrt{x}}\,dx &= 1 \\ 2c &= 1 \\ c &= 1/2. \end{aligned}$$

The pdf in part (c) is unbounded.

$$\textbf{(ii)}\quad \begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_0^x \frac{1}{2\sqrt{t}}\,dt \\ &= \left[\sqrt{t}\right]_0^x = \sqrt{x}, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ \sqrt{x}, & 0 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

(iii)

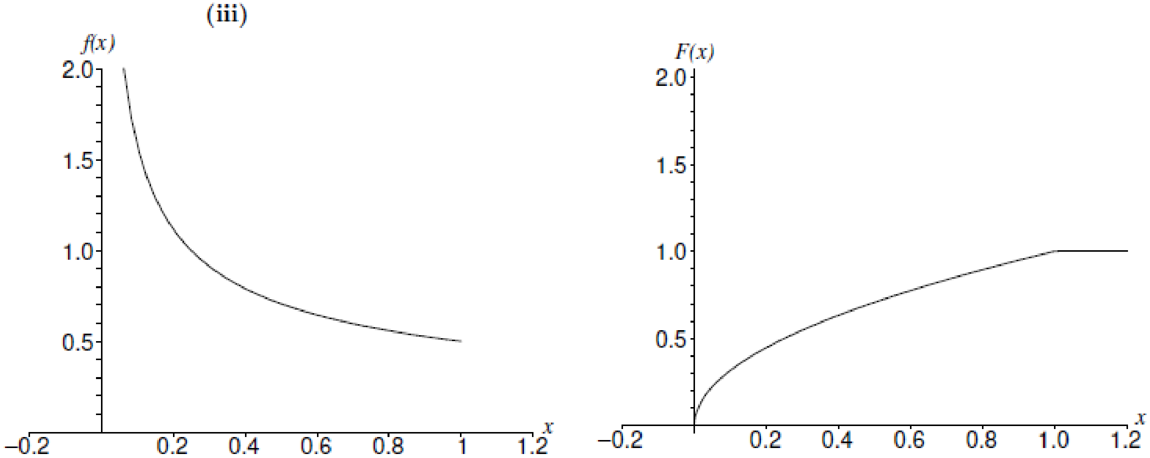


Figure 3.1–8: **(c)** Continuous distribution pdf and cdf

$$\textbf{(iv)}\quad \begin{aligned} \mu &= \int_0^1 (x)(1/2)/\sqrt{x}\,dx = \frac{1}{3}; \\ E(X^2) &= \int_0^1 (x^2)(1/2)/\sqrt{x}\,dx = \frac{1}{5}; \\ \sigma^2 &= \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}. \end{aligned}$$

$$\textbf{3.1–10 (a)}\quad \begin{aligned} \int_1^\infty \frac{c}{x^2}\,dx &= 1 \\ \left[\frac{-c}{x}\right]_1^\infty &= 1 \\ c &= 1; \end{aligned}$$

(b) $E(X) = \displaystyle\int_1^\infty \frac{x}{x^2}\,dx = [\ln x]_1^\infty$, which is unbounded.

3.1–12 **(a)**

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x^3+1)/2, & -1 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

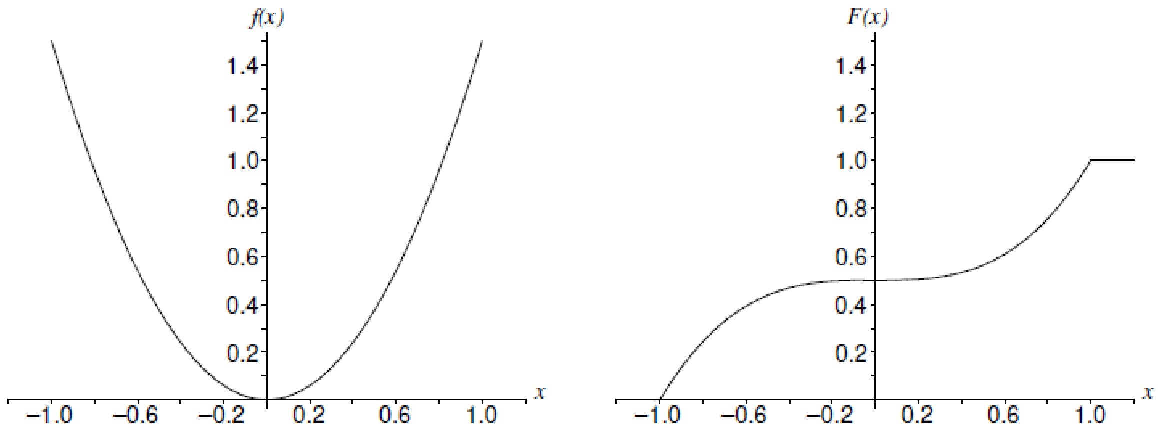


Figure 3.1–12: **(a)** $f(x) = (3/2)x^2$ and $F(x) = (x^3+1)/2$

(b)

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x+1)/2, & -1 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

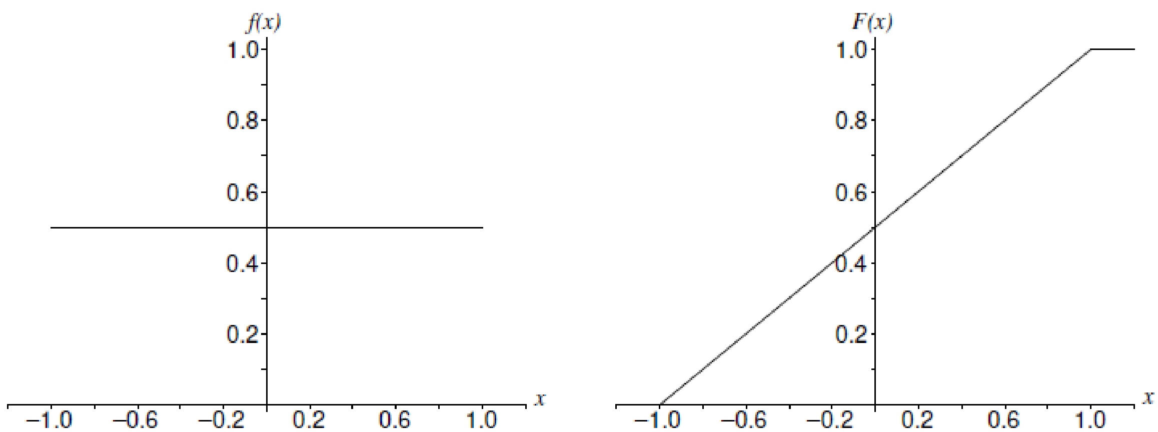


Figure 3.1–12: **(b)** $f(x) = 1/2$ and $F(x) = (x+1)/2$

(c)

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x+1)^2/2, & -1 \le x < 0, \\ 1-(1-x)^2/2, & 0 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

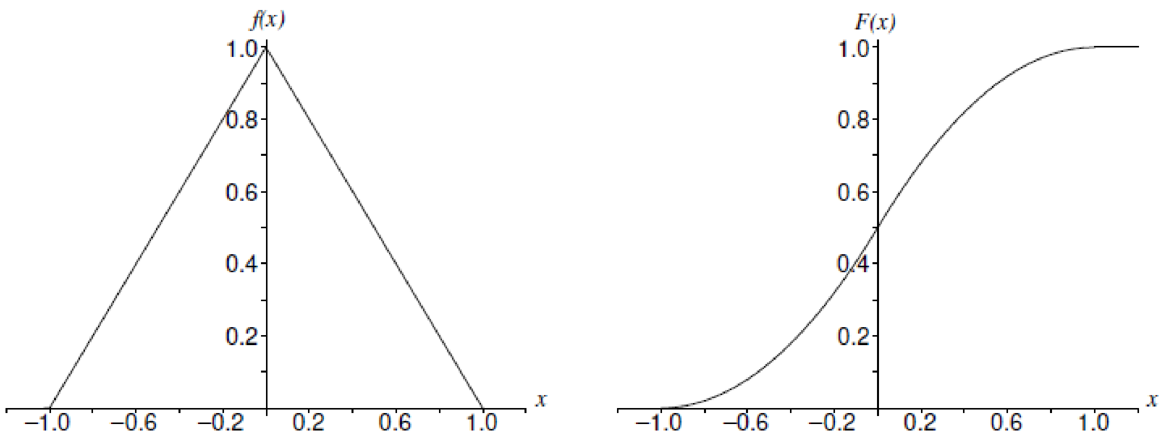


Figure 3.1–12: **(c)** $f(x)$ and $F(x)$ for Exercise **3.1-12(c)**

3.1–14 (b)

$$F(x) = \begin{cases} 0, & -\infty < x \le 0, \\ \dfrac{x}{2}, & 0 < x \le 1, \\ \dfrac{1}{2}, & 1 < x \le 2, \\ \dfrac{x}{2} - \dfrac{1}{2} & 2 \le x < 3, \\ 1, & 3 \le x < \infty; \end{cases}$$

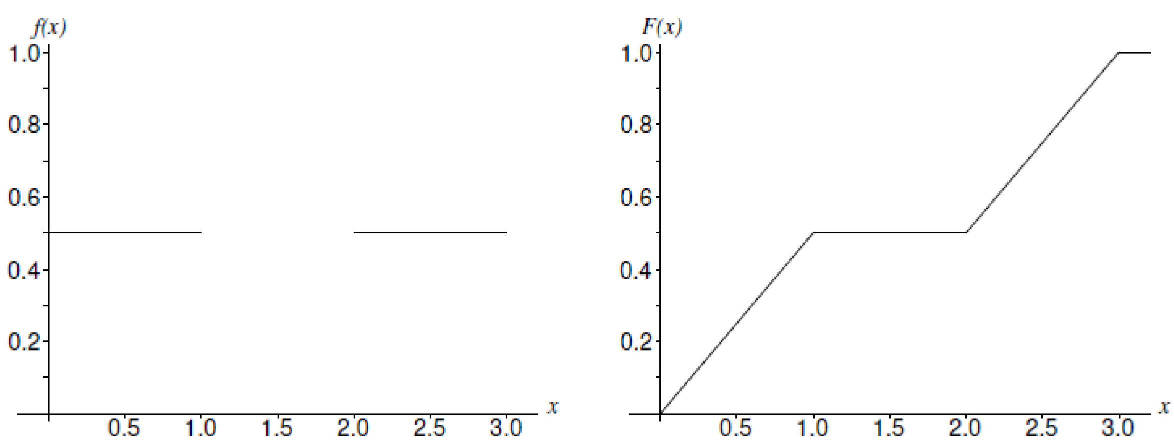


Figure 3.1–14: $f(x)$ and $F(x)$ for Exercise **3.1-14(a)** and **(b)**

3-1-1) Let us consider the uniform distribution with pdf,

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

Mean, $\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{a} x(0)\,dx + \int_{a}^{b} \frac{x}{b-a}\,dx + \int_{b}^{\infty} x(0)\,dx$$

$$= 0 + \left[\frac{x^2}{2}\left(\frac{1}{b-a}\right)\right]_a^b + 0$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{a+b}{2}$$

Variance :- $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x) - 2\mu x f(x) + \mu^2 f(x)\,dx$$

$$= \left[\frac{x^3}{3(b-a)} - \frac{2\mu x^2}{2(b-a)} + \frac{\mu^2 x}{(b-a)}\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)(b^2-a^2)}{(b-a)} + \frac{(a+b)^2}{4}\left(\frac{b-a}{b-a}\right)$$

$$= \frac{a^2+ab+b^2}{3} + \left(-(a+b)^2\right) + \frac{(a+b)^2}{4}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

M.g.f :- $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, \quad -h<t<h.$

$$\Rightarrow \quad M(t) = \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \left[\frac{e^{tx}}{t} \cdot \frac{1}{(b-a)} \right]_a^b$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}$$

if $t = 0$, $\quad M(t) = \int_a^b e^{(0)x} \cdot \frac{1}{b-a}\, dx$

$$= \int_a^b \frac{1}{b-a}\, dx$$

$$= \frac{b-a}{b-a}$$

$$= \underline{1}.$$

So, M.g.f $= \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)} & , t \neq 0 \\ 1 & , t = 0. \end{cases}$

3-1-2) given, $f(x) = 1/2, \quad -1 \le x \le 1.$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$x < -1$: $F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{x} (0)\, dx = 0$

$-1 \le x \le 1$: $F(x) = \int_{-\infty}^{x} f(x)\, dx = 0 + \int_{-1}^{x} f(x)\, dx$

$$= \int_{-1}^{x} 1/2\, dx = \left[\frac{x}{2}\right]_{-1}^{x} = \frac{x+1}{2}.$$

$x > 1$: $F(x) = \int_{-\infty}^{x} f(x)\, dx = 0 + \int_{-1}^{1} f(x)\, dx + \int_{1}^{x} f(x)\, dx$

$$= 0 + \frac{2}{2} + \int_{1}^{x} (0)\, dx$$

$$= \underline{1}.$$

Mean, $\mu = \frac{a+b}{2} = \frac{-1+1}{2} = 0$

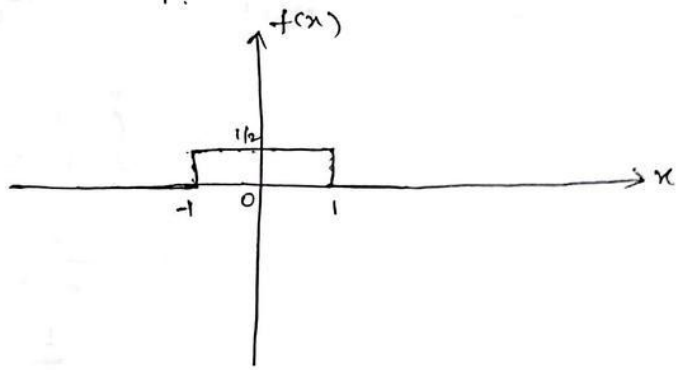
variance, $\sigma^2 = \frac{(b-a)^2}{12} = \frac{(1-(-1))^2}{12}$

$= \frac{2^2}{12} = 4/12$

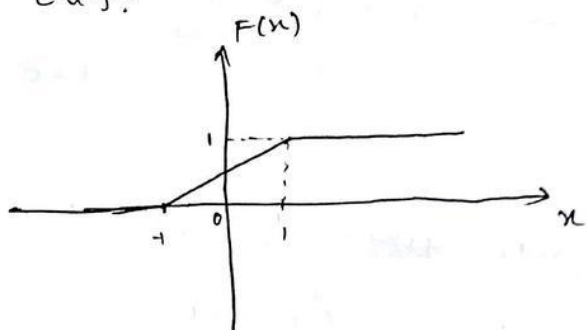
$= \underline{\underline{1/3}}$

graphs:-

p.d.f:-



c.d.f:-



3-1-3) $X \sim U(0,10)$

a) $\Rightarrow \quad f(x) = \frac{1}{10}, \quad 0 \le x \le 10.$

b) $p(X \geq 8) = \int_{8}^{\infty} f(x)\,dx$

$= \int_{8}^{10} \frac{1}{10}\,dx + \int_{10}^{\infty} (0)\,dx$

$= \left(\frac{2}{10}\right) + 0$

$= 1/5$

(c) $P(2 \le X < 8) = \int_2^8 f(x)\,dx$

$= \int_2^8 \frac{1}{10}\,dx = \left(\frac{8-2}{10}\right)$

$= 6/10$

$= 3/5$

(d) $E(X) = \mu = \frac{a+b}{2}$

$= \frac{0+10}{2}$

$= 5$

(e) $Var(X) = \frac{(b-a)^2}{12}$

$= \frac{(10-0)^2}{12} = \frac{10^2}{12}$

$= 100/12$

$= 25/3$

3.2 THE EXPONENTIAL, GAMMA, AND CHI-SQUARE DISTRIBUTIONS

Exercises

3.2-1. What are the pdf, the mean, and the variance of X if the moment-generating function of X is given by the following?

(a) $M(t) = \dfrac{1}{1-3t}, \quad t < 1/3.$

(b) $M(t) = \dfrac{3}{3-t}, \quad t < 3.$

3.2-2. Telephone calls arrive at a doctor's office according to a Poisson process on the average of two every 3 minutes. Let X denote the waiting time until the first call that arrives after 10 A.M.

(a) What is the pdf of X?

(b) Find $P(X > 2)$.

3.2-3. Let X have an exponential distribution with mean $\theta > 0$. Show that

$$P(X > x + y \mid X > x) = P(X > y).$$

3.2-4. Let $F(x)$ be the cdf of the continuous-type random variable X, and assume that $F(x) = 0$ for $x \le 0$ and $0 < F(x) < 1$ for $0 < x$. Prove that if

$$P(X > x + y \mid X > x) = P(X > y),$$

then

$$F(x) = 1 - e^{-\lambda x}, \qquad 0 < x.$$

HINT: Show that $g(x) = 1 - F(x)$ satisfies the functional equation

$$g(x + y) = g(x)g(y),$$

which implies that $g(x) = a^{cx}$.

3.2-5. There are times when a shifted exponential model is appropriate. That is, let the pdf of X be

$$f(x) = \frac{1}{\theta} e^{-(x-\delta)/\theta}, \qquad \delta < x < \infty.$$

(a) Define the cdf of X.

(b) Calculate the mean and variance of X.

3.2-6. A certain type of aluminum screen 2 feet in width has, on the average, three flaws in a l00-foot roll.

(a) What is the probability that the first 40 feet in a roll contain no flaws?

(b) What assumption did you make to solve part (a)?

3.2-7. Find the moment-generating function for the gamma distribution with parameters α and θ.

HINT: In the integral representing $E(e^{tX})$, change variables by letting $y = (1 - \theta t)x/\theta$, where $1 - \theta t > 0$.

3.2-8. If X has a gamma distribution with $\theta = 4$ and $\alpha = 2$, find $P(X < 5)$.

3.2-9. If the moment-generating function of a random variable W is

$$M(t) = (1 - 7t)^{-20},$$

find the pdf, mean, and variance of W.

3.2-10. Use the moment-generating function of a gamma distribution to show that $E(X) = \alpha\theta$ and $\text{Var}(X) = \alpha\theta^2$.

3.2-11. If X is $\chi^2(17)$, find

(a) $P(X < 7.564)$.

(b) $P(X > 27.59)$.

(c) $P(6.408 < X < 27.59)$.

(d) $\chi^2_{0.95}(17)$.

(e) $\chi^2_{0.025}(17)$.

3.2-12. Let X equal the number of alpha particle emissions of carbon-14 that are counted by a Geiger counter each second. Assume that the distribution of X is Poisson with mean 16. Let W equal the time in seconds before the seventh count is made.

(a) Give the distribution of W.

(b) Find $P(W \le 0.5)$. HINT: Use Equation 3.2-1 with $\lambda w = 8$.

3.2-13. If X is $\chi^2(23)$, find the following:

(a) $P(14.85 < X < 32.01)$.

(b) Constants a and b such that $P(a < X < b) = 0.95$ and $P(X < a) = 0.025$.

(c) The mean and variance of X.

(d) $\chi^2_{0.05}(23)$ and $\chi^2_{0.95}(23)$.

3.2-14. If X is $\chi^2(12)$, find constants a and b such that

$$P(a < X < b) = 0.90 \text{ and } P(X < a) = 0.05.$$

3.2-15. Let the distribution of X be $\chi^2(r)$.

(a) Find the point at which the pdf of X attains its maximum when $r \ge 2$. This is the mode of a $\chi^2(r)$ distribution.

(b) Find the points of inflection for the pdf of X when $r \ge 4$.

(c) Use the results of parts (a) and (b) to sketch the pdf of X when $r = 4$ and when $r = 10$.

3.2-16. Cars arrive at a tollbooth at a mean rate of 5 cars every 10 minutes according to a Poisson process. Find the probability that the toll collector will have to wait longer than 26.30 minutes before collecting the eighth toll.

3.2-17. If 15 observations are taken independently from a chi-square distribution with 4 degrees of freedom, find the probability that at most 3 of the 15 observations exceed 7.779.

3.2-18. Say the serum cholesterol level (X) of U.S. males ages 25–34 follows a translated gamma distribution with pdf

$$f(x) = \frac{x - 80}{50^2} e^{-(x-80)/50}, \qquad 80 < x < \infty.$$

(a) What are the mean and the variance of this distribution?

(b) What is the mode?

(c) What percentage have a serum cholesterol level less than 200? HINT: Integrate by parts.

3.2-19. A bakery sells rolls in units of a dozen. The demand X (in 1000 units) for rolls has a gamma distribution with parameters $\alpha = 3, \theta = 0.5$, where θ is in units of days per 1000 units of rolls. It costs \$2 to make a unit that sells for \$5 on the first day when the rolls are fresh. Any leftover units are sold on the second day for \$1. How many units should be made to maximize the expected value of the profit?

3.2-20. The initial value of an appliance is \$700 and its dollar value in the future is given by

$$v(t) = 100\,(2^{3-t} - 1), \qquad 0 \le t \le 3,$$

where t is time in years. Thus, after the first three years, the appliance is worth nothing as far as the warranty is concerned. If it fails in the first three years, the warranty pays $v(t)$. Compute the expected value of the payment on the warranty if T has an exponential distribution with mean 5.

3.2-21. A loss (in \$100,000) due to fire in a building has a pdf $f(x) = (1/6)e^{-x/6}$, $0 < x < \infty$. Given that the loss is greater than 5, find the probability that it is greater than 8.

3.2-22. Let X have a logistic distribution with pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty.$$

Show that

$$Y = \frac{1}{1+e^{-X}}$$

has a $U(0,1)$ distribution.

HINT: Find $G(y) = P(Y \le y) = P\left(\frac{1}{1+e^{-X}} \le y\right)$ when $0 < y < 1$.

3.2-23. Some dental insurance policies cover the insurer only up to a certain amount, say, M. (This seems to us to be a dumb type of insurance policy because most people should want to protect themselves against large losses.) Say the dental expense X is a random variable with pdf $f(x) = (0.001)e^{-x/1000}$, $0 < x < \infty$. Find M so that $P(X < M) = 0.08$.

3.2-24. Let the random variable X be equal to the number of days that it takes a high-risk driver to have an accident. Assume that X has an exponential distribution. If $P(X < 50) = 0.25$, compute $P(X > 100 \mid X > 50)$.

3-2-1) (a) $M(t) = \frac{1}{1-3t}$, $t < 1/3$.

If we compare this with M.g.f of exponential distribution, $M(t) = \frac{1}{1-\theta t}$, $t < 1/\theta$

$$\theta = 3.$$

so, $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $x \geq 0$.

$$= \frac{1}{3} e^{-x/3}, \quad x \geq 0.$$

$$\mu = \theta = 3.$$

$$\sigma^2 = \theta^2 = 3^2 = 9.$$

(b) $M(t) = \frac{3}{3-t}$, $t < 3$

$$= \frac{1}{1-\left(\frac{1}{3}\right)t}, \quad t < 1/(1/3).$$

here, $\theta = 1/3$.

so, $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $x \geq 0$

$$= 3e^{-3x}, \quad x \geq 0.$$

$$\mu = \theta = 1/3$$

$$\sigma^2 = \theta^2 = (1/3)^2 = 1/9.$$

3-2-27)

given, calls arrive according to a poisson process with $\lambda = 2/3$ per minute.

X denotes waiting time until first call arrives after 10 A.M.

so, X is exponential distribution with

$$\theta = 1/\lambda = 1/\left(\tfrac{2}{3}\right) = 3/2.$$

(a) $$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0$$

$$= \frac{2}{3} \cdot e^{-\frac{2}{3}x}, \quad x \geq 0$$

(b) $$P(X > 2)$$

$$= \int_2^\infty f(x)\,dx$$

$$= \int_2^\infty \frac{2}{3} e^{\frac{-2x}{3}}\,dx$$

$$= \left(\frac{2}{3}\right) \left[\frac{e^{-2x/3}}{(-2/3)}\right]_2^\infty$$

$$= -1 \left[\frac{1}{e^{2x/3}}\right]_2^\infty$$

$$= -1 \left[0 - \frac{1}{e^{4/3}}\right]$$

$$= 1/e^{4/3} \approx 0.263$$

3-2-3)

$X \sim$ exponential distribution.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(X > x+y \mid X > x) = \frac{P((X > x+y) \cap (X > x))}{P(X > x)}$$

if X is greater than $x+y$ then, intersection of $X > x+y$ and $X > x$ will be $X > x+y$

$$= \frac{P(X > x+y)}{P(X > x)}$$

$$= \int_{x+y}^{\infty} f(x)\,dx \Big/ \int_{x}^{\infty} f(x)\,dx.$$

$$\int_{x}^{\infty} f(x)\,dx = \int_{x}^{\infty} \frac{1}{\theta} e^{-x/\theta}\,dx$$

$$= \left(\frac{1}{\theta}\right)\left[\frac{e^{-x/\theta}}{(-1/\theta)}\right]_x^{\infty}$$

$$= -1\left[e^{-x/\theta}\right]_x^{\infty}$$

$$= -1\left[0 - e^{-x/\theta}\right]$$

$$= e^{-x/\theta}.$$

$$\text{so, } P(X > x+y \mid X > x) = \frac{e^{-(x+y)/\theta}}{e^{-x/\theta}}$$

$$= e^{-y/\theta}$$

$$= \int_{y}^{\infty} f(x)\,dx$$

$$= P(X > y)$$

Ex. 3.2

A.4 let $g(x) = 1 - F(x)$.

then, $P(X > x+y \mid X > x) = \dfrac{P(X > x+y)}{P(X > x)}$

$= \dfrac{1 - F(x+y)}{1 - F(x)}$ $\quad (\because P(X \leq x) = F(x))$

$= \dfrac{g(x+y)}{g(x)}$

& $g(y) = P(X > x+y \mid X > x) \Rightarrow g(x)g(y) = g(x)g(y)$.

$\Rightarrow g(x) = 1 - a^{cx}$.

Distribution f^n can be given as.

$F(x) = 1 - a^{cx}$

$= 1 - \left(a^{\frac{c}{-\lambda}}\right)^{-\lambda x}$

$= 1 - e^{-\lambda x}, \quad \lambda > 0 \quad \left(\text{as } c = \dfrac{-\lambda}{\ln a}\right)$.

3.3 THE NORMAL DISTRIBUTION

Exercises

3.3-1. If Z is $N(0,1)$, find

(a) $P(0.53 < Z \le 2.06)$. **(b)** $P(-0.79 \le Z < 1.52)$.
(c) $P(Z > -1.77)$. **(d)** $P(Z > 2.89)$.
(e) $P(|Z| < 1.96)$. **(f)** $P(|Z| < 1)$.
(g) $P(|Z| < 2)$. **(h)** $P(|Z| < 3)$.

3.3-2. If Z is $N(0,1)$, find

(a) $P(0 \le Z \le 0.87)$. **(b)** $P(-2.64 \le Z \le 0)$.
(c) $P(-2.13 \le Z \le -0.56)$. **(d)** $P(|Z| > 1.39)$.
(e) $P(Z < -1.62)$. **(f)** $P(|Z| > 1)$.
(g) $P(|Z| > 2)$. **(h)** $P(|Z| > 3)$.

3.3-3. If Z is $N(0,1)$, find values of c such that

(a) $P(Z \ge c) = 0.025$. **(b)** $P(|Z| \le c) = 0.95$.
(c) $P(Z > c) = 0.05$. **(d)** $P(|Z| \le c) = 0.90$.

3.3-4. Find the values of **(a)** $z_{0.10}$, **(b)** $-z_{0.05}$, **(c)** $-z_{0.0485}$, and **(d)** $z_{0.9656}$.

3.3-5. If X is normally distributed with a mean of 6 and a variance of 25, find

(a) $P(6 \le X \le 12)$. **(b)** $P(0 \le X \le 8)$.
(c) $P(-2 < X \le 0)$. **(d)** $P(X > 21)$.
(e) $P(|X - 6| < 5)$. **(f)** $P(|X - 6| < 10)$.
(g) $P(|X - 6| < 15)$. **(h)** $P(|X - 6| < 12.41)$.

3.3-6. If the moment-generating function of X is $M(t) = \exp(166t + 200t^2)$, find

(a) The mean of X. **(b)** The variance of X.
(c) $P(170 < X < 200)$. **(d)** $P(148 \le X \le 172)$.

3.3-7. If X is $N(650, 625)$, find
(a) $P(600 \le X < 660)$.
(b) A constant $c > 0$ such that $P(|X - 650| \le c) = 0.9544$.

3.3-8. Let the distribution of X be $N(\mu, \sigma^2)$. Show that the points of inflection of the graph of the pdf of X occur at $x = \mu \pm \sigma$.

3.3-9. Find the distribution of $W = X^2$ when

(a) X is $N(0, 4)$,

(b) X is $N(0, \sigma^2)$.

3.3-10. If X is $N(\mu, \sigma^2)$, show that the distribution of $Y = aX + b$ is $N(a\mu + b, a^2\sigma^2)$, $a \neq 0$. Hint: Find the cdf $P(Y \le y)$ of Y, and in the resulting integral, let $w = ax + b$ or, equivalently, $x = (w - b)/a$.

3.3-11. A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is $N(21.37, 0.16)$.

(a) Let X denote the weight of a single mint selected at random from the production line. Find $P(X > 22.07)$.

(b) Suppose that 15 mints are selected independently and weighed. Let Y equal the number of these mints that weigh less than 20.857 grams. Find $P(Y \le 2)$.

3.3-12. If the moment-generating function of X is given by $M(t) = e^{500t+5000t^2}$, find $P[27{,}060 \le (X - 500)^2 \le 50{,}240]$.

3.3-13. The serum zinc level X in micrograms per deciliter for males between ages 15 and 17 has a distribution that is approximately normal with $\mu = 90$ and $\sigma = 15$. Compute the conditional probability $P(X > 120 \,|\, X > 105)$.

3.3-14. The strength X of a certain material is such that its distribution is found by $X = e^Y$, where Y is $N(10, 1)$. Find the cdf and pdf of X, and compute $P(10{,}000 < X < 20{,}000)$. Note: $F(x) = P(X \le x) = P(e^Y \le x) = P(Y \le \ln x)$ so that the random variable X is said to have **a lognormal distribution**.

3.3-15. The "fill" problem is important in many industries, such as those making cereal, toothpaste, beer, and so on. If an industry claims that it is selling 12 ounces of its product in a container, it must have a mean greater than 12 ounces, or else the FDA will crack down, although the FDA will allow a very small percentage of the containers to have less than 12 ounces.

(a) If the content X of a container has a $N(12.1, \sigma^2)$ distribution, find σ so that $P(X < 12) = 0.01$.

(b) If $\sigma = 0.05$, find μ so that $P(X < 12) = 0.01$.

3.3-16. The graphs of the moment-generating functions of three normal distributions—$N(0, 1)$, $N(-1, 1)$, and $N(2, 1)$—are given in Figure 3.3-3(a). Identify them.

3.3-17. Figure 3.3-3(b) shows the graphs of the following three moment-generating functions near the origin:

$$g_1(t) = \frac{1}{1 - 4t}, \qquad t < 1/4,$$

$$g_2(t) = \frac{1}{(1 - 2t)^2}, \qquad t < 1/2,$$

$$g_3(t) = e^{4t + t^2/2}.$$

Why do these three graphs look so similar around $t = 0$?

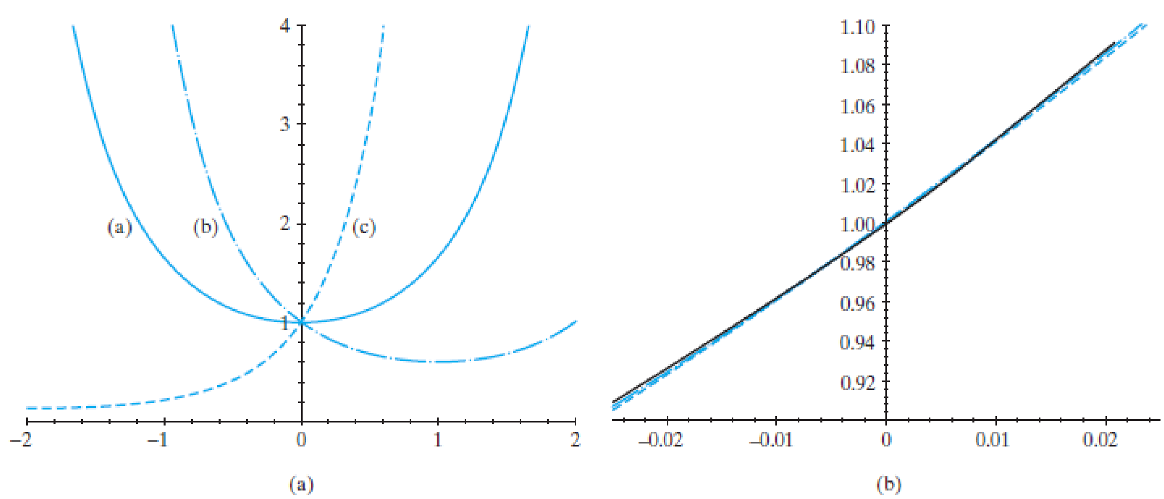


Figure 3.3-3 Moment-generating functions

Ex. 3.3

A.1 (a) m.g.f of any random var. with exp. distribution is

$$M(t) = \frac{1}{1-\theta t} \quad t < 1/\theta \quad , \theta = 3 .$$

p.d.f of X is , $f(x) = \frac{1}{\theta} e^{-x/\theta} = \frac{e^{-x/3}}{3}$, $0 \le x < \infty$.

$\mu_x = \theta = 3$

$Var(X) = \sigma^2 = \theta^2 = 9$

(b) m.g.f of X, $M(t) = \frac{3}{3-t}$ (for exp. distribution with $\theta = 1/3$).

p.d.f of X is, $f(x) = \frac{e^{-x/\theta}}{\theta} = 3e^{-3x}$ $0 \le x < \infty$

$\mu_x = \theta = 1/3$

$Var(x) = \sigma^2 = \theta^2 = 1/9$

A.2 $Z \sim N(0,1)$ then

(a) $P(0 \le Z \le 0.87) = \phi(0.87) - \phi(0) = 0.8078 - 0.5 = 0.3078$

(b) $P(-2.64 \le Z \le 0) = \phi(0) - \phi(-2.64) = 0.5 - \phi(1-\phi(2.64)) = 0.4949$

(c) $P(-2.13 \le Z \le -0.56) = P(0.56 \le Z \le 2.13) = \phi(2.13) - \phi(0.56)$

$\therefore P(-2.13 \le Z \le 0.56) = 0.9834 - 0.7123 = 0.2711$

(d) $P(|Z| > 1.39) = 2P(Z > 1.39) = 2(1 - \phi(1.39)) = 2(1 - 0.9177) = 0.1646$

(e) $P(Z < -1.62) = 1 - \phi(1.62) = 1 - 0.9474 = 0.0526$

(f) $P(|Z| > 1) = 2P(Z \ge 1) = 2(1 - \phi(1)) = 2(1 - 0.8413) = 0.3174$.

(g) $P(|Z| > 2) = 2(1 - \phi(2)) = 2(1 - 0.9772) = 0.0456$

(h) $P(|Z| > 3) = 2(1 - \phi(3)) = 2(1 - 0.9987) = 0.0026$.

A.3 Given $Z \sim N(0,1)$

(a) $P(Z \geq c) = 0.025 \Rightarrow c = 1.96$ ← (from table of std. normal dist)

(b) $P(|Z| \leq c) = 0.95 = 1 - P(|Z| > c) = 1 - 2P(Z > c)$

$\Rightarrow P(Z > c) = 0.025. \Rightarrow c = 1.96$

(c) $P(Z > c) = 0.05 \Rightarrow c = 1.645$

(d) $P(|Z| \leq c) = 0.9 \Rightarrow 1 - P(|Z| > c) = 1 - 2P(Z > c) = 0.9.$

$\Rightarrow P(Z > c) = 0.05 \Rightarrow c = 1.645$

A.5 $X \sim N(6, 25)$ (given)

$Z = \dfrac{X-6}{5} \sim N(0,1)$

(a) $P(6 \le X \le 12) = P\left(\frac{6-6}{5} \le \frac{X-6}{5} \le \frac{12-6}{5}\right) = P(0 \le Z \le 1.2)$

$= \phi(1.2) - \phi(0) = 0.8849 - 0.5 = 0.3849.$

(b) $P(0 \le X \le 8) = P(-1.2 \le Z \le 0.4) = \phi(0.4) - \phi(-1.2)$

$= \phi(0.4) + \phi(1.2) - 1 = 0.6554 + 0.8849 - 1$

$= 0.5403$

(c) $P(-2 < X \le 0) = P(1.2 \le Z < 1.6) = \phi(1.6) - \phi(1.2)$

$= 0.9452 - 0.8849 = 0.0603$

(d) $P(X > 21) = P(Z > 3) = 1 - \phi(3) = 1 - 0.9987 = 0.0013$

(e) $P(|X-6| < 5) = P(X-6 < 5) - P(X-6 \le -5)$

$= P(Z < 1) - P(Z \le -1)$

$= \phi(1) - \phi(-1) = 0.8413 - 0.1587$

$= 0.6826.$

(f) $P(|X-6| < 10) = P(X-6 < 10) - P(X-6 \le -10)$

$= P(Z < 2) - P(Z \le -2) = \phi(2) - \phi(-2)$

$= 0.9772 - 0.0228 = 0.9544.$

(g) $P(|X-6| < 15) = P(Z < 3) - P(Z \le -3) = \phi(3) - \phi(-3)$

$= 0.9987 - 0.0013$

$= 0.9974$

(h) $P(|X-6| \le 12.41) = P(Z < 2.48) - P(Z < -2.48)$

$= \phi(2.48) - \phi(-2.48)$

$= 0.9934 - 0.0066 = 0.9868.$

3.3.6 (a) Given $M(t) = \exp(166t + 200t^2)$

$X \sim N(166, 400)$

(a) $\because X \sim N(166, 400)$! $\therefore$ mean $\mu = 166$

(b) the variance $\sigma^2 = 400$

(c) $Z = \dfrac{X-166}{20} \sim N(0,1)$

$\therefore P(170 < X < 200) = P\left(\dfrac{170-166}{20} < \dfrac{X-166}{20} < \dfrac{200-166}{20}\right)$

$= P(0.2 < Z < 1.7)$

$= \Phi(1.7) - \Phi(0.2) =$

$= 0.9554 - 0.5793 = 0.3761$

(d) $P(148 \leq X \leq 172) = P\left(\dfrac{148-166}{20} < \dfrac{X-166}{20} < \dfrac{172-166}{20}\right)$

$= P(-0.9 < Z < 0.3)$

$= \Phi(0.3) - \Phi(0.9) =$

$= 0.6179 - (1 - 0.8159)$

$= 0.4338$

3.3.7 (a) $X \sim N(650, 625)$, then $Z = \frac{X-650}{25}$

$P(600 \leq X \leq 660) = P\left[\frac{600-65}{25} \leq \frac{X-650}{25} < \frac{660-650}{25}\right]$

$= P(-2 \leq Z < 0.4)$

$= \phi(0.4) - (1 - \phi(2))$

$= 06554 - (1 - 0.9772)$

$= 0.6326$

(b) $P(|X-650| \leq c) = 0.9544$

$P\left(-\frac{c}{25} \leq \frac{X-650}{25} < \frac{c}{25}\right) = 0.9544$

$\phi\left(\frac{c}{25}\right) - \left[1 - \phi\left(\frac{c}{25}\right)\right] = 0.9544$

$\phi\left(\frac{c}{25}\right) = 0.9772$

From the table for normal distribution, $\frac{c}{25} = 2 \quad \therefore c = 50$

A.9 $X \sim N(\mu, \sigma^2)$, $\sigma^2 > 0 \Rightarrow V = \frac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$

$\therefore$ p.d.f of V

$$P(V \le v) = \int_0^v \frac{1}{\sqrt{\pi}\sqrt{2}} y^{\frac{1}{2}-1} e^{-y/2} dy \quad , \quad 0 < v < \infty$$

(a) $W = X^2$, $X \sim N(0, 4)$.

$\Rightarrow V = \frac{X^2}{4} \sim \chi^2(1)$ let $W = X^2 = 4V$.

$$P(W \le w) = P(4V \le w) = P\left(V \le \frac{w}{4}\right).$$

$$= \int_0^{w/4} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} dy. \quad -(1)$$

$$= \int_0^{w} \frac{1}{\sqrt{2\pi}} \left(\frac{z}{4}\right)^{-1/2} e^{-z/8} \frac{dz}{4} \quad \left(\text{let } z = 4y \Rightarrow \frac{dz}{4} = dy\right)$$

$$= \int_0^{w} \frac{1}{\Gamma(1/2)(8)^{1/2}} (z)^{-1/2} e^{-z/8} dz$$

which is distribution f^n of Γ-distribution with $\alpha = \frac{1}{2}$ & $\theta = 8$, such is f^n $W = X^2$.

(b) $W = X^2$, $X \sim N(0, \sigma^2)$.

let $X \sim N(0, \sigma^2) \Rightarrow V = \frac{X^2}{\sigma^2}$ ~~follows~~ $\sim \chi^2(1)$.

let $W = X^2. \rightarrow W = \sigma^2 V$.

$$P(W \le w) = P(\sigma^2 V \le w)$$

$$= P\left(V \le \frac{w}{\sigma^2}\right).$$

$$= \int_0^{w/\sigma^2} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} dy$$

$$= \int_0^{w} \frac{1}{\sqrt{2\pi}} \left(\frac{z}{\sigma^2}\right)^{-1/2} e^{-z/2\sigma^2} \frac{dz}{\sigma^2} \quad \left(\text{let } z = \sigma^2 y \Rightarrow \frac{dz}{\sigma^2} = dy\right)$$

$$= \int_0^{w} \frac{1}{\Gamma(1/2)(2\sigma^2)^{1/2}} z^{1/2 - 1} e^{-\frac{z}{2\sigma^2}}$$

which is gamma distribution with parameters $\alpha = \frac{1}{2}$, $\theta = 2\sigma^2$.

A.10 $X \sim N(\mu, \sigma^2)$, then

$$P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \quad , -\infty < x < \infty$$

let $Y = aX + b$, $a \neq 0$.

$$P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right)$$

$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

$$= \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(\frac{w-b}{a}-\mu\right)^2}{2\sigma^2}} \frac{dw}{a} \quad \left(\begin{array}{l}\text{let } w = at+b \\ \Rightarrow \frac{dw}{a} = dt\end{array}\right)$$

$$= \int_{-\infty}^{y} \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{(w-(a\mu+b))^2}{2(a\sigma)^2}} dw$$

which is the distribution f^n of $N(a\mu+b, a^2\sigma^2)$.

$\Rightarrow Y = aX+b \sim N(a\mu+b, a^2\sigma^2)$.

3.3.11 (a) $X \sim N(21.37, 0.16)$ $\therefore Z = \frac{X - 21.37}{0.4}$

$$P(X > 22.07) = P\left(\frac{X - 21.37}{0.4} > \frac{22.07 - 21.37}{0.4}\right)$$

$$= P(Z > 1.75)$$

$$= 1 - \Phi(1.75) = 1 - 0.9599 = 0.0401$$

(b) $$P(X < 20.857) = P\left(\frac{X - 21.37}{0.4} < \frac{20.857 - 21.37}{0.4}\right)$$

$$= P(Z < -1.28) = \Phi(-1.28)$$

$$= 1 - 0.8980 = 0.1020$$

Let $Y \rightarrow$ no. of units from the sample of size 15 that weight less than 20.857 gm.

$$\therefore P(Y \leq 2) = P(Y=0) + P(Y=1) + P(Y=2)$$

$$= \binom{15}{0}(0.1020)^0(0.8980)^{15} + \binom{15}{1}(0.1020)^1(0.8$$

$$\binom{15}{2}(0.1020)^2(0.8980)^{13}$$

$$= 0.8082$$

3.3.12 Let $V = \frac{(X-500)^2}{10000}$ $\because X \sim N(500, 10000)$

$P(27{,}060 \leq (X-500)^2 \leq 50{,}240] = P\left[\frac{27060}{10{,}000} \leq \frac{(X-500)^2}{10000} \leq \frac{50{,}240}{10000}\right] =$

$= P(2.706 \leq V \leq 5.024$

Using Chi square distribution table with $r=1$ degree of freedom

$P(27060 \leq (X-500)^2 \leq 50240] = P[2.706 \leq V \leq 5.024]$

$= F(5.024) - F(2.706)$

$= 0.975 - 0.900$

$= 0.075$

3.3.13 $P(X > 120 \mid X > 105) = \dfrac{P[(X > 120) \cap (X > 105)]}{P(X > 105)}$

$$= \frac{P(X > 120)}{P(X > 105)} \quad \text{——} \quad (1)$$

$X \sim N(90, 15^2) \quad \therefore Z = \dfrac{X-90}{15}$

$$\therefore P(X > 20) = P\left(\frac{X-90}{15} > \frac{120-90}{15}\right)$$

$$= P(Z > 2) = 1 - \phi(2)$$

$$= 1 - 0.9772 = 0.0228$$

$$P(X > 105) = P\left(\frac{X-90}{15} > \frac{105-90}{15}\right)$$

$$= P(Z > 1) = 1 - \phi(1)$$

$$= 1 - 0.8413 = 0.1587$$

$\therefore$ Putting the values in (1), we get

$$P(X > 120 \mid X > 105) = \frac{0.0228}{0.1587} = 0.1437$$

3.3.15 (a) $X \sim N(12.1, \sigma^2)$ $\quad Z = \dfrac{X-12.1}{\sigma} \sim N(0,1)$

$$P(X<12) = P\left(\frac{X-12.1}{\sigma} < \frac{12-12.1}{\sigma}\right)$$

$$= P\left(Z < -\frac{0.1}{\sigma}\right) = P\left(Z > \frac{0.1}{\sigma}\right)$$

From the table of Standard normal distribution

for $\quad P\left(Z > \dfrac{0.1}{\sigma}\right) = 0.01$

$$\frac{0.1}{\sigma} = 2.326 \Rightarrow \sigma = 0.043$$

(b) $X \sim N(\mu, 0.05^2)$ $\therefore Z = \dfrac{X-\mu}{0.05} \sim N(0,1)$

$$P(X<12) = P\left(\frac{X-\mu}{0.05} < \frac{12-\mu}{0.05}\right) = P\left(Z < \frac{12-\mu}{0.05}\right)$$

$$= P\left(Z > \frac{\mu-12}{0.05}\right)$$

From the table of Standard normal distribution

for $\quad P\left(Z > \dfrac{\mu-12}{0.05}\right) = 0.01$

$$\frac{\mu-12}{0.05} = 2.326 \Rightarrow \mu = 12.116$$

A.16 (a) For m.g.f of normal distribution,

$M'(0) = \mu$ (slope of line tangent to m.g.f at 0)

(a) For $N(0,1)$ ~~or~~ $N(\mu, \sigma^2)$. is reqd. distribution.

$\therefore M'(0) = 0 \Rightarrow$ slope at $(0,1)$ is 0. for ~~neg.~~ tangent to mgf

(b) for $N(-1,1)$ ~~$= N(\mu,\sigma^2)$~~ is reqd. distribution as ~~$M'(0)$ $\mu=-1$ is the~~ slope of line tangent to mgf in point $(0,1)$ is negative for f^n (b.)

(C) Slope of line tangent to mgf (c) in $(0,1)$ is +ve

$\Rightarrow$ Corresponding distribution is $N(2,1)$ with $\mu = 2$.

A.17 mgf, $g_1(t) = \frac{1}{1-4t}$, $t < \frac{1}{4}$

$g_1'(t) = \frac{4}{(1-4t)^2} \Rightarrow g_1'(0) = 4$

m.g.f, $g_2(t) = \frac{1}{(1-2t)^2}$, $t < \frac{1}{2}$

$g_2'(t) = \frac{4}{(1-2t)^3} \Rightarrow g_2'(0) = 4$

mgf, $g_3(t) = e^{4t + \frac{t^2}{2}}$

$g_3'(t) = (4+t)\, e^{4t+\frac{t^2}{2}} \Rightarrow g_3'(0) = (4+0)e^0 = 4$

$g_i'(0)$ is the slope of line tangent to m.g.f $g_i(t)$ at $t=0$

as $g_i'(0) = 4$ for $i = 1,2,3$, all 3 f^n have same slope ~~at~~ of tangent at $t=0$. Hence graphs look similiar around $t=0$.

3.4* ADDITIONAL MODELS

Exercises

3.4-1. Let the life W (in years) of the usual family car have a Weibull distribution with $\alpha = 2$. Show that β must equal 10 for $P(W > 5) = e^{-1/4} \approx 0.7788$. HINT: $P(W > 5) = e^{-H(5)}$.

3.4-2. Suppose that the length W of a man's life does follow the Gompertz distribution with $\lambda(w) = a(1.1)^w = ae^{(\ln 1.1)w}$, $P(63 < W < 64) = 0.01$. Determine the constant a and $P(W \le 71 \mid 70 < W)$.

3.4-3. Let Y_1 be the smallest observation of three independent random variables W_1, W_2, W_3, each with a Weibull distribution with parameters α and β. Show that Y_1 has a Weibull distribution. What are the parameters of this latter distribution? HINT:

$$\begin{aligned} G(y_1) &= P(Y_1 \le y_1) = 1 - P(y_1 < W_i, i = 1, 2, 3) \\ &= 1 - [P(y_1 < W_1)]^3. \end{aligned}$$

3.4-4. A frequent force of mortality used in actuarial science is $\lambda(w) = ae^{bw} + c$. Find the cdf and pdf associated with this **Makeham's law**.

3.4-5. From the graph of the first cdf of X in Figure 3.4-4, determine the indicated probabilities:

(a) $P(X < 0)$. **(b)** $P(X < -1)$. **(c)** $P(X \le -1)$.

(d) $P(X < 1)$. **(e)** $P\left(-1 \le X < \frac{1}{2}\right)$. **(f)** $P(-1 < X \le 1)$.

3.4-6. Determine the indicated probabilities from the graph of the second cdf of X in Figure 3.4-4:

(a) $P\left(-\frac{1}{2} \le X \le \frac{1}{2}\right)$. **(b)** $P\left(\frac{1}{2} < X < 1\right)$. **(c)** $P\left(\frac{3}{4} < X < 2\right)$.

(d) $P(X > 1)$. **(e)** $P(2 < X < 3)$. **(f)** $P(2 < X \le 3)$.

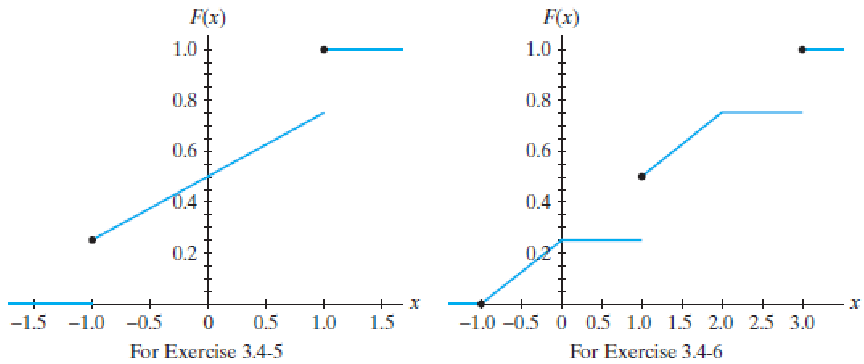


Figure 3.4-4 Mixed distribution functions

3.4-7. Let X be a random variable of the mixed type having the cdf

$$F(x) = \begin{cases} 0, & x < 0, \\ \dfrac{x^2}{4}, & 0 \le x < 1, \\ \dfrac{x+1}{4}, & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

(a) Carefully sketch the graph of $F(x)$.

(b) Find the mean and the variance of X.

(c) Find $P(1/4 < X < 1)$, $P(X = 1)$, $P(X = 1/2)$, and $P(1/2 \le X < 2)$.

3.4-8. Find the mean and variance of X if the cdf of X is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \left(\dfrac{2}{3}\right)e^{-x}, & 0 \le x. \end{cases}$$

3.4-9. Consider the following game: A fair die is rolled. If the outcome is even, the player receives a number of dollars equal to the outcome on the die. If the outcome is odd, a number is selected at random from the interval $[0, 1)$ with a balanced spinner, and the player receives that fraction of a dollar associated with the point selected.

(a) Define and sketch the cdf of X, the amount received.

(b) Find the expected value of X.

3.4-10. The weekly gravel demand X (in tons) follows the pdf

$$f(x) = \left(\frac{1}{5}\right)e^{-x/5}, \qquad 0 < x < \infty.$$

However, the owner of the gravel pit can produce at most only 4 tons of gravel per week. Compute the expected value of the tons sold per week by the owner.

3.4-11. The lifetime X of a certain device has an exponential distribution with mean five years. However, the device is not observed on a continuous basis until after three years. Hence, we actually observe $Y = \max(X, 3)$. Compute $E(Y)$.

3.4-12. Let X have an exponential distribution with $\theta = 1$; that is, the pdf of X is $f(x) = e^{-x}$, $0 < x < \infty$. Let T be defined by $T = \ln X$, so that the cdf of T is

$$G(t) = P(\ln X \le t) = P(X \le e^t).$$

(a) Show that the pdf of T is

$$g(t) = e^t e^{-e^t}, \qquad -\infty < x < \infty,$$

which is the pdf of an extreme-value distribution.

(b) Let W be defined by $T = \alpha + \beta \ln W$, where $-\infty < \alpha < \infty$ and $\beta > 0$. Show that W has a Weibull distribution.

3.4-13. A loss X on a car has a mixed distribution with $p = 0.95$ on zero and $p = 0.05$ on an exponential distribution with a mean of \$5000. If the loss X on a car is greater than the deductible of \$500, the difference $X - 500$ is paid to the owner of the car. Considering zero (if $X \le 500$) as a possible payment, determine the mean and the standard deviation of the payment.

3.4-14. A customer buys a \$1000 deductible policy on her \$31,000 car. The probability of having an accident in which the loss is greater than \$1000 is 0.03, and then that loss, as a fraction of the value of the car minus the deductible, has the pdf $f(x) = 6(1 - x)^5$, $0 < x < 1$.

(a) What is the probability that the insurance company must pay the customer more than \$2000?

(b) What does the company expect to pay?

3.4-15. A certain machine has a life X that has an exponential distribution with mean 10. The warranty is such that 100% of the price is returned if the machine fails in the first year, and 50% of the price is returned for a failure during the second year, and nothing is returned after that. If the machine cost \$2500, what are the expected value and the standard deviation of the return on the warranty?

3.4-16. A certain machine has a life X that has an exponential distribution with mean 10. The warranty is such that $\$m$ is returned if the machine fails in the first year, $(0.5)m$ of the price is returned for a failure during the second year, and nothing is returned after that. If the machine cost \$2500, find m so that the expected payment is \$200.

3.4-17. Some banks now compound daily, but report only on a quarterly basis. It seems to us that it would be easier to compound every instant, for then a dollar invested at an annual rate of i for t years would be worth e^{ti}. [You might find it interesting to prove this statement by taking the limit of $(1 + i/n)^{nt}$ as $n \to \infty$.] If X is a random rate with pdf $f(x) = ce^{-x}$, $0.04 < x < 0.08$, find the pdf of the value of one dollar after three years invested at the rate of X.

3.4-18. The time X to failure of a machine has pdf $f(x) = (x/4)^3 e^{-(x/4)^4}$, $0 < x < \infty$. Compute $P(X > 5 \mid X > 4)$.

3.4-19. Suppose the birth weight (X) in grams of U.S. infants has an approximate Weibull model with pdf

$$f(x) = \frac{3x^2}{3500^3} e^{-(x/3500)^3}, \qquad 0 < x < \infty.$$

Given that a birth weight is greater than 3000, what is the conditional probability that it exceeds 4000?

3.4-20. Let X be the failure time (in months) of a certain insulating material. The distribution of X is modeled by the pdf

$$f(x) = \frac{2x}{50^2} e^{-(x/50)^2}, \qquad 0 < x < \infty.$$

Find

(a) $P(40 < X < 60)$,

(b) $P(X > 80)$.

3.4-21. In a medical experiment, a rat has been exposed to some radiation. The experimenters believe that the rat's survival time X (in weeks) has the pdf

$$f(x) = \frac{3x^2}{120^3} e^{-(x/120)^3}, \qquad 0 < x < \infty.$$

(a) What is the probability that the rat survives at least 100 weeks?

(b) Find the expected value of the survival time. Hint: In the integral representing $E(X)$, let $y = (x/120)^3$ and get the answer in terms of a gamma function.

Ex. 3.4

A.2 As $\lambda(w) = \alpha(1.1)^w$, ~~dist b.~~ distribution f^n is

$$G(w) = 1 - e^{\left(-\frac{a}{\ln(1.1)}(1.1)^w + \frac{a}{\ln 1.1}\right)}$$

$$= 1 - e^{\frac{a}{\ln(1.1)}\left(1-(1.1)^w\right)}.$$

3.4.3 We know,

$$G(y_1) = P(Y_1 \le y_1)$$
$$= 1 - P(W_1 > y_1, W_2 > y_2, W_3 > y_1)$$

$\because$ W_1, W_2, W_3 are independent

$$\therefore G(y_1) = 1 - (P(y_1 < W))^3$$
$$= 1 - (e^{-H(y_1)})^3$$
$$= 1 - e^{-3H(y_1)}$$

$$H(y_1) = \left(\frac{y_1}{\beta}\right)^{\alpha} = 3\left(\frac{y_1}{\beta}\right)^{\alpha}$$

$$= \left(\frac{y_1}{3^{-1/\alpha}\beta}\right)^{\alpha}$$

$$\therefore \alpha_1 = \alpha \qquad \beta_1 = 3^{-1/\alpha}\beta$$

3.4.4 (a) $H(w) = \int_0^w \lambda(t)\,dt$

$$= \int_0^w (ae^{bt} + c)\,dt$$

$$= \frac{a}{b} e^{bt} + ct \Big|_0^w = \frac{a}{b}\left(e^{bw} - 1\right) + cw$$

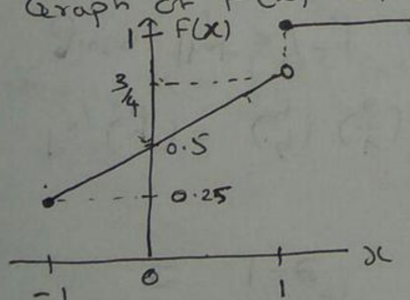
$\therefore G(w) = 1 - e^{-H(w)}$, $g(w) = G'(w) = H'(w)e^{-H(w)}$

$\therefore G(w) = 1 - e^{-H(w)} = 1 - \exp\left(-\frac{a}{b}\left(e^{bw} - 1\right) - cw\right)$

$g(w) = G'(w) = H'(w)e^{-H(w)}$

$$= \left(ae^{bw} + c\right) \exp\left(-\frac{a}{b}\left(e^{bw} - 1\right) - cw\right)$$

A.5 Graph of $F(x)$ is,



(a) from fig, $P(X<0) = \frac{1}{2}$

(b) from fig, $P(X<-1) = 0$

(c) from fig, $P(X \leq -1) = \frac{1}{4}$

(d) from fig, $P(X<1) = \frac{3}{4}$

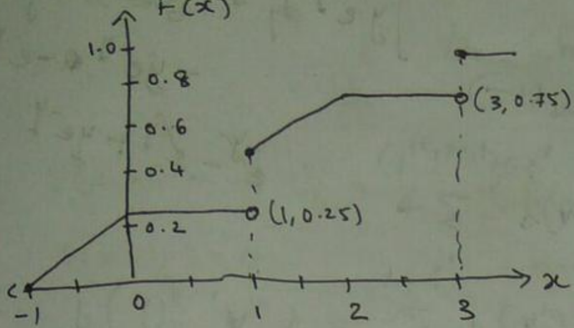
(e) $P(-1 \leq X < \frac{1}{2}) = P(X<\frac{1}{2}) - P(X<-1)$

$$= \frac{5}{8} - 0 = \frac{5}{8}$$

(f) $P(-1 < X \leq 1) = P(X \leq 1) - P(X \leq -1)$

$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

A-6 Graph of $F(x)$ is,



(a) $P(-\frac{1}{2} \le x \le \frac{1}{2}) = P(x \le \frac{1}{2}) - P(x < -\frac{1}{2})$

$= \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$

(b) $P(\frac{1}{2} < x < 1) = P(x < 1) - P(x \le \frac{1}{2})$

$= \frac{1}{4} - \frac{1}{4} = 0$

(c) $P(\frac{3}{4} < x < 2) = P(x < 2) - P(x \le \frac{3}{4})$

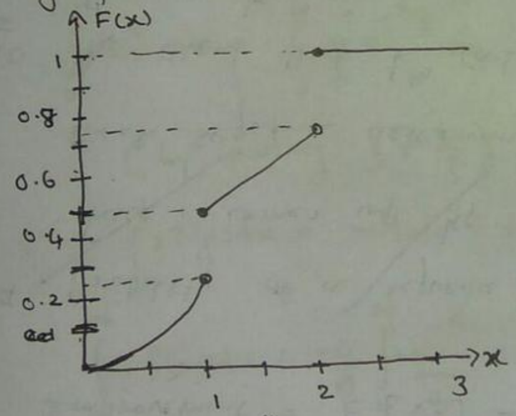
$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

(d) $P(x > 1) = 1 - P(x \le 1) = 1 - \frac{1}{2} = \frac{1}{2}$

(e) $P(2 < x < 3) = P(x < 3) - P(x \le 2) = \frac{3}{4} - \frac{3}{4} = 0$

(f) $P(2 < x \le 3) = P(x \le 3) - P(x < 2) = 1 - \frac{3}{4} = \frac{1}{4}$

A.7 graph of $F(x)$ is,



(b) $F'(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{1}{4} & 1 \le x < 2 \\ 0 & 2 \le x \end{cases}$

$P(x=0) = 0$

$P(x=1) = 1/4$

$P(x=2) = 1/4$

mean, $\mu = \int_{-\infty}^{\infty} x F'(x)\,dx = \int_0^1 \frac{x^2}{2}\,dx + \int_1^2 \frac{x}{4}\,dx + 1 \cdot F'(1) + 2 \cdot F'(2)$

$= \frac{x^3}{6}\Big|_0^1 + \frac{x^2}{8}\Big|_1^2 + 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right)$

$= \frac{31}{24}.$

$E(x^2) = \int_0^1 x^2 \left(\frac{x}{2}\right) dx + \int_1^2 x^2 \left(\frac{1}{4}\right) dx + 1^2 F'(1) + 2^2 F'(2).$

$= \frac{x^4}{8}\Big|_0^1 + \frac{x^3}{12}\Big|_1^2 + \frac{1}{4} + 1 = \frac{47}{24}.$

$\sigma^2 = E(x^2) - \mu^2 = \frac{47}{24} - \left(\frac{31}{24}\right)^2 = \frac{1128 - 961}{576} = \frac{167}{576}$ (variance)

(c) $P(1/4 < X < 1) = P(X < 1) - P(X < 1/4)$

$= \frac{1}{4} - \frac{1}{64}$ (from graph).

$= \frac{15}{64}$

$P(x = 1) = \frac{1}{2} - \frac{1}{4} = 1/4$ (from graph)

$P(x = 1/2) = 0$ (from graph)

$P(\frac{1}{2} \le X < 2) = P(X < 2) - P(X \le 1/2)$

$= \frac{3}{4} - 1/16$ (from graph)

$= \frac{11}{16}.$

3.4.8 Mean: $E(x) = \int_{-\infty}^{\infty} x F'(x)\, dx$

$$= \int_0^{\infty} x \left(\frac{2}{3} e^{-x}\right) dx$$

$$= -2/3 \int_0^{\infty} x\, d\left(e^{-x}\right)$$

$$= -\frac{2}{3} x e^{-x} \Big|_0^{\infty} + \frac{2}{3} \int_0^{\infty} e^{-x}\, dx$$

$$= 0 - 2/3\, e^{-x} \Big|_0^{\infty} = 2/3$$

Variance $\sigma^2 = E(x^2) - (E(x))^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 F'(x)\, dx = -\frac{2}{3} \int_0^{\infty} x^2\, d(e^{-x})$$

$$= \frac{-2}{3} x^2 e^{-x} \Big|_0^{\infty} + \frac{4}{3} \int_0^{\infty} x e^{-x}\, dx$$

$$= 0 - 4/3 \int_0^{\infty} x\, d(e^{-x}) = 4/3$$

$$\therefore \sigma^2 = E(x^2) - (E(x))^2 = 4/3 - \left(\frac{2}{3}\right)^2$$

$$= 4/3 - 4/9 = 8/9$$

A.10 Given X = weekly gravel demand with p.d.f, $f(x) = \frac{e^{-x/5}}{5}$

Most production of gravel/week = 4 tons. $\Rightarrow x \le 4$.

Expected value of tons sold/week = T

$$T = \int_0^4 x f(x)\,dx + \int_4^\infty 4 f(x)\,dx \quad -①$$

$$\int_0^4 x f(x)\,dx = \int_0^4 \frac{x}{5} e^{-x/5}\,dx = \int_0^{4/5} 5y e^{-y}\,dy = -5[ye^{-y} + e^{-y}]_0^{4/5}$$

(by putting $y = \frac{x}{5}$; $dx = 5dy$)

$$\therefore \int_0^4 x f(x)\,dx = -9e^{-4/5} + 5. \quad - ②$$

$$\int_4^\infty 4 f(x)\,dx = -4\int_4^\infty \frac{-e^{-x/5}}{5}\,dx = -4\left[e^{-x/5}\right]_4^\infty = 4e^{-4/5}. \quad -③$$

$\therefore$ From eq. ② & ③

$$T = -9e^{-4/5} + 5 + 4e^{-4/5} = (1 - e^{-4/5})5 = 5(1 - 0.449) = 2.75.$$

Expected sale/week = 2.75 Tons.

A.11 $Y = \max(X, 3)$, $Y \le 3$

$$\therefore E(Y) = \int_0^3 x f(x)\,dx + \int_3^\infty 3 f(x)\,dx. \quad \& \quad f(x) = \frac{e^{-x/5}}{5}.$$

$$\& \int_0^3 x\left(\frac{1}{5}e^{-x/5}\right)dx = -\int_0^3 x\, d(e^{-x/5}) = -\left[x\left[e^{-x/5}\right]_0^3 + \int_0^3 e^{-x/5}\,dx\right].$$

$$\text{or } \int_0^3 x\left(\frac{e^{-x/5}}{5}\right)dx = -3e^{-3/5} - 5e^{-3/5} + 5(1) = -8e^{-0.6} + 5$$

$$\int_3^\infty 3 f(x)\,dx = 3\int_3^\infty \frac{e^{-x/5}}{5}\,dx = 3\left[e^{-x/5}\right]_3^\infty = +3e^{-0.6}$$

$$\therefore E(Y) = -8e^{-0.6} + 5 + 3e^{-0.6} = (1 - e^{-0.6})5 = \underline{2.26.}$$

A.12 (a) $G(t) = P(T \le t)$, $T = \ln X$,

$\Rightarrow G(t) = P(\ln X \le t) = P(X \le e^t) = F(e^t) = 1 - e^{(-e^t)}$

p.d.f of T is,

$g(t) = G'(t) = e^t e^{-e^t} = e^{t-e^t}$.

(b) $T = \alpha + \beta \ln W$, $G(t) = 1 - e^{-e^t}$, distribution f^n of W is

$J(w) = P(W \le w) = P(T \le \alpha + \beta \ln W)$

$= G(\alpha + \beta \ln W) = 1 - \exp(e^{-\alpha+\beta \ln W})$

$\therefore H(w) = e^{\alpha + \beta \ln W} = e^{\alpha} \cdot w^{\beta} = \left(\frac{w}{e^{-\alpha/\beta}}\right)^{\beta}$.

which is weibull distribution with parameters β & $e^{-\alpha/\beta}$.

A.13 Expected payment is, $E(x) = 0(0.95) + 0.05 \int_{500}^{\infty} (x-500) f(x)\,dx$

$$+ 0(0.05) \int_{500}^{500} (x-500) f(x)\,dx$$

as payment is zero when $x \le 500$ with prob. 0.05.

$$f(x) = \frac{1}{5000} e^{-\frac{x}{5000}} \quad (\text{p.d.f of } x,\ x > 0)$$

$$\therefore E(x) = 0.05 \int_{500}^{\infty} \frac{(x-500)\, e^{-\frac{x}{5000}}}{5000}\,dx = -0.05 \int_{500}^{\infty} (x-500)\, d e^{-\frac{x}{5000}}$$

$$= \left[-0.05(x-500)\, e^{-\frac{x}{5000}}\right]_{500}^{\infty} + 0.05 \int_{500}^{\infty} e^{-\frac{x}{5000}}\,dx$$

$$= 0 + 5000(0.5) \int_{500}^{\infty} \frac{e^{-\frac{x}{5000}}}{5000}\,dx$$

$$= -250 \left(e^{-\frac{x}{5000}}\right)_{500}^{\infty} = -250\left(0 - e^{-0.1}\right) = 250 e^{-0.1}$$

$$= 226.209355 \approx 226.21$$

$$\text{Var.}(X) = \sigma^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = 0.05 \int_{500}^{\infty} (x-500)^2 f(x)\,dx = 0.05 \int_{500}^{\infty} \frac{e^{-x/5000}}{5000} (x-500)^2\,dx$$

$$= \left[-0.05\,(x-500)^2 \left(e^{-\frac{x}{5000}}\right)\right]_{500}^{\infty} + (0.05) \int_{500}^{\infty} 2(x-500)\, e^{-\frac{x}{5000}}\,dx.$$

$$= 0 + (2 \times 5000)\,(0.05) \int_{500}^{\infty} \frac{(x-500)}{5000} e^{-x/5000}\,dx$$

$$= 10^4\, E(x) = 2262093.55$$

$$\therefore \sigma^2 = E(x^2) - (E(x))^2 = 2210922.581$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{2210922.581} = 1486.92.$$

A.14 (a) $P(x > 2000) = P(x > 1000)\left(1 - F\left(\frac{2}{30}\right)\right)$

$$= 0.03\left(1 - F\left(\tfrac{2}{30}\right)\right)$$

$$= 0.03\left(1 - \int_0^{2/30} f(x)\,dx\right)$$

& $\displaystyle\int_0^{2/30} f(x)\,dx = \int_0^{2/30} 6(1-x)^5\,dx = \int_{28/30}^{1} 6y^5\,dy = \left(y^6\right)_{28/30}^{1} = \cancel{3\left(1 - \left(\tfrac{28}{30}\right)^6\right)}$

$\therefore P(x > 2000) = 0.03\left(1 - \cancel{3}\left(1 - \left(\frac{28}{30}\right)^6\right)\right) = \cancel{0.03}\left(\frac{28}{30}\right)^6 = 0.0198$

(b) Payment is zero when $x < 1000$ with prob. 0.03.

$$E(x) = 30000\left(0.97(0) + (0.03)\int_0^1 x f(x)\,dx\right)$$

$$= 900\int_0^1 6x(1-x)^5\,dx = 5400\int_0^1 (1-y)y^5\,dy \quad \left(\text{put } 1-x=y,\ dx=-dy\right)$$

$$\therefore E(x) = 5400\left(\frac{y^6}{6} - \frac{y^7}{7}\right)_0^1$$

$$= 5400\left(\frac{1}{42}\right) = \frac{900}{7} = 128.57.$$

A.15 $E(x) = 2500 \int_0^1 f(x)\,dx + 1250 \int_1^2 f(x)\,dx + 0 \cdot \int_2^\infty f(x)\,dx.$

$$\int f(x)\,dx = \int \frac{e^{-x/10}}{10}\,dx = -e^{-x/10} + C. \quad — (1)$$

$$\therefore E(x) = 2500 \left(-e^{-x/10}\right)_0^1 + 1250 \left(-e^{-x/10}\right)_1^2$$

$$= 2500\left(1 - e^{-0.1}\right) + 1250\left(e^{-0.1} - e^{-0.2}\right).$$

$$= 2500 - 1250\,e^{-0.1} - 1250\,e^{-0.2}$$

$$= 1250\left(2 - e^{-0.1} - e^{-0.2}\right)$$

$$= 345.54.$$

$$E(x^2) = 2500^2 \int_0^1 f(x)\,dx + 1250^2 \int_1^2 f(x)\,dx + 0 \cdot \int_2^\infty f(x)\,dx.$$

$$= 2500^2\left(1 - e^{-0.1}\right) + 1250^2\left(e^{-0.1} - e^{-0.2}\right) \quad — (\text{from } (1)).$$

$$= 729309.38$$

$$\sigma^2 = E(x^2) - (E(x))^2 = 729309.38 - 345.54^2 = 609911.49$$

$$\sigma = \sqrt{\sigma^2} = 780.97.$$

A.16 X = life time of certain machine.

X follows exponential distribution with mean 10., $\lambda = 10$.

p.d.f of X is, $f(x) = \frac{1}{\lambda} e^{-x/\lambda} = \frac{e^{-x/10}}{10}$.

$$E(x) = m\int_0^1 f(x)\,dx + \frac{m}{2}\int_1^2 f(x)\,dx + 0\cdot\int_2^\infty f(x)\,dx$$

$$= m\int_0^1 \frac{e^{-x/10}}{10}\,dx + \frac{m}{2}\int_1^2 \frac{e^{-x/10}}{10}\,dx.$$

$$= m\left[-e^{-\frac{x}{10}}\right]_0^1 + \frac{m}{2}\left[-e^{-\frac{x}{10}}\right]_1^2$$

$$= m\left[1-e^{-0.1}\right] + \frac{m}{2}\left[e^{-0.1}-e^{-0.2}\right]$$

$$= 0.0952m + 0.0431m$$

$$= 0.1383m$$

given ~~$E(x) = 200\$$~~ expected value $= 200\$$

machine cost $= 2500\$$

$\Rightarrow 2500 \times E(x) = 200 \Rightarrow m = \frac{2}{25} \times \frac{1}{0.1383}$.

$\therefore$ $\underline{m = 0.5785}$.

A.17 Period of deposit $= 3$ yrs, rate of deposit $= X$,

worth of $Y\$ = e^{3X}$.

$$F(x) = \int_{0.04}^{x} f(t)\,dt = -c\left[e^{-t}\right]_{0.04}^{x} = c\left(e^{-0.04} - e^{-x}\right)$$

$0.04 < x < 0.08$.

$X = \frac{\ln Y}{3}$, $\Rightarrow G(Y) = F\left(\frac{1}{3}\ln Y\right) = ce^{-0.04} - ce^{\ln y^{-1/3}} = ce^{-0.04} - cy^{-1/3}$.

p.d.f of Y is,

$$g(y) = G'(y) = +\frac{1}{3}cy^{-4/3} \qquad e^{0.12} < y < e^{0.24}.$$

To find c, $\int_{e^{0.12}}^{e^{0.24}} g(y)\,dy = 1$.

$$\& \int_{e^{0.12}}^{e^{0.24}} g(y)\,dy = \cancel{\frac{c}{3}\left(-3\int_{e^{0.12}}^{e^{0.24}} y^{-4/3}\right)dy} = -cy^{-1/3}\Big|_{e^{0.12}}^{e^{0.24}}$$

$$\Rightarrow c\left(e^{-0.04} - e^{-0.08}\right) = 1 \Rightarrow c = \frac{1}{e^{-0.04} - e^{-0.08}} = 26.54.$$

A.18 $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ (formula)

$\therefore P(X>5|X>4) = \dfrac{P(X>5)}{P(X>4)}$ & $P(X>x) = 1-P(X\le x) = 1-F(x)$

$\cancel{\Rightarrow P(X>5|X>4)} = \dfrac{\cancel{1-F(5)}}{1-F(4)}$ ~~& P(X~~

$P(X>x) = \displaystyle\int_x^\infty f(t)\,dt = \int_x^\infty \left(\frac{t}{4}\right)^3 e^{-\left(\frac{t}{4}\right)^4} dt = 4\left[-e^{-\left(\frac{t}{4}\right)^4}\right]_x^\infty$

$\therefore P(X>x) = 4e^{-\left(\frac{x}{4}\right)^4}$

$\therefore P(X>5|X>4) = e^{-\left(\frac{5}{4}\right)^4 + \left(\frac{4}{4}\right)^4} = e^{-\frac{369}{256}} = 0.237.$

A.19 $P(X>4000|X>3000) = \dfrac{P(X>4000)}{P(X>3000)} = \dfrac{1-F(4000)}{1-F(3000)}$ — (1)

$F(x) = \displaystyle\int_{-\infty}^x f(t)\,dt$ where $f(t) = \dfrac{3t^2}{3500^3} e^{-\left(\frac{t}{3500}\right)^3}$. $0<x<\infty$.

$\therefore \cancel{F(x) = \int_0^x \left(\frac{3t^2}{3500^3}dt\right) e^{-\left(\frac{t}{3500}\right)^3}}$

$F(x) = \displaystyle\int_0^x \frac{3t^2}{3500^3} e^{-\left(\frac{t}{3500}\right)^3} dt = -\int_0^x e^{-\left(\frac{t}{3500}\right)^3} d\left(-\left(\frac{t}{3500}\right)^3\right)$

$= \left[-e^{-\left(\frac{t}{3500}\right)^3}\right]_0^x$

$= 1 - e^{-\frac{x^3}{3500^3}}$

$\therefore P(X>4000|X>3000) = \dfrac{e^{-\left(\frac{4000}{3500}\right)^3}}{e^{-\left(\frac{3000}{3500}\right)^3}}$ (from eq (1))

$= e^{-\left(\frac{8}{7}\right)^3 + \left(\frac{6}{7}\right)^3}$

$= e^{-\frac{296}{343}}$

$= 0.4219.$

A.20 By def. $P(X \le x) = F(x)$. , $F(x)$ is distribution of X.

with p.d.f of x, $f(x) = \frac{2t}{50^2} e^{-\frac{t^2}{50^2}}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \frac{2t}{50^2} e^{-\frac{t^2}{50^2}}\,dt = \left[-e^{-\frac{t^2}{50^2}}\right]_{0}^{x} = 1 - e^{-\frac{x^2}{2500}}$$

(a) $P(40 < X < 60) = P(X \le 60) - P(X \le 40)$

$= F(60) - F(40)$

$= e^{-0.64} - e^{-1.44}$

$= 0.29$

(b) $P(X > 80) = 1 - P(X \le 80) = 1 - F(80) = e^{-\left(\frac{8}{5}\right)^2} \cancel{= e^{-2.56}}$

$\therefore P(X > 80) = e^{-2.56} = 0.077.$

Chapter 3

3.1-3 **(a)** $f(x) = 1/10,\ 0 < x < 10$; **(b)** 0.2; **(c)** 0.6; **(d)** $\mu = 5$; (e) $\sigma^2 = 25/3$.

3.1-5 **(a)** $G(w) = (w - a)/(b - a),\ a \le w \le b$;
(b) $U(a, b)$.

3.1-7 **(a)** **(i)** 3; **(ii)** $F(x) = x^4,\ 0 \le x \le 1$; **(iv)** $\mu = 4/5, \sigma^2 = 2/75$;
(b) **(i)** 3/16; **(ii)** $F(x) = (1/8)x^{3/2},\ 0 \le x \le 4$; **(iv)** $\mu = 12/5, \sigma^2 = 192/175$;
(c) **(i)** 1/4; **(ii)** $F(x) = x^{1/4},\ 0 \le x \le 1$; **(iv)** $\mu = 1/5, \sigma^2 = 16/225$.

3.1-9 **(b)**

$$F(x) = \begin{cases} 0, & x < 0, \\ x(2 - x), & 0 \le x < 1, \\ 1, & 1 \le x. \end{cases}$$

(c) **(i)** 3/4, **(ii)** 1/2, **(iii)** 0, **(iv)** 1/16.

3.1-11 **(a)** $d = 2$; **(b)** $E(Y) = 2$; **(c)** $E(Y^2) = +\infty$.

3.1-13 $f(x) = \dfrac{e^{-x}}{(1 + e^{-x})^2} = \dfrac{e^{-x}}{(1 + e^{-x})^2}\dfrac{e^{2x}}{e^{2x}} = \dfrac{e^{x}}{(e^{x} + 1)^2} = f(-x)$.

3.1-15 **(a)** $1/e$; **(b)** $1/e^{19/8}$.

3.1-17 \$740.74.

3.1-19 **(a)** $\mu = \$28{,}571.43$, $\sigma = \$15{,}971.91$; **(b)** 0.6554.

3.1-21 **(a)** $\int_{-\infty}^{\infty} \sum_{i=1}^{k} c_i f_i(x)\, dx = \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} f_i(x)\, dx = \sum_{i=1}^{k} c_i = 1$;
(b) $\mu = \sum_{i=1}^{k} c_i \mu_i,\ \sigma^2 = \sum_{i=1}^{k} c_i(\sigma_i^2 + \mu_i^2) - \mu^2$.

3.2-1 **(a)** $f(x) = (1/3)e^{-x/3}$, $0 < x < \infty$; $\mu = 3$; $\sigma^2 = 9$;
(b) $f(x) = 3e^{-3x}$, $0 < x < \infty$; $\mu = 1/3$; $\sigma^2 = 1/9$.

3.2-3 $P(X > x + y \mid X > x) = \dfrac{P(X > x + y)}{P(X > x)} = \dfrac{e^{-(x+y)/\theta}}{e^{-x/\theta}} = P(X > y)$.

3.2-5 **(a)** $F(x) = 1 - e^{-(x-\delta)/\theta}$, $\delta \le x < \infty$;
(b) $\theta + \delta$; θ^2.

3.2-9 $f(x) = \dfrac{1}{\Gamma(20)7^{20}} x^{19} e^{-x/7}$, $0 \le x < \infty$; $\mu = 140$; $\sigma^2 = 980$.

3.2-11 **(a)** 0.025; **(b)** 0.05; **(c)** 0.94; **(d)** 8.672; **(e)** 30.19.

3.2-13 **(a)** 0.80; **(b)** $a = 11.69, b = 38.08$; **(c)** $\mu = 23$, $\sigma^2 = 46$; **(d)** 35.17, 13.09.

3.2-15 **(a)** $r - 2$; **(b)** $x = r - 2 \pm \sqrt{2r - 4}$, $r \ge 4$.

3.2-17 0.9444.

3.2-19 1.96, or 1,960 units per day, yields an expected profit of $3,304.96.

3.2-21 $e^{-1/2}$.

3.2-23 $M = 83.38$.

3.3-1 **(a)** 0.2784; **(b)** 0.7209; **(c)** 0.9616; **(d)** 0.0019; **(e)** 0.9500; **(f)** 0.6826; **(g)** 0.9544; **(h)** 0.9974.

3.3-3 **(a)** 1.96; **(b)** 1.96; **(c)** 1.645; **(d)** 1.645.

3.3-5 **(a)** 0.3849; **(b)** 0.5403; **(c)** 0.0603; **(d)** 0.0013; **(e)** 0.6826; **(f)** 0.9544; **(g)** 0.9974; **(h)** 0.9869.

3.3-7 **(a)** 0.6326; **(b)** 50.

3.3-9 **(a)** Gamma ($\alpha = 1/2$, $\theta = 8$); **(b)** Gamma ($\alpha = 1/2$, $\theta = 2\sigma^2$).

3.3-11 **(a)** 0.0401; **(b)** 0.8159.

3.3-13 0.1437.

3.3-15 **(a)** $\sigma = 0.043$; **(b)** $\mu = 12.116$.

3.3-17 The three respective distributions are exponential with $\theta = 4$, $\chi^2(4)$, and $N(4, 1)$. Each has a mean of 4, so the slopes of the mgfs equal 4 at $t = 0$.

3.4-1 $e^{-(5/10)^2} = e^{-1/4} = 0.7788$.

3.4-3 Weibull with parameters α and $\beta/3^{1/\alpha}$.

3.4-5 **(a)** 0.5; **(b)** 0; **(c)** 0.25; **(d)** 0.75; **(e)** 0.625; **(f)** 0.75.

3.4-7 **(b)** $\mu = 31/24, \sigma^2 = 167/567$; **(c)** 15/64; 1/4; 0; 11/16.

3.4-9 **(a)**

$$F(x) = \begin{cases} 0, & x < 0, \\ x/2, & 0 \le x < 1, \\ 1/2, & 1 \le x < 2, \\ 4/6, & 2 \le x < 4, \\ 5/6, & 4 \le x < 6, \\ 1, & 6 \le x. \end{cases}$$

(b) \$2.25.

3.4-11 $3 + 5\,e^{-3/5} = 5.744$.

3.4-13 $\mu = \$226.21$, $\sigma = \$1{,}486.92$.

3.4-15 $\mu = \$345.54$, $\sigma = \$780.97$.

3.4-17 $g(y) = \dfrac{c}{3y^{4/3}}$ for $e^{0.12} < y < e^{0.24}$; $c = 26.54414$.

3.4-19 0.4219.

3.4-21 **(a)** $e^{-(125/216)}$; **(b)** $120 * \Gamma(4/3) = 107.1575$.

Chapter 4

Bivariate Distributions

4.1 BIVARIATE DISTRIBUTIONS OF THE DISCRETE TYPE

Exercises

4.1-1. For each of the following functions, determine the constant c so that $f(x,y)$ satisfies the conditions of being a joint pmf for two discrete random variables X and Y:

(a) $f(x,y) = c(x+2y), \quad x = 1,2, \quad y = 1,2,3.$

(b) $f(x,y) = c(x+y), \quad x = 1,2,3, \quad y = 1,\ldots,x.$

(c) $f(x,y) = c,$ x and y are integers such that $6 \le x+y \le 8,\ 0 \le y \le 5.$

(d) $f(x,y) = c\left(\frac{1}{4}\right)^x\left(\frac{1}{3}\right)^y, \quad x = 1,2,\ldots, \quad y = 1,2,\ldots.$

4.1-2. Roll a pair of four-sided dice, one red and one black, each of which has possible outcomes 1, 2, 3, 4 that have equal probabilities. Let X equal the outcome on the red die, and let Y equal the outcome on the black die.

(a) On graph paper, show the space of X and Y.

(b) Define the joint pmf on the space (similar to Figure 4.1-1).

(c) Give the marginal pmf of X in the margin.

(d) Give the marginal pmf of Y in the margin.

(e) Are X and Y dependent or independent? Why or why not?

4.1-3. Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1,2, \quad y = 1,2,3,4.$$

(a) Find $f_X(x)$, the marginal pmf of X.

(b) Find $f_Y(y)$, the marginal pmf of Y.

(c) Find $P(X > Y)$.

(d) Find $P(Y = 2X)$.

(e) Find $P(X + Y = 3)$.

(f) Find $P(X \le 3 - Y)$.

(g) Are X and Y independent or dependent? Why or why not?

(h) Find the means and the variances of X and Y.

4.1-4. Select an (even) integer randomly from the set {0, 2, 4, 6, 8}. Then select an integer randomly from the set {0, 1, 2, 3, 4}. Let X equal the integer that is selected from the first set and let Y equal the sum of the two integers.

(a) Show the joint pmf of X and Y on the space of X and Y.

(b) Compute the marginal pmfs.

(c) Are X and Y independent? Why or why not?

4.1-5. Roll a pair of four-sided dice, one red and one black. Let X equal the outcome on the red die and let Y equal the sum of the two dice.

(a) On graph paper, describe the space of X and Y.

(b) Define the joint pmf on the space (similar to Figure 4.1-1).

(c) Give the marginal pmf of X in the margin.

(d) Give the marginal pmf of Y in the margin.

(e) Are X and Y dependent or independent? Why or why not?

4.1-6. The torque required to remove bolts in a steel plate is rated as very high, high, average, and low, and these occur about 30%, 40%, 20%, and 10% of the time, respectively. Suppose $n = 25$ bolts are rated; what is the probability of rating 7 very high, 8 high, 6 average, and 4 low? Assume independence of the 25 trials.

4.1-7. A particle starts at (0, 0) and moves in one-unit independent steps with equal probabilities of 1/4 in each of the four directions: north, south, east, and west. Let S equal the east–west position and T the north–south position after n steps.

(a) Define the joint pmf of S and T with $n = 3$. On a two–dimensional graph, give the probabilities of the joint pmf and the marginal pmfs (similar to Figure 4.1-1).

(b) What are the marginal distributions of X and Y?

4.1-8. In a smoking survey among boys between the ages of 12 and 17, 78% prefer to date nonsmokers, 1% prefer to date smokers, and 21% don't care. Suppose seven such boys are selected randomly. Let X equal the number who prefer to date nonsmokers and Y equal the number who prefer to date smokers.

(a) Determine the joint pmf of X and Y. Be sure to include the support of the pmf.

(b) Find the marginal pmf of X. Again include the support.

4.1-9. A manufactured item is classified as good, a "second," or defective with probabilities 6/10, 3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.

(a) Give the joint pmf of X and Y, $f(x,y)$.

(b) Sketch the set of integers (x,y) for which $f(x,y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?

(c) Find $P(X = 10, Y = 4)$.

(d) Give the marginal pmf of X.

(e) Find $P(X \le 11)$.

4-1-1)

(a) $f(x,y) = c(x+2y)$

$S: \; x = 1,2$
$\quad\; y = 1,2,3.$

$$\sum_{(x,y)\in S}\sum f(x,y) = 1$$

$$\Rightarrow \sum_{x=1}^{2}\sum_{y=1}^{3} c(x+2y) = 1$$

$$= \sum_{x=1}^{2}\left[cx(3) + 2.c.\frac{(3)(3+1)}{2}\right]$$

$$= \sum_{x=1}^{2}\left[3cx + 12c\right]$$

$$= 3c + 6c + 12c(2)$$

$$= 9c + 24c$$

$$= 33c = 1$$

$$\Rightarrow c = 1/33.$$

(b) $f(x,y) = c(x+y)$

$S: \; x = 1,2,3$
$\quad\; y = 1,\ldots x.$

$$\sum_{(x,y)\in S}\sum f(x,y) = 1$$

$$\Rightarrow \sum_{x=1}^{3}\sum_{y=1}^{x} c(x+y) = \sum_{x=1}^{3}\left[cx(x) + \frac{c(x)(x+1)}{2}\right]$$

$$= c\sum_{x=1}^{3} x^2 + \frac{x^2+x}{2}$$

$$= c\sum_{x=1}^{3}\frac{3x^2+x}{2}$$

$$= c\left[\frac{3+1}{2} + \frac{12+2}{2} + \frac{27+3}{2}\right] = c\left[2+7+15\right] = 24c$$

$$\Rightarrow 24c = 1$$

$$\Rightarrow c = 1/24$$

c) $f(x,y) = c$

$S: \quad 6 \le x+y \le 8$

$0 \le y \le 5$

$\Rightarrow \quad y = 0, 1, 2, \ldots 5$

$y = 0 \Rightarrow x = 6, 7, 8$

$y = 1 \Rightarrow x = 5, 6, 7$

$y = 2 \Rightarrow x = 4, 5, 6$

$\vdots$

$y = 5 \Rightarrow x = 1, 2, 3.$

So, total no. of pairs of (x,y) in $S = 5 \times 3$
$= 15.$

and $f(x,y)$ is c at all these points.

So, $\displaystyle\sum\sum_{(x,y)\in S} f(x,y) = \sum\sum_{(x,y)\in S} c$

$= c(15) = 1$

$\Rightarrow c = 1/15$

(d) $f(x,y) = c \cdot \left(\frac{1}{4}\right)^x \left(\frac{1}{3}\right)^y$

$S: \quad x = 1, 2, \ldots$
$\quad\; y = 1, 2, \ldots$

$$\sum\sum_{(x,y)\in S} f(x,y) = \sum_{y=1}^{\infty}\sum_{x=1}^{\infty} c\left(\frac{1}{4}\right)^x\left(\frac{1}{3}\right)^y = 1$$

$$\Rightarrow c \cdot \sum_{y=1}^{\infty}\left(\frac{1}{3}\right)^y \cdot \sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x$$

$$= c \cdot \sum_{y=1}^{\infty}\left(\frac{1}{3}\right)^y \left(\frac{1/4}{1-1/4}\right) = \frac{1}{3}c \cdot \sum_{y=1}^{\infty}\left(\frac{1}{3}\right)^y$$

$$= \frac{1}{3}c\left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{1}{3}\cdot c \cdot \frac{1}{2} = \frac{c}{6}$$

$\Rightarrow \frac{c}{6} = 1$

$\Rightarrow c = 6$

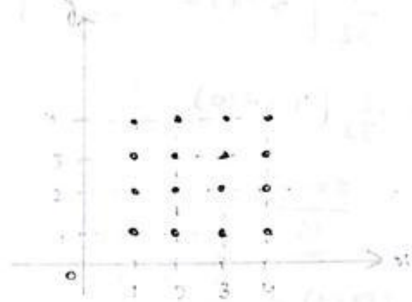
4-1-2) X: outcome on red die

$\Rightarrow$ x can be 1, 2, 3, 4.

Y: outcome on black die

$\Rightarrow$ y can be 1, 2, 3, 4.

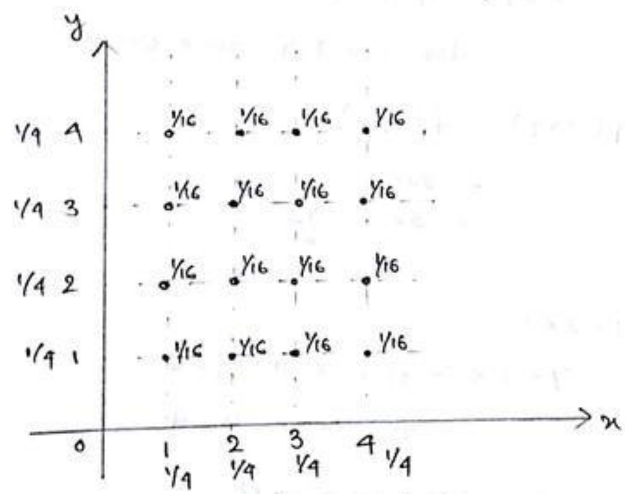
(a) S: $x = 1, 2, 3, 4$. $y = 1, 2, 3, 4$.



points in the graph indicate S.

joint pmf $f(x, y) = P(X = x, Y = y)$.

b, c, d, e)



X & Y are independent because $f(x, y) = f_X(x) \cdot f_Y(y)$ for every $(x, y) \in S$.

3. $f(x,y) = \frac{x+y}{32}$, $x=1,2$ $\quad y=1,2,3,4$

a) $f_x(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} = \frac{4x+10}{32} = \frac{2x+5}{16}$, $x=1,2$

b) $f_y(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32}$, $y=1,2,3,4$

c) Find $P(x>y)$

$$\frac{2+1}{32} = \frac{3}{32}$$

d) Find $P(Y=2X)$

$$\frac{2+4}{32} + \frac{1+2}{32} = \frac{9}{32}$$

e) Find $P(X+Y=3)$

$$\frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32} = \frac{3}{16}$$

f) Find $P(x \leq (3-y))$

$3-1=2$, $x=1,2$ $\qquad \frac{1+1}{32} + \frac{2+1}{32} + \frac{1+2}{32} = \frac{2+3+3}{32} = \frac{8}{32} = 1/4$

$3-2=1$, $x=1$

g) dependent, $P(X=1, Y=1) = P(X=1)P(Y=1)$

$$\frac{1+1}{32} = \left(\frac{2(1)+5}{16}\right)\left(\frac{3+2(1)}{32}\right)$$

$$\frac{1}{16} = \left(\frac{7}{16}\right)\left(\frac{6}{32}\right)$$

$$\frac{1}{16} = \frac{42}{512}$$

$$0.0625 \neq 0.08203$$

h) $E(x) = \sum_{x=1}^{2} x\left(\frac{2x+5}{16}\right) = \frac{2+5}{16} + \frac{8+10}{16} = \frac{25}{16}$ $\quad E(y) = \sum_{y=1}^{4} y\left(\frac{3+2y}{32}\right) = \frac{3+2}{32} + \frac{6+8}{32} + \frac{9+15}{32} + \frac{12+32}{32} = \frac{90}{32}$

$E(x^2) = \sum_{x=1}^{2} x^2\left(\frac{2x+5}{16}\right) = \frac{2+5}{16} + \frac{16+20}{16} = \frac{43}{16}$ $\quad E(y^2) = \sum_{y=1}^{4} y^2\left(\frac{3+2y}{32}\right) = \frac{5}{32} + \frac{12+16}{32} + \frac{27+54}{32} + \frac{48+128}{32} = \frac{290}{32}$

$\sigma_x^2 = E(x^2) - (E(x))^2 = \frac{43}{16} - \left(\frac{25}{16}\right)^2 = \frac{63}{256}$ $\quad \sigma_y^2 = E(y^2) - (E(y))^2 = \frac{290}{32} - \left(\frac{90}{32}\right)^2 = 1.152$

4.2 THE CORRELATION COEFFICIENT

Exercises

4.2-1. Let the random variables X and Y have the joint pmf

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1,2, \quad y = 1,2,3,4.$$

Find the means μ_X and μ_Y, the variances σ_X^2 and σ_Y^2, and the correlation coefficient ρ.

4.2-2. Let X and Y have the joint pmf defined by $f(0,0) = f(1,2) = 0.2, f(0,1) = f(1,1) = 0.3$.

(a) Depict the points and corresponding probabilities on a graph.

(b) Give the marginal pmfs in the "margins."

(c) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, Cov$(X, Y)$, and ρ.

(d) Find the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-3. Roll a fair four-sided die twice. Let X equal the outcome on the first roll, and let Y equal the sum of the two rolls.

(a) Determine $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, Cov$(X, Y)$, and ρ.

(b) Find the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2-4. Let X and Y have a trinomial distribution with parameters $n = 3, p_X = 1/6$, and $p_Y = 1/2$. Find

(a) $E(X)$.

(b) $E(Y)$.

(c) Var(X).

(d) Var(Y).

(e) Cov(X, Y).

(f) ρ.

Note that $\rho = -\sqrt{p_X p_Y/[(1-p_X)(1-p_Y)]}$ in this case. (Indeed, the formula holds in general for the trinomial distribution; see Example 4.3-3.)

4.2-5. Let X and Y be random variables with respective means μ_X and μ_Y, respective variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. Fit the line $y = a + bx$ by the method of least squares to the probability distribution by minimizing the expectation

$$K(a,b) = E[(Y - a - bX)^2]$$

with respect to a and b. Hint: Consider $\partial K/\partial a = 0$ and $\partial K/\partial b = 0$, and solve simultaneously.

4.2-6. The joint pmf of X and Y is $f(x,y) = 1/6$, $0 \le x + y \le 2$, where x and y are nonnegative integers.

(a) Sketch the support of X and Y.

(b) Record the marginal pmfs $f_X(x)$ and $f_Y(y)$ in the "margins."

(c) Compute Cov(X, Y).

(d) Determine ρ, the correlation coefficient.

(e) Find the best-fitting line and draw it on your figure.

4.2-7. Let the joint pmf of X and Y be

$$f(x,y) = 1/4, \quad (x,y) \in S = \{(0,0),(1,1),(1,-1),(2,0)\}.$$

(a) Are X and Y independent?

(b) Calculate Cov(X, Y) and ρ.

This exercise also illustrates the fact that dependent random variables can have a correlation coefficient of zero.

4.2-8. A certain raw material is classified as to moisture content X (in percent) and impurity Y (in percent). Let X and Y have the joint pmf given by

	x			
y	1	2	3	4
2	0.10	0.20	0.30	0.05
1	0.05	0.05	0.15	0.10

(a) Find the marginal pmfs, the means, and the variances.

(b) Find the covariance and the correlation coefficient of X and Y.

(c) If additional heating is needed with high moisture content and additional filtering with high impurity such that the additional cost is given by the function $C = 2X + 10Y^2$ in dollars, find $E(C)$.

4.2-9. A car dealer sells X cars each day and always tries to sell an extended warranty on each of these cars. (In our opinion, most of these warranties are not good deals.) Let Y be the number of extended warranties sold; then $Y \le X$. The joint pmf of X and Y is given by

$$f(x,y) = c(x+1)(4-x)(y+1)(3-y),$$
$$x = 0,1,2,3, \quad y = 0,1,2, \quad \text{with } y \le x.$$

(a) Find the value of c.

(b) Sketch the support of X and Y.

(c) Record the marginal pmfs $f_X(x)$ and $f_Y(y)$ in the "margins."

(d) Are X and Y independent?

(e) Compute μ_X and σ_X^2.

(f) Compute μ_Y and σ_Y^2.

(g) Compute $\text{Cov}(X, Y)$.

(h) Determine ρ, the correlation coefficient.

(i) Find the best-fitting line and draw it on your figure.

4.2-10. If the correlation coefficient ρ exists, show that ρ satisfies the inequality $-1 \leq \rho \leq 1$. Hint: Consider the discriminant of the nonnegative quadratic function that is given by $h(v) = E\{[(X - \mu_X) + v(Y - \mu_Y)]^2\}$.

4.3 CONDITIONAL DISTRIBUTIONS

Exercises

4.3-1. Let X and Y have the joint pmf

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1,2, \qquad y = 1,2,3,4.$$

(a) Display the joint pmf and the marginal pmfs on a graph like Figure 4.3-1(a).

(b) Find $g(x \mid y)$ and draw a figure like Figure 4.3-1(b), depicting the conditional pmfs for $y = 1,2,3,$ and 4.

(c) Find $h(y \mid x)$ and draw a figure like Figure 4.3-1(c), depicting the conditional pmfs for $x = 1$ and 2.

(d) Find $P(1 \le Y \le 3 \mid X = 1)$, $P(Y \le 2 \mid X = 2)$, and $P(X = 2 \mid Y = 3)$.

(e) Find $E(Y \mid X = 1)$ and $\text{Var}(Y \mid X = 1)$.

4.3-2. Let the joint pmf $f(x,y)$ of X and Y be given by the following:

(x,y)	$f(x,y)$
(1, 1)	3/8
(2, 1)	1/8
(1, 2)	1/8
(2, 2)	3/8

Find the two conditional probability mass functions and the corresponding means and variances.

4.3-3. Let W equal the weight of laundry soap in a 1-kilogram box that is distributed in Southeast Asia. Suppose that $P(W < 1) = 0.02$ and $P(W > 1.072) = 0.08$. Call a box of soap light, good, or heavy depending on whether $\{W < 1\}$, $\{1 \le W \le 1.072\}$, or $\{W > 1.072\}$, respectively. In $n = 50$ independent observations of these boxes, let X equal the number of light boxes and Y the number of good boxes.

(a) What is the joint pmf of X and Y?

(b) Give the name of the distribution of Y along with the values of the parameters of this distribution.

(c) Given that $X = 3$, how is Y distributed conditionally?

(d) Determine $E(Y \mid X = 3)$.

(e) Find ρ, the correlation coefficient of X and Y.

4.3-4. The alleles for eye color in a certain male fruit fly are (R, W). The alleles for eye color in the mating female fruit fly are (R, W). Their offspring receive one allele for eye color from each parent. If an offspring ends up with either (W, W), (R, W), or (W, R), its eyes will look white. Let X equal the number of offspring having white eyes. Let Y equal the number of white-eyed offspring having (R, W) or (W, R) alleles.

(a) If the total number of offspring is $n = 400$, how is X distributed?

(b) Give the values of $E(X)$ and $\text{Var}(X)$.

(c) Given that $X = 300$, how is Y distributed?

(d) Give the value of $E(Y \mid X = 300)$ and the value of $\text{Var}(Y \mid X = 300)$.

4.3-5. Let X and Y have a trinomial distribution with $n = 2$, $p_X = 1/4$, and $p_Y = 1/2$.

(a) Give $E(Y \mid x)$.

(b) Compare your answer in part (a) with the equation of the line of best fit in Example 4.2-2. Are they the same? Why or why not?

4.3-6. An insurance company sells both homeowners' insurance and automobile deductible insurance. Let X be the deductible on the homeowners' insurance and Y the deductible on automobile insurance. Among those who take both types of insurance with this company, we find the following probabilities:

	x		
y	100	500	1000
1000	0.05	0.10	0.15
500	0.10	0.20	0.05
100	0.20	0.10	0.05

(a) Compute the following probabilities: $P(X = 500)$, $P(Y = 500)$, $P(Y = 500 \mid X = 500)$, $P(Y = 100 \mid X = 500)$.

(b) Compute the means μ_X, μ_Y, and the variances σ_X^2, σ_Y^2.

(c) Compute the conditional means $E(X \mid Y = 100)$, $E(Y \mid X = 500)$.

(d) Compute $\mathrm{Cov}(X, Y)$.

(e) Find the correlation coefficient, ρ.

4.3-7. Using the joint pmf from Exercise 4.2-3, find the value of $E(Y \mid x)$ for $x = 1, 2, 3, 4$. Do the points $[x, E(Y \mid x)]$ lie on the best-fitting line?

4.3-8. A fair six-sided die is rolled 30 independent times. Let X be the number of ones and Y the number of twos.

(a) What is the joint pmf of X and Y?

(b) Find the conditional pmf of X, given $Y = y$.

(c) Compute $E(X^2 - 4XY + 3Y^2)$.

4.3-9. Let X and Y have a uniform distribution on the set of points with integer coordinates in $S = \{(x, y) : 0 \le x \le 7, x \le y \le x + 2\}$. That is, $f(x, y) = 1/24$, $(x, y) \in S$, and both x and y are integers. Find

(a) $f_X(x)$.

(b) $h(y \mid x)$.

(c) $E(Y \mid x)$.

(d) $\sigma_{Y|x}^2$.

(e) $f_Y(y)$.

4.3-10. Let $f_X(x) = 1/10$, $x = 0, 1, 2, \ldots, 9$, and $h(y \mid x) = 1/(10 - x)$, $y = x, x + 1, \ldots, 9$. Find

(a) $f(x, y)$.

(b) $f_Y(y)$.

(c) $E(Y \mid x)$.

4.3-11. Choose a random integer X from the interval $[0, 4]$. Then choose a random integer Y from the interval $[0, x]$, where x is the observed value of X. Make assumptions about the marginal pmf $f_X(x)$ and the conditional pmf $h(y \mid x)$ and compute $P(X + Y > 4)$.

4.4 BIVARIATE DISTRIBUTIONS OF THE CONTINUOUS TYPE

Exercises

4.4-1. Let $f(x, y) = (3/16)xy^2$, $0 \le x \le 2$, $0 \le y \le 2$, be the joint pdf of X and Y.

(a) Find $f_X(x)$ and $f_Y(y)$, the marginal probability density functions.

(b) Are the two random variables independent? Why or why not?

(c) Compute the means and variances of X and Y.

(d) Find $P(X \le Y)$.

4.4-2. Let X and Y have the joint pdf $f(x, y) = x + y$, $0 \le x \le 1$, $0 \le y \le 1$.

(a) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$ and show that $f(x, y) \not\equiv f_X(x)f_Y(y)$. Thus, X and Y are dependent.

(b) Compute **(i)** μ_X, **(ii)** μ_Y, **(iii)** σ_X^2, and **(iv)** σ_Y^2.

4.4-3. Let $f(x, y) = 2e^{-x-y}$, $0 \le x \le y < \infty$, be the joint pdf of X and Y. Find $f_X(x)$ and $f_Y(y)$, the marginal pdfs of X and Y, respectively. Are X and Y independent?

4.4-4. Let $f(x, y) = 3/2$, $x^2 \le y \le 1$, $0 \le x \le 1$, be the joint pdf of X and Y.

(a) Find $P(0 \le X \le 1/2)$.

(b) Find $P(1/2 \le Y \le 1)$.

(c) Find $P(X \ge 1/2, Y \ge 1/2)$.

(d) Are X and Y independent? Why or why not?

4.4-5. For each of the following functions, determine the value of c for which the function is a joint pdf of two continuous random variables X and Y.

(a) $f(x, y) = cxy, \quad 0 \le x \le 1, \quad x^2 \le y \le x.$

(b) $f(x, y) = c(1 + x^2 y), \quad 0 \le x \le y \le 1.$

(c) $f(x, y) = cye^x, \quad 0 \le x \le y^2, \quad 0 \le y \le 1.$

(d) $f(x, y) = c\sin(x + y), \quad 0 \le x \le \pi/2, \quad 0 \le y \le \pi/2.$

4.4-6. Using Example 4.4-2,

(a) Determine the variances of X and Y.

(b) Find $P(-X \le Y)$.

4.4-7. Let $f(x, y) = 4/3$, $0 < x < 1$, $x^3 < y < 1$, zero elsewhere.

(a) Sketch the region where $f(x, y) > 0$.

(b) Find $P(X > Y)$.

4.4-8. Using the background of Example 4.4-4, calculate the means and variances of X and Y.

4.4-9. Two construction companies make bids of X and Y (in \$100,000's) on a remodeling project. The joint pdf of X and Y is uniform on the space $2 < x < 2.5, 2 < y < 2.3$. If X and Y are within 0.1 of each other, the companies will be asked to rebid; otherwise, the low bidder will be awarded the contract. What is the probability that they will be asked to rebid?

4.4-10. Let T_1 and T_2 be random times for a company to complete two steps in a certain process. Say T_1 and T_2 are measured in days and they have the joint pdf that is uniform over the space $1 < t_1 < 10$, $2 < t_2 < 6$, $t_1 + 2t_2 < 14$. What is $P(T_1 + T_2 > 10)$?

4.4-11. Let X and Y have the joint pdf $f(x, y) = cx(1 - y)$, $0 < y < 1$, and $0 < x < 1 - y$.

(a) Determine c.

(b) Compute $P(Y < X \mid X \le 1/4)$.

4.4-12. Show that in the bivariate situation, E is a linear or distributive operator. That is, for constants a_1 and a_2, show that

$$E[a_1 u_1(X, Y) + a_2 u_2(X, Y)] = a_1 E[u_1(X, Y)] + a_2 E[u_2(X, Y)].$$

4.4-13. Let X and Y be random variables of the continuous type having the joint pdf

$$f(x, y) = 2, \qquad 0 \le y \le x \le 1.$$

Draw a graph that illustrates the domain of this pdf.

(a) Find the marginal pdfs of X and Y.

(b) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, $\text{Cov}(X, Y)$, and ρ.

(c) Determine the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.4-14. Let X and Y be random variables of the continuous type having the joint pdf

$$f(x, y) = 8xy, \qquad 0 \le x \le y \le 1.$$

Draw a graph that illustrates the domain of this pdf.

(a) Find the marginal pdfs of X and Y.

(b) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, $\text{Cov}(X, Y)$, and ρ.

(c) Determine the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.4-15. An automobile repair shop makes an initial estimate X (in thousands of dollars) of the amount of money needed to fix a car after an accident. Say X has the pdf

$$f(x) = 2\,e^{-2(x-0.2)}, \qquad 0.2 < x < \infty.$$

Given that $X = x$, the final payment Y has a uniform distribution between $x - 0.1$ and $x + 0.1$. What is the expected value of Y?

4.4-16. For the random variables defined in Example 4.4-3, calculate the correlation coefficient directly from the definition

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

4.4-17. Let $f(x, y) = 1/40$, $0 \le x \le 10$, $10 - x \le y \le 14 - x$, be the joint pdf of X and Y.

(a) Sketch the region for which $f(x, y) > 0$.

(b) Find $f_X(x)$, the marginal pdf of X.

(c) Determine $h(y \mid x)$, the conditional pdf of Y, given that $X = x$.

(d) Calculate $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

4.4-18. Let $f(x, y) = 1/8$, $0 \le y \le 4$, $y \le x \le y + 2$, be the joint pdf of X and Y.

(a) Sketch the region for which $f(x, y) > 0$.

(b) Find $f_X(x)$, the marginal pdf of X.

(c) Find $f_Y(y)$, the marginal pdf of Y.

(d) Determine $h(y \mid x)$, the conditional pdf of Y, given that $X = x$.

(e) Determine $g(x \mid y)$, the conditional pdf of X, given that $Y = y$.

(f) Compute $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

(g) Compute $E(X \mid y)$, the conditional mean of X, given that $Y = y$.

(h) Graph $y = E(Y \mid x)$ on your sketch in part (a). Is $y = E(Y \mid x)$ linear?

(i) Graph $x = E(X \mid y)$ on your sketch in part (a). Is $x = E(X \mid y)$ linear?

4.4-19. Let X have a uniform distribution $U(0, 2)$, and let the conditional distribution of Y, given that $X = x$, be $U(0, x^2)$.

(a) Determine $f(x, y)$, the joint pdf of X and Y.

(b) Calculate $f_Y(y)$, the marginal pdf of Y.

(c) Compute $E(X \mid y)$, the conditional mean of X, given that $Y = y$.

(d) Find $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

4.4-20. Let X have a uniform distribution on the interval $(0, 1)$. Given that $X = x$, let Y have a uniform distribution on the interval $(0, x + 1)$.

(a) Find the joint pdf of X and Y. Sketch the region where $f(x, y) > 0$.

(b) Find $E(Y \mid x)$, the conditional mean of Y, given that $X = x$. Draw this line on the region sketched in part (a).

(c) Find $f_Y(y)$, the marginal pdf of Y. Be sure to include the domain.

4.5 THE BIVARIATE NORMAL DISTRIBUTION

Exercises

4.5-1. Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3$, $\mu_Y = 10$, $\sigma_X^2 = 25$, $\sigma_Y^2 = 9$, and $\rho = 3/5$. Compute

(a) $P(-5 < X < 5)$.

(b) $P(-5 < X < 5 \mid Y = 13)$.

(c) $P(7 < Y < 16)$.

(d) $P(7 < Y < 16 \mid X = 2)$.

4.5-2. Show that the expression in the exponent of Equation 4.5-2 is equal to the function $q(x, y)$ given in the text.

4.5-3. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 2.8$, $\mu_Y = 110$, $\sigma_X^2 = 0.16$, $\sigma_Y^2 = 100$, and $\rho = 0.6$. Compute

(a) $P(106 < Y < 124)$.

(b) $P(106 < Y < 124 \mid X = 3.2)$.

4.5-4. Let X and Y have a bivariate normal distribution with $\mu_X = 70$, $\sigma_X^2 = 100$, $\mu_Y = 80$, $\sigma_Y^2 = 169$, and $\rho = 5/13$. Find

(a) $E(Y \mid X = 72)$.

(b) $\mathrm{Var}(Y \mid X = 72)$.

(c) $P(Y \le 84, \mid X = 72)$.

4.5-5. Let X denote the height in centimeters and Y the weight in kilograms of male college students. Assume that X and Y have a bivariate normal distribution with parameters $\mu_X = 185$, $\sigma_X^2 = 100$, $\mu_Y = 84$, $\sigma_Y^2 = 64$, and $\rho = 3/5$.

(a) Determine the conditional distribution of Y, given that $X = 190$.

(b) Find $P(86.4 < Y < 95.36 \mid X = 190)$.

4.5-6. For a freshman taking introductory statistics and majoring in psychology, let X equal the student's ACT mathematics score and Y the student's ACT verbal score. Assume that X and Y have a bivariate normal distribution with $\mu_X = 22.7$, $\sigma_X^2 = 17.64$, $\mu_Y = 22.7$, $\sigma_Y^2 = 12.25$, and $\rho = 0.78$.

(a) Find $P(18.5 < Y < 25.5)$.

(b) Find $E(Y \mid x)$.

(c) Find $\mathrm{Var}(Y \mid x)$.

(d) Find $P(18.5 < Y < 25.5 \mid X = 23)$.

(e) Find $P(18.5 < Y < 25.5 \mid X = 25)$.

(f) For $x = 21, 23$, and 25, draw a graph of $z = h(y \mid x)$ similar to Figure 4.5-1.

4.5-7. For a pair of gallinules, let X equal the weight in grams of the male and Y the weight in grams of the female. Assume that X and Y have a bivariate normal distribution with $\mu_X = 415$, $\sigma_X^2 = 611$, $\mu_Y = 347$, $\sigma_Y^2 = 689$, and $\rho = -0.25$. Find

(a) $P(309.2 < Y < 380.6)$.

(b) $E(Y \mid x)$.

(c) $\mathrm{Var}(Y \mid x)$.

(d) $P(309.2 < Y < 380.6 \mid X = 385.1)$.

4.5-8. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 10$, $\sigma_X^2 = 9$, $\mu_Y = 15$, $\sigma_Y^2 = 16$, and $\rho = 0$. Find

(a) $P(13.6 < Y < 17.2)$.

(b) $E(Y \mid x)$.

(c) $\mathrm{Var}(Y \mid x)$.

(d) $P(13.6 < Y < 17.2 \mid X = 9.1)$.

4.5-9. Let X and Y have a bivariate normal distribution. Find two different lines, $a(x)$ and $b(x)$, parallel to and equidistant from $E(Y \mid x)$, such that

$$P[a(x) < Y < b(x) \mid X = x] = 0.9544$$

for all real x. Plot $a(x)$, $b(x)$, and $E(Y \mid x)$ when $\mu_X = 2$, $\mu_Y = -1$, $\sigma_X = 3$, $\sigma_Y = 5$, and $\rho = 3/5$.

4.5-10. In a college health fitness program, let X denote the weight in kilograms of a male freshman at the beginning of the program and Y denote his weight change during a semester. Assume that X and Y have a bivariate normal distribution with $\mu_X = 72.30$, $\sigma_X^2 = 110.25$, $\mu_Y = 2.80$, $\sigma_Y^2 = 2.89$, and $\rho = -0.57$. (The lighter students tend to gain weight, while the heavier students tend to lose weight.) Find

(a) $P(2.80 \le Y \le 5.35)$.

(b) $P(2.76 \le y \le 5.34 \mid X = 82.3)$.

4.5-11. For a female freshman in a health fitness program, let X equal her percentage of body fat at the beginning of the program and Y equal the change in her percentage of body fat measured at the end of the program. Assume that X and Y have a bivariate normal distribution with $\mu_X = 24.5$, $\sigma_X^2 = 4.8^2 = 23.04$, $\mu_Y = -0.2$, $\sigma_Y^2 = 3.0^2 = 9.0$, and $\rho = -0.32$. Find

(a) $P(1.3 \le Y \le 5.8)$.

(b) $\mu_{Y|x}$, the conditional mean of Y, given that $X = x$.

(c) $\sigma_{Y|x}^2$, the conditional variance of Y, given that $X = x$.

(d) $P(1.3 \le Y \le 5.8 \mid X = 18)$.

4.5-12. Let

$$f(x, y) = \left(\frac{1}{2\pi}\right) e^{-(x^2+y^2)/2} \left[1 + xye^{-(x^2+y^2-2)/2}\right],$$
$$-\infty < x < \infty, -\infty < y < \infty.$$

Show that $f(x, y)$ is a joint pdf and the two marginal pdfs are each normal. Note that X and Y can each be normal, but their joint pdf is not bivariate normal.

4.5-13. An obstetrician does ultrasound examinations on her patients between their 16th and 25th weeks of pregnancy to check the growth of each fetus. Let X equal the widest diameter of the fetal head, and let Y equal the length of the femur, both measurements in mm. Assume that X and Y have a bivariate normal distribution with $\mu_X = 60.6$, $\sigma_X = 11.2$, $\mu_Y = 46.8$, $\sigma_Y = 8.4$, and $\rho = 0.94$.

(a) Find $P(40.5 < Y < 48.9)$.

(b) Find $P(40.5 < Y < 48.9 \mid X = 68.6)$.

Chapter 4

4.1-1 **(a)** 1/33; **(b)** 1/24; **(c)** 1/18; **(d)** 6.

4.1-3 **(a)** $f_X(x) = (2x+5)/16,\ x = 1,2;$
(b) $f_Y(y) = (2y+3)/32,\ y = 1,2,3,4;$
(c) 3/32; **(d)** 9/32; **(e)** 3/16; **(f)** 1/4; **(g)** Dependent;
(h) $\mu_X = 25/16$; $\mu_Y = 45/16$; $\sigma_X^2 = 63/256$; $\sigma_Y^2 = 295/256$.

4.1-5 **(b)** $f(x,y) = 1/16,\ x = 1,2,3,4;\ y = x+1, x+2, x+3, x+4;$
(c) $f_X(x) = 1/4,\ x = 1,2,3,4;$
(d) $f_Y(y) = (4 - |y-5|)/16,\ y = 2,3,4,5,6,7,8;$
(e) Dependent because the space is not rectangular.

4.1-7 **(b)** $b(6,1/2)$, $b(6,1/2)$.

4.1-9 **(a)** $f(x,y) = \dfrac{15!}{x!\,y!\,(15-x-y)!}\left(\dfrac{6}{10}\right)^x\left(\dfrac{3}{10}\right)^y\left(\dfrac{1}{10}\right)^{15-x-y},$
$0 \le x+y \le 15;$
(b) No, because the space is not rectangular;
(c) 0.0735;
(d) X is $b(15, 0.6)$;
(e) 0.9095.

4.2-1 $\mu_X = 25/16$; $\mu_Y = 45/16$; $\sigma_X^2 = 63/256$; $\sigma_Y^2 = 295/256$;
$\text{Cov}(X,Y) = -5/256$; $\rho = -\sqrt{2{,}065}/1{,}239 = -0.0367$.

4.2-3 **(a)** $\mu_X = 5/2$; $\mu_Y = 5$; $\sigma_X^2 = 5/4$; $\sigma_Y^2 = 5/2$; $\text{Cov}(X,Y) = 5/4$; $\rho = \sqrt{2}/2$;
(b) $y = x + 5/2$.

4.2-5 $a = \mu_Y - \mu_X b,\ b = \text{Cov}(X,Y)/\sigma_X^2$.

4.2-7 **(a)** No; **(b)** $\text{Cov}(X,Y) = 0,\ \rho = 0$.

4.2-9 **(a)** $c = 1/154$;
(c) $f_X(0) = 6/77$, $f_X(1) = 21/77$, $f_X(2) = 30/77$, $f_X(3) = 20/77$;
$f_Y(0) = 30/77$, $f_Y(1) = 32/77$, $f_Y(2) = 15/77$;
(d) No;
(e) $\mu_X = 141/77, \sigma_X^2 = 4836/5929$;
(f) $\mu_Y = 62/77, \sigma_Y^2 = 3240/5929$;
(g) $\text{Cov}(X,Y) = 1422/5929$;
(h) $\rho = 79\sqrt{12090}/24180$;
(i) $y = 215/806 + (237/806)x$.

4.3-1 **(d)** 9/14, 7/18, 5/9; **(e)** 20/7, 55/49.

4.3-3 **(a)** $f(x, y) = \dfrac{50!}{x!\,y!\,(50-x-y)!}(0.02)^x(0.90)^y(0.08)^{50-x-y}$, $0 \le x + y \le 50$;
(b) Y is $b(50, 0.90)$;
(c) $b(47, 0.90/0.98)$;
(d) 2115/49; (e) $\rho = -3/7$.

4.3-5 **(a)** $E(Y \mid x) = 2(2/3) - (2/3)x$, $x = 1, 2$; **(b)** Yes.

4.3-7 $E(Y \mid x) = x + 5/2$, $x = 1, 2, 3, 4$; yes.

4.3-9 **(a)** $f_X(x) = 1/8$, $x = 0, 1, \ldots, 7$;
(b) $h(y \mid x) = 1/3$, $y = x, x+1, x+2$, for $x = 0, 1, \ldots, 7$;
(c) $E(Y \mid x) = x + 1$, $x = 0, 1, \ldots, 7$;
(d) $\sigma_Y^2 = 2/3$;
(e)

$$f_Y(y) = \begin{cases} 1/24, & y = 0, 9, \\ 2/24, & y = 1, 8, \\ 3/24, & y = 2, 3, 4, 5, 6, 7. \end{cases}$$

4.3-11 $f_X(x) = 1/5$ and $h(y \mid x) = 1/[5(x+1)]$, for $x = 0, 1, 2, 3, 4$, and $y = 0 \ldots x$; $P(X + Y > 4) = 13/50$.

4.4-1 **(a)** $f_X(x) = x/2$, $0 \le x \le 2$; $f_Y(y) = 3y^2/8$, $0 \le y \le 2$;
(b) Yes, because $f_X(x)f_Y(y) = f(x, y)$;
(c) $\mu_X = 4/3$; $\mu_Y = 3/2$; $\sigma_X^2 = 2/9$; $\sigma_Y^2 = 3/20$;
(d) 3/5.

4.4-3 $f_X(x) = 2e^{-2x}$, $0 < x < \infty$; $f_Y(y) = 2e^{-y}(1 - e^{-y})$, $0 < y < \infty$; no.

4.4-5 **(a)** $c = 24$; **(b)** $c = 30/17$; **(c)** $c = 2/(e - 2)$;
(d) $c = 1/2$.

4.4-7 **(b)** 1/3.

4.4-9 11/30.

4.4-11 **(a)** $c = 8$; **(b)** 29/93.

4.4-13 **(a)** $f_X(x) = 4x(1 - x^2)$, $0 \le x \le 1$; $f_Y(y) = 4y^3$, $0 \le y \le 1$;
(b) $\mu_X = 8/15$; $\mu_Y = 4/5$; $\sigma_X^2 = 11/225$; $\sigma_Y^2 = 2/75$; $\text{Cov}(X, Y) = 4/225$; $\rho = 2\sqrt{66}/33$;
(c) $y = 20/33 + (4/11)x$.

4.4-15 $E(Y \mid x) = x$ and $E(X) = 0.700$; thus, \$700.

4.4-17 **(b)** $f_X(x) = 1/10$, $0 \le x \le 10$;
(c) $h(y \mid x) = 1/4$, $10 - x \le y \le 14 - x$ for $0 \le x \le 10$;
(d) $E(Y \mid x) = 12 - x$.

4.4-19 **(a)** $f(x, y) = 1/(2x^2)$, $0 < x < 2$, $0 < y < x^2$;
(b) $f_Y(y) = (2 - \sqrt{y})/(4\sqrt{y})$, $0 < y < 4$;
(c) $E(X \mid y) = [2\sqrt{y}\,\ln(2/\sqrt{y})]/[2 - \sqrt{y}]$;
(d) $E(Y \mid x) = x^2/2$.

4.5-1 **(a)** 0.6006; **(b)** 0.7888; **(c)** 0.8185; **(d)** 0.9371.

(d) $E(Y \mid x) = x^2/2$.

4.5-1 **(a)** 0.6006; **(b)** 0.7888; **(c)** 0.8185; **(d)** 0.9371.

4.5-3 **(a)** 0.5746; **(b)** 0.7357.

4.5-5 **(a)** $N(86.4, 40.96)$; **(b)** 0.4192.

4.5-7 **(a)** 0.8248;
(b) $E(Y \mid x) = 457.1735 - 0.2655x$;
(c) $\text{Var}(Y \mid x) = 645.9375$;
(d) 0.8079.

4.5-9 $a(x) = x - 11,\ b(x) = x + 5$.

4.5-11 **(a)** 0.2857; **(b)** $\mu_{Y|x} = -0.2x + 4.7$;
(c) $\sigma^2_{Y|x} = 8.0784$; **(d)** 0.4230.

4.5-13 **(a)** 0.3721; **(b)** 0.1084.

Chapter 5

Distributions of Functions of Random Variables

5.1 Functions of One Random Variable

Exercises

5.1-1. Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$. Find the pdf of $Y = X^2$.

5.1-2. Let X have the pdf $f(x) = xe^{-x^2/2}$, $0 < x < \infty$. Find the pdf of $Y = X^2$.

5.1-3. Let X have a gamma distribution with $\alpha = 3$ and $\theta = 2$. Determine the pdf of $Y = \sqrt{X}$.

5.1-4. The pdf of X is $f(x) = 2x$, $0 < x < 1$.

(a) Find the cdf of X.

(b) Describe how an observation of X can be simulated.

(c) Simulate 10 observations of X.

5.1-5. The pdf of X is $f(x) = \theta x^{\theta-1}$, $0 < x < 1, 0 < \theta < \infty$. Let $Y = -2\theta \ln X$. How is Y distributed?

5.1-6. Let X have a **logistic distribution** with pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty.$$

Show that

$$Y = \frac{1}{1+e^{-X}}$$

has a $U(0, 1)$ distribution.

5.1-7. A sum of \$50,000 is invested at a rate R, selected from a uniform distribution on the interval (0.03, 0.07). Once R is selected, the sum is compounded instantaneously for a year, so that $X = 50000\,e^R$ dollars is the amount at the end of that year.

(a) Find the cdf and pdf of X.

(b) Verify that $X = 50000\,e^R$ is defined correctly if the compounding is done instantaneously. HINT: Divide the year into n equal parts, calculate the value of the amount at the end of each part, and then take the limit as $n \to \infty$.

5.1-8. The lifetime (in years) of a manufactured product is $Y = 5X^{0.7}$, where X has an exponential distribution with mean 1. Find the cdf and pdf of Y.

5.1-9. Statisticians frequently use the **extreme value distribution** given by the cdf

$$F(x) = 1 - \exp\left[-e^{(x-\theta_1)/\theta_2}\right], \qquad -\infty < x < \infty.$$

A simple case is when $\theta_1 = 0$ and $\theta_2 = 1$, giving

$$F(x) = 1 - \exp\left[-e^{x}\right], \qquad -\infty < x < \infty.$$

Let $Y = e^X$ or $X = \ln Y$; then the support of Y is $0 < y < \infty$.

(a) Show that the distribution of Y is exponential when $\theta_1 = 0$ and $\theta_2 = 1$.

(b) Find the cdf and the pdf of Y when $\theta_1 \neq 0$ and $\theta_2 > 0$.

(c) Let $\theta_1 = \ln \beta$ and $\theta_2 = 1/\alpha$ in the cdf and pdf of Y. What is this distribution?

(d) As suggested by its name, the extreme value distribution can be used to model the longest home run, the deepest mine, the greatest flood, and so on. Suppose the length X (in feet) of the maximum of someone's home runs was modeled by an extreme value distribution with $\theta_1 = 550$ and $\theta_2 = 25$. What is the probability that X exceeds 500 feet?

5.1-10. Let X have the uniform distribution $U(-1, 3)$. Find the pdf of $Y = X^2$.

5.1-11. Let X have a Cauchy distribution. Find

(a) $P(X > 1)$.

(b) $P(X > 5)$.

(c) $P(X > 10)$.

5.1-12. Let $f(x) = 1/[\pi(1 + x^2)]$, $-\infty < x < \infty$, be the pdf of the Cauchy random variable X. Show that $E(X)$ does not exist.

5.1-13. If the distribution of X is $N(\mu, \sigma^2)$, then $M(t) = E(e^{tX}) = \exp(\mu t + \sigma^2 t^2/2)$. We then say that $Y = e^X$ has a **lognormal distribution** because $X = \ln Y$.

(a) Show that the pdf of Y is

$$g(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp[-(\ln y - \mu)^2/2\sigma^2], \quad 0 < y < \infty.$$

(b) Using $M(t)$, find **(i)** $E(Y) = E(e^X) = M(1)$, **(ii)** $E(Y^2) = E(e^{2X}) = M(2)$, and **(iii)** $\text{Var}(Y)$.

5.1-14. Let X be $N(0, 1)$. Find the pdf of $Y = |X|$, a distribution that is often called the **half-normal**. HINT: Here $y \in S_y = \{y : 0 < y < \infty\}$. Consider the two transformations $x_1 = -y, -\infty < x_1 < 0$, and $x_2 = y, 0 < y < \infty$.

5.1-15. Let $Y = X^2$.

(a) Find the pdf of Y when the distribution of X is $N(0, 1)$.

(b) Find the pdf of Y when the pdf of X is $f(x) = (3/2)x^2$, $-1 < x < 1$.

Ex. 5.1 170

Sol.

A.1 Random var. X has p.d.f $f(x) = 4x^3$, $0<x<1$

Domain of X is $0<x<1$, $\Rightarrow$ Domain of $Y = X^2$ is $0<y<1$.

For $0<x<1$, one-to-one transformation is,

$x = v(y) = \sqrt{y}$

So p.d.f of y is

$g(y) = f[v(y)]\,|v'(y)| = 4(\sqrt{y})^3\left(\frac{1}{2\sqrt{y}}\right)$ as $v'(y) = \frac{\partial}{\partial y}\sqrt{y}$.

$\therefore\ g(y) = 2y$, $0<y<1$.

A.2 let pdf of X be, $f(x) = xe^{-x^2/2}$, $0<x<\infty$.

Domain of X is $0<x<\infty$, $\Rightarrow$ domain of $Y = X^2$ is $0<y<\infty$.

For each y in domain, distribution f^n of Y is.

$G(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$

$\therefore\ G(y) = \int_0^{\sqrt{y}} f(x)\,dx = \int_0^{\sqrt{y}} x e^{-x^2/2}\,dx = 1 - e^{-y/2}$.

Hence p.d.f of Y is,

$g(y) = G'(y) = \frac{1}{2}e^{-y/2}$, $0<y<\infty$,

Hence Y has exponential distribution with $\theta = 2$.

A.3 p.d.f of Gamma distribution with $\alpha = 3$ & $\theta = 2$ is

$$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}.$$

$$= \frac{1}{16} x^2 e^{-x/2}, \quad 0 < x < \infty$$

So domain of $Y = \sqrt{X}$ is $0 < y < \infty$ ($\because 0 < x < \infty$)

One-to-one transformation is,

$$x = y^2 \Rightarrow \frac{dx}{dy} = 2y.$$

p.d.f of Y is, $g(y) = f(y^2) \left| \frac{dx}{dy} \right|$

$$= \frac{1}{16} (y^2)^2 e^{-(y^2)/2} (2y)$$

$$= \frac{1}{8} y^5 e^{-y^2/2}, \quad 0 < y < \infty$$

A.4 p.d.f of X, $f(x) = \theta x^{\theta - 1}$, $0 < x < 1$, $0 < \theta < \infty$.

Domain of X is $0 < x < 1$ & $0 < \theta < \infty$.

Random var. $Y = -2\theta \ln X$ also has domain $0 < y < \infty$.

For y in domain of Y, one-to-one transform is,

$$x = v(y) = e^{-\frac{y}{2\theta}}.$$

p.d.f of Y is $g(y) = f(v(y)) |v'(y)|$

$$= \theta \left(e^{-y/2\theta} \right)^{\theta - 1} \left| -\frac{1}{2\theta} e^{-y/2\theta} \right|.$$

$$= \frac{1}{2} e^{-y/2}$$

So Y has exponential distribution with $\theta = 2$.

A.6 X has logistic distribution with p.d.f,

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

given $Y = \frac{1}{e^{-X}+1}$.

As Y is increasing f^n of X & domain of X is $-\infty < x < \infty$, solving $y = \frac{1}{1+e^{-x}}$ for x, we get

$$x = v(y) = \ln \frac{y}{1-y}, \quad 0 < y < 1$$

So, p.d.f of Y is,

$$g(y) = f(v(y))\,|v'(y)| = f\left(\ln \frac{y}{1-y}\right)\left|\frac{\partial}{\partial y} \ln\left(\frac{y}{1-y}\right)\right|.$$

$$\therefore\ g(y) = \frac{e^{-\ln \frac{y}{1-y}}}{\left(1+e^{-\ln \frac{y}{1-y}}\right)^2} \cdot \left| \left(\frac{1-y}{y}\right) \cdot \frac{(1-y) - y(-1)}{(1-y)^2} \right|$$

$$= \frac{\left(\frac{1-y}{y}\right)}{\left(1+\left(\frac{1-y}{y}\right)\right)^2} \left|\frac{1}{y(1-y)}\right|$$

$$= 1, \quad 0 < y < 1.$$

Hence, p.d.f of Y is $g(y) = 1$ for $0<y<1$. which is also the p.d.f of $U(0,1)$.

Hence $Y = U(0,1)$.

1.7 (a) $R \Rightarrow U(0.03, 0.07)$.

prob. density f^n of uniform distribution is,

$$f(r) = \frac{1}{b-a} = \frac{1}{0.07-0.03} = 25.$$

Cumulative distribution f^n of random var. X is,

$$P(X \le x) = P(50000e^R \le x)$$

$$= P\left(R \le \cancel{\log} \ln \frac{x}{50000}\right)$$

$$= \int_{0.03}^{\ln \frac{x}{50000}} 25\,dx.$$

$$= 25\,[x]\Big|_{0.03}^{\ln\frac{x}{50000}}$$

$$= 25\left(\ln\left(\frac{x}{50000}\right) - 0.03\right), \quad 0.03 < \ln\frac{x}{50000} < 0.07.$$

Prob. density f^n of X is,

$$g(x) = G'(x) = \frac{d}{dx}P(X \le x) = \frac{25}{x}.$$

$$\therefore g(x) = f_X(x) = \frac{25}{x}, \quad 50000e^{0.03} \le x \le 50000e^{0.07}.$$

(b) For each n part of year, interest is $\frac{R}{n}$.

Capitalizing the earnings at end of each part of year, total amount is obtained as.

$$X = 50000\left(1+\frac{R}{n}\right)^n, \text{ take } \lim_{n\to\infty}.$$

$$\therefore X = \lim_{n\to\infty} 50000\left(1+\frac{R}{n}\right)^n.$$

$$= \lim_{n\to\infty} 50000\left(1 + n\frac{R}{n} + \frac{n(n-1)}{2!}\cdot\frac{R^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{R^3}{n^3} + \cdots\right)$$

$$= 50000\left(1 + R + \frac{R^2}{2!} + \frac{R^3}{3!} + \cdots\right)$$

$$= 50000\,e^R.$$

A.8 X has exponential ~~pdf~~ distribution, with mean 1,

p.d.f of X, $f(x) = e^{-x}$, $0 < x < \infty$

Random var. $Y = 5X^{0.7}$ has domain $0 < y < \infty$.

one to one transformation is obtained as

$$x = v(y) = (0.2y)^{10/7}.$$

p.d.f of Y is, $g(y) = f(v(y))|v'(y)|$

$$\therefore g(y) = e^{-(0.2y)^{10/7}} \left| \frac{10}{7}(0.2y)^{3/7}(0.2) \right|$$

$$= \frac{2}{7}\left(\frac{y}{5}\right)^{3/7} e^{-\left(\frac{y}{5}\right)^{10/7}}, \quad 0 < y < \infty$$

Distribution f^n of Y.

$$G(y) = \int_0^y g(y)\,dy = \int_0^y \frac{2}{7}\left(\frac{y}{5}\right)^{3/7} e^{-\left(\frac{y}{5}\right)^{10/7}} dy.$$

$$= \int_0^y e^{-\left(\frac{y}{5}\right)^{10/7}} d\left(\left(\frac{y}{5}\right)^{10/7}\right)$$

$$= 1 - e^{-\left(\frac{y}{5}\right)^{10/7}}, \quad 0 < y < \infty.$$

also $G(y) = 0$, $y \leq 0$.

A.10 $X = U(-1,3)$, p.d.f of X is

$$f(x) = \frac{1}{b-a} = \frac{1}{3-(-1)} = \frac{1}{4}, \quad -1<x<3$$

$Y = X^2$ has domain $0 \le y < 9$ as $-1 < x < 3$.

One to one transformation is,

$$x_1 = -\sqrt{y} \qquad -1 < x_1 < 0 \quad \Rightarrow \frac{dx_1}{dy} = -\frac{1}{2\sqrt{y}}$$

$$x_2 = \sqrt{y} \qquad 0 < x_2 < 3. \quad \Rightarrow \frac{dx_2}{dy} = \frac{1}{2\sqrt{y}}$$

p.d.f of Y is, for $0<y<1$.

$$g(y) = f(x_1(y))\left|\frac{dx_1}{dy}\right| + f(x_2(y))\left|\frac{dx_2}{dy}\right| = \frac{1}{4\sqrt{y}}$$

Similarly, for $1<y<9$ $\quad g(y) = f(x_2(y))\left|\frac{dx_2}{dy}\right| = \frac{1}{8\sqrt{y}}$.

p.d.f of Y.

A.11 (a) X has Cauchy distribution, distribution f^n of X is,

$$F(x) = \frac{1}{2} + \frac{\tan^{-1}x}{\pi}, \quad -\infty < x < \infty$$

$$P(X>1) = 1 - F(1) = \frac{1}{2} - \frac{\tan^{-1}(1)}{\pi} = \frac{1}{2} - \frac{\pi}{4}\cdot\frac{1}{\pi} = \frac{1}{4}.$$

(b) $P(X>5) = 1 - F(5) = \frac{1}{2} - \frac{\tan^{-1}5}{\pi} = 0.0628.$

(c) $P(X>10) = 1 - F(10) = \frac{1}{2} - \frac{\tan^{-1}10}{\pi} = 0.0317$

A.12 X has Cauchy distribution, p.d.f of X is,

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} \frac{x\,dx}{\pi(1+x^2)}.$$

$$\therefore E[X] = \frac{1}{2\pi} \ln(1+x^2)\Big|_{-\infty}^{\infty}$$

as $\lim_{x \to \pm\infty} \ln(1+x^2)$ do not exist, $E[X]$ do not exist.

A.13 (a) $X = N(\mu, \sigma^2)$, p.d.f of X is,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

$Y = e^x$ has domain $0 < y < \infty$.

$v(y) = \ln y$ & p.d.f of Y is,

$$g(y) = f[v(y)]|v'(y)| = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}, \quad 0 < y < \infty.$$

(b) $M(t) = e^{\left(\mu t + \frac{\sigma^2}{2}t^2\right)}$, m.g.f of X,

$$E(Y) = E(e^X) = M(1) = e^{\left(\mu + \frac{\sigma^2}{2}\right)}$$

$$E(Y^2) = E(e^{2X}) = M(2) = e^{(2\mu + 2\sigma^2)}$$

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = e^{2(\mu+\sigma^2)} - e^{2\left(\mu + \frac{\sigma^2}{2}\right)}.$$

A.14 $X = N(0,1)$. p.d.f of X is,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , \quad -\infty < x < \infty$$

Random var. $Y = |X|$ has domain $0 \leq y < \infty$.

Two-to-one transformation is,

$x_1 = -y$ for $-\infty < x_1 < 0 \Rightarrow \frac{dx_1}{dy} = -1$

$x_2 = y$ for $0 \leq x_2 < \infty \Rightarrow \frac{dx_2}{dy} = 1$

p.d.f of Y is, $g(y)$

$$g(y) = f(x_1(y)) \left|\frac{dx_1}{dy}\right| + f(x_2(y)) \left|\frac{dx_2}{dy}\right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(-y)^2}{2}} |-1| + \frac{1}{\sqrt{2\pi}} e^{-\frac{(y)^2}{2}} |1|$$

$$= \sqrt{\frac{2}{\pi}} e^{-y^2/2} \quad , 0 < y < \infty$$

A.15 (a) $X = N(0,1) \Rightarrow$ p.d.f of X is, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $0 < x < \infty$.

Random Var $Y = X^2$ has domain $0 \le y < \infty$.

Two to one transformation is,

$x_1 = -\sqrt{y}$ for $-\infty < x_1 < \infty \Rightarrow \frac{dx_1}{dy} = -\frac{1}{2\sqrt{y}}$.

$x_2 = \sqrt{y}$ for $0 \le x_2 \le \infty \Rightarrow \frac{dx_2}{dy} = \frac{1}{2\sqrt{y}}$

p.d.F of Y is, $g(y)$.

$$g(y) = f(x_1)\left|\frac{dx_1}{dy}\right| + f(x_2)\left|\frac{dx_2}{dy}\right|$$

$$\therefore g(y) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2}\left|-\frac{1}{2\sqrt{y}}\right| + \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2}\left|\frac{1}{2\sqrt{y}}\right| .$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad , 0 \le \cancel{\infty}\, y < \infty .$$

(b) p.d.f of X is,

$$f(x) = \frac{3}{2}x^2 \quad , \quad -1 < x < 1.$$

2-to-1 transformation is, for $Y = X^2$ in domain $0 \leq y < 1$.

$$x_1 = -\sqrt{y} \quad -1 < x_1 < 0 \quad \Rightarrow \left|\frac{dx_1}{dy}\right| = \frac{1}{2\sqrt{y}}$$

$$x_2 = \sqrt{y} \quad 0 \leq x_2 < 1 \quad \Rightarrow \frac{dx_2}{dy} = \frac{1}{2\sqrt{y}}.$$

p.d.f of Y is, $g(y)$

$$g(y) = f(x_1)\left|\frac{dx_1}{dy}\right| + f(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{3}{2}(-\sqrt{y})^2\left(\frac{1}{2\sqrt{y}}\right) + \frac{3}{2}(\sqrt{y})^2\left(\frac{1}{2\sqrt{y}}\right)$$

$$= \frac{3}{2}\sqrt{y} \quad , \quad 0 < y < 1.$$

5.2 TRANSFORMATIONS OF TWO RANDOM VARIABLES

Exercises

5.2-1. Let X_1, X_2 denote two independent random variables, each with a $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_2 + X_1$. Note that the support of Y_1, Y_2 is $0 < y_1 < y_2 < \infty$. Also, find the marginal pdf of each of Y_1 and Y_2. Are Y_1 and Y_2 independent?

5.2-2. Let X_1 and X_2 be independent chi-square random variables with r_1 and r_2 degrees of freedom, respectively. Let $Y_1 = (X_1/r_1)/(X_2/r_2)$ and $Y_2 = X_2$.

(a) Find the joint pdf of Y_1 and Y_2.

(b) Determine the marginal pdf of Y_1 and show that Y_1 has an F distribution. (This is another, but equivalent, way of finding the pdf of F.)

5.2-3. Find the mean and the variance of an F random variable with r_1 and r_2 degrees of freedom by first finding $E(U)$, $E(1/V)$, $E(U^2)$, and $E(1/V^2)$.

5.2-4. Let the distribution of W be $F(9, 24)$. Find the following:

(a) $F_{0.05}(9, 24)$.

(b) $F_{0.95}(9, 24)$.

(c) $P(0.277 \le W \le 2.70)$.

5.2-5. Let the distribution of W be $F(8, 4)$. Find the following:

(a) $F_{0.01}(8, 4)$.

(b) $F_{0.99}(8, 4)$.

(c) $P(0.198 \le W \le 8.98)$.

5.2-6. Let X_1 and X_2 have independent gamma distributions with parameters α, θ and β, θ, respectively. Let $W = X_1/(X_1 + X_2)$. Use a method similar to that given in the derivation of the F distribution (Example 5.2-4) to show that the pdf of W is

$$g(w) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, w^{\alpha-1}(1-w)^{\beta-1}, \qquad 0 < w < 1.$$

We say that W has a beta distribution with parameters α and β. (See Example 5.2-3.)

5.2-7. Let X_1 and X_2 be independent chi-square random variables with r_1 and r_2 degrees of freedom, respectively. Show that

(a) $U = X_1/(X_1 + X_2)$ has a beta distribution with $\alpha = r_1/2$ and $\beta = r_2/2$.

(b) $V = X_2/(X_1 + X_2)$ has a beta distribution with $\alpha = r_2/2$ and $\beta = r_1/2$.

5.2-8. Let X have a beta distribution with parameters α and β. (See Example 5.2-3.)

(a) Show that the mean and variance of X are, respectively,

$$\mu = \frac{\alpha}{\alpha+\beta} \qquad \text{and} \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.$$

(b) Show that when $\alpha > 1$ and $\beta > 1$, the mode is at $x = (\alpha - 1)/(\alpha + \beta - 2)$.

5.2-9. Determine the constant c such that $f(x) = cx^3(1-x)^6$, $0 < x < 1$, is a pdf.

5.2-10. When α and β are integers and $0 < p < 1$, we have

$$\int_0^p \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, y^{\alpha-1}(1-y)^{\beta-1}\,dy = \sum_{y=\alpha}^{n} \binom{n}{y} p^y (1-p)^{n-y},$$

where $n = \alpha + \beta - 1$. Verify this formula when $\alpha = 4$ and $\beta = 3$. HINT: Integrate the left member by parts several times.

5.2-11. Evaluate

$$\int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)}\, y^3(1-y)^2\,dy$$

(a) Using integration.

(b) Using the result of Exercise 5.2-10.

5.2-12. Let W_1, W_2 be independent, each with a Cauchy distribution. In this exercise we find the pdf of the sample mean, $(W_1 + W_2)/2$.

(a) Show that the pdf of $X_1 = (1/2)W_1$ is

$$f(x_1) = \frac{2}{\pi\,(1 + 4x_1^2)}, \qquad -\infty < x_1 < \infty.$$

(b) Let $Y_1 = X_1 + X_2 = \overline{W}$ and $Y_2 = X_1$, where $X_2 = (1/2)W_2$. Show that the joint pdf of Y_1 and Y_2 is

$$g(y_1, y_2) = f(y_1 - y_2)f(y_2), \quad -\infty < y_1 < \infty,$$

$$-\infty < y_2 < \infty.$$

(c) Show that the pdf of $Y_1 = \overline{W}$ is given by the **convolution formula**,

$$g_1(y_1) = \int_{-\infty}^{\infty} f(y_1 - y_2)f(y_2)\,dy_2.$$

(d) Show that

$$g_1(y_1) = \frac{1}{\pi\,(1 + y_1^2)}, \qquad -\infty < y_1 < \infty.$$

That is, the pdf of $\overline{W}$ is the same as that of an individual W.

5.2-13. Let X_1, X_2 be independent random variables representing lifetimes (in hours) of two key components of a device that fails when and only when both components fail. Say each X_i has an exponential distribution with mean 1000. Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$, so that the space of Y_1, Y_2 is $0 < y_1 < y_2 < \infty$.

(a) Find $G(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$.

(b) Compute the probability that the device fails after 1200 hours; that is, compute $P(Y_2 > 1200)$.

5.2-14. A company provides earthquake insurance. The premium X is modeled by the pdf

$$f(x) = \frac{x}{5^2}\,e^{-x/5}, \qquad 0 < x < \infty,$$

while the claims Y have the pdf

$$g(y) = \frac{1}{5}\,e^{-y/5}, \qquad 0 < y < \infty.$$

If X and Y are independent, find the pdf of $Z = X/Y$.

5.2-15. In Example 5.2-6, verify that the given transformation maps $\{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}$ onto $\{(z_1, z_2) : -\infty < z_1 < \infty, -\infty < z_2 < \infty\}$, except for a set of points that has probability 0. Hint: What is the image of vertical line segments? What is the image of horizontal line segments?

5.2-16. Let W have an F distribution with parameters r_1 and r_2. Show that $Z = 1/[1 + (r_1/r_2)W]$ has a beta distribution.

Ex. 5.2

A.1 X_1, X_2 are independent, χ^2

joint p.d.f of X_1 & X_2 is,

$$f(x_1, x_2) = \left(\frac{e^{-x_1/2}}{2}\right)\left(\frac{e^{-\frac{x_2}{2}}}{2}\right)$$

$$= \frac{1}{4} e^{-\frac{x_1+x_2}{2}}, \quad 0 < x_1, x_2 < \infty.$$

Consider $Y_1 = X_1$ & $Y_2 = X_2 + X_1$ or $X_2 = Y_2 - Y_1$.

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Joint p.d.f of y_1 & y_2 is,

$$g(y_1, y_2) = |J| f(x_1(y_1, y_2), x_2(y_1, y_2))$$

$$= |J| f(\cancel{x_1}(y_1), \cancel{x_2}(y_2 - y_1))$$

$$= \frac{1}{4} e^{-\frac{(y_1) + (y_2 - y_1)}{2}}$$

$$= \frac{1}{4} e^{-y_2/2} \qquad 0 < y_1 < y_2 < \infty. \left(\because \begin{matrix} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{matrix}\right)$$

marginal p.d.f of Y_1 is, $g_1(y_1) = \frac{1}{4}\int_{y_1}^{\infty} e^{-y_2/2}\, dy_2.$

$$\therefore g_1(y_1) = \frac{1}{2} e^{-y_1/2}, \quad 0 < y_1 < \infty$$

marginal pdf of y_2 is $g_2(y_2) = \frac{1}{4}\int_0^{y_2} e^{-y_2/2}\, dy_1.$

$$\therefore g_2(y_2) = \frac{y_2}{4} e^{-y_2/2} \qquad 0 < y_2 < \infty$$

as $g(y_1, y_2) \neq g_1(y_1) g_2(y_2)$ & domain $(0 < y_1 < y_2 < \infty)$ of $Y_1 Y_2$, is not bounded by horizontal & vertical line segments, $\Rightarrow Y_1$ & Y_2 are dependent.

A.2 (a) X_1 & X_2 are independent χ^2 distributed random variables.

with r_1 & r_2 degrees of freedom, respectively.

Joint pdf of x_1 & x_2 is

$$f(x_1, x_2) = f(x_1)\, f(x_2) = \left(\frac{1}{\Gamma(\frac{r_1}{2})2^{\frac{r_1}{2}}} x_1^{\frac{r_1}{2}-1} e^{-\frac{x_1}{2}}\right)\left(\frac{x_2^{\frac{r_2}{2}-1} e^{-\frac{x_2}{2}}}{\Gamma(\frac{r_2}{2})2^{r_2/2}}\right)$$

$$\therefore f(x_1, x_2) = \frac{x_1^{\frac{r_1}{2}-1}\, x_2^{\frac{r_2}{2}-1}\, e^{-\frac{x_1+x_2}{2}}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \qquad 0 < x_1, x_2 < \infty.$$

let $Y_1 = \frac{X_1}{r_1} \Big/ \frac{X_2}{r_2}$ & $Y_2 = X_2 \equiv X_1 = \frac{r_1}{r_2} Y_1 Y_2,\ X_2 = Y_2.$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{r_1}{r_2} y_2 & \frac{r_1}{r_2} y_1 \\ 0 & 1 \end{vmatrix} = \frac{r_1}{r_2} y_2.$$

Joint pdf of Y_1 & Y_2 is,

$$g(y_1, y_2) = |J|\, f(x_1(y_1, y_2), x_2(y_1, y_2))$$

$$= \left(\frac{r_1}{r_2} y_2\right) f\left(\frac{r_1}{r_2} y_1 y_2, y_2\right).$$

$$= \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} y_1^{\frac{r_1}{2}-1}\, y_2^{\frac{r_1+r_2}{2}}\, e^{-\frac{y_2\left(\frac{r_1}{r_2} y_1 + 1\right)}{2}}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2}) \cdot 2^{\frac{r_1+r_2}{2}}} \qquad \begin{pmatrix} 0 < y_1 < \infty \\ 0 < y_2 < \infty \end{pmatrix}$$

(b) Marginal p.d.f of Y_1 is

$$g_1(y_1) = \int_0^\infty g(y_1, y_2)\, dy_2 = \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} y_1^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \int_0^\infty \frac{y_2^{\frac{r_1+r_2}{2}}\, e^{-\frac{y_2\left(\frac{r_1}{r_2} y_1 + 1\right)}{2}}}{2^{\frac{r_1+r_2}{2}}}\, dy_2$$

let $v = \left(1 + \frac{r_1}{r_2} y_1\right) y_2 \Rightarrow dy_2 = \frac{dv}{1 + \frac{r_1}{r_2} y_1}$

$$g_1(y_1) = \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} y_1^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \cdot \frac{\Gamma\left(\frac{r_1+r_2}{2}\right) y_1^{\frac{r_1}{2}-1}}{\left(1 + \frac{r_1}{r_2} y_1\right)^{\frac{r_1+r_2}{2}}} \int_0^\infty \frac{v^{\frac{r_1+r_2}{2}} e^{-\frac{v}{2}}}{\Gamma\left(\frac{r_1+r_2}{2}\right) 2^{\frac{r_1+r_2}{2}}}\, dv.$$

The integrand is p.d.f of the χ^2 distribution with (r_1+r_2) degrees of freedom. & integral $= 1$.

$$\therefore g_1(y_1) = \frac{\left(\frac{r_1}{r_2}\right)\Gamma\left(\frac{r_1+r_2}{2}\right) y_1^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)\left(1+\frac{r_1}{r_2}y_1\right)^{\frac{r_1+r_2}{2}}}, \quad 0<y_1<\infty.$$

which is p.d.f of F distribution with r_1 & r_2 degrees of freedom.

$\Rightarrow Y_1$ has ~~p.d.f~~ F distribution with r_1 & r_2 degrees of freedom.

A.4 (a) From F-distribution table, $F_{0.05}(9,24) = 2.3$.

(b) $F_{0.95}(9,24) = \dfrac{1}{F_{0.05}(24,9)} = \dfrac{1}{2.9} = 0.3448$

(From F-distribution table, $F_{0.05}(24,9) = 2.9$)

(c) W is $F(9,24)$, from F-distribution table,

$P(W \leq 2.7) = 0.975.$

& $F_{0.975}(9,24) = \dfrac{1}{F_{0.025}(24,9)} = \dfrac{1}{3.61} = 0.277.$

So, $P(W \leq 0.277) = 0.025$

$P(0.277 \leq W \leq 2.7) = P(W \leq 2.7) - P(W \leq 0.277)$

$= 0.975 - 0.025$

$= 0.95.$

A.5 (a) From F-distribution table, $F_{0.01}(8,4) = 14.8$.

(b) $F_{0.99}(8,4) = \dfrac{1}{F_{0.01}(4,8)} = \dfrac{1}{7.01} = 0.1427$

(from F-distribution table).

(c) W is $F(8,4)$. from F distribution table. that.

$P(W \leq 8.98) = 0.975 \rightarrow$ (from F-distribution table).

& $F_{0.975}(8,4) = \dfrac{1}{F_{0.025}(4,8)} = \dfrac{1}{5.05} = 0.198$

(from F-dist. table).

So $P(W \leq 0.198) = 0.025$.

$$P(0.198 \leq W \leq 8.98) = P(W \leq 8.98) - P(W \leq 0.198)$$
$$= 0.975 - 0.025$$
$$= 0.95.$$

A.9 To make $f(x) = cx^3(1-x)^6 \quad 0<x<1$ is a p.d.f.

we need c such that

$$\int_0^1 f(x)\,dx = 1.$$

$$\therefore 1 = \int_0^1 cx^3(1-x)^6\,dx$$

$$= c\left[\frac{x^{10}}{10} - \frac{2}{3}x^9 + \frac{15}{8}x^8 - \frac{20}{7}x^7 + \frac{5}{2}x^6 - \frac{6}{5}x^5 + \frac{1}{4}x^4\right]_0^1$$

$$= c\left(\frac{1}{10} - \frac{2}{3} + \frac{15}{8} - \frac{20}{7} + \frac{5}{2} - \frac{6}{5} + \frac{1}{4}\right)$$

$$= \frac{c}{840}$$

$$\Rightarrow \boxed{c = 840}$$

(11) (a) Using Integration -

$$I = \int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} y^3(1-y)^2 dy$$

$\Gamma(3) = 2 \quad \Gamma(4) = 6 \quad \Gamma(7) = 720$

$$\Rightarrow I = \int_0^{0.4} \frac{720}{6 \times 2} y^3(1-y)^2 dy$$

$$I = 60 \int_0^{0.4} y^3(y^2+1-2y)\, dy$$

$$I = 60 \int_0^{0.4} (y^5 - 2y^4 + y^3)\, dy$$

$$I = 60 \left(\frac{y^6}{6} - \frac{2y^5}{5} + \frac{y^4}{4} \right)_0^{0.4}$$

$$\boxed{I = 0.1792}$$

(b) Using Result of question (10)

$$\int_0^p \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy = \sum_{y=\alpha}^{n} \binom{n}{y} p^y (1-p)^{n-y}$$

$n = \alpha+\beta-1$

$\alpha = 4,\ \beta = 3,\ n = 6,\ p = 0.4$

$$I = \int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} y^3(1-y)^2\,dy = \sum_{y=4}^{6} \binom{6}{y}(0.4)^y(0.6)^{6-y}$$

$$I = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6)^1 + (0.4)^6$$

$I = 0.1792$

~~0.177~~

A.14 X & Y are independent, their joint p.d.f is,

$$h(x,y) = f(x)\,g(y) = \left(\frac{x}{5^2} e^{-\frac{x}{5}}\right)\left(\frac{1}{5} e^{-y/5}\right) = \frac{x}{5^3} e^{-\frac{x+y}{5}}$$

$$0 < x < \infty,\quad 0 < y < \infty$$

let $Z = \frac{X}{Y}$, ~~support~~ ~~is~~ $0 < z < \infty$,

$$F(z) = P(Z \le z)$$

$$= P\left(\frac{X}{Y} \le zY\right)$$

$$= \int_0^\infty \int_0^{xy} h(x,y)\,dx\,dy = \int_0^\infty \left[\left[\int_0^{xy} \frac{x}{125} e^{-x/5}\,dx\right) e^{-y/5}\,dy.\right]$$

p.d of Z is, $f(z) = F'(z)$

$$\therefore f(z) = \int_0^\infty \frac{zy}{125} e^{-\frac{xy}{5}}\, y e^{-y/5}\,dy. = \frac{z}{125}\int_0^\infty y^2 e^{-\frac{(x+1)y}{5}}\,dy.$$

$$\therefore f(z) = \frac{z}{125}\int_0^\infty y^2 e^{-\frac{(x+1)y}{5}}\,dy$$

$$= \frac{z}{125}\int_0^\infty y^2\left(-\frac{5}{z+1}\right) d\left(e^{-\frac{(x+1)y}{5}}\right)$$

$$= \frac{-z}{25(z+1)}\left[\left[y^2 e^{-\frac{(x+1)y}{5}}\right]_0^\infty - 2\int_0^\infty e^{-\frac{(x+1)y}{5}}\, y\,dy\right]$$

$$= \frac{2z}{25(z+1)}\int_0^\infty y\left(-\frac{5}{z+1}\right) de^{-\frac{(x+1)y}{5}}.$$

$$= -\frac{2z}{5(z+1)^2}\left[\left[ye^{-\frac{(x+1)y}{5}}\right]_0^\infty - \int_0^\infty e^{-\frac{(x+1)y}{5}}\,dy\right]$$

$$= \frac{2z}{5(z+1)^2}\left(-\frac{5}{(z+1)}\right)\left[e^{-\frac{(x+1)y}{5}}\right]_0^\infty$$

$$= \frac{2z}{(z+1)^3}, \quad 0 < z < \infty.$$

A.15 $S = \{(x_1, x_2) : 0 \leq x_1 < 1, 0 \leq x_2 < 1\}$ is

Given $Z_1 = \sqrt{-2\ln X_1} \cos(2\pi X_2)$ &

$Z_2 = \sqrt{-2\ln X_1} \sin(2\pi X_2)$

$\Rightarrow$ The boundaries of S are not a part of support of X_1 & X_2

For $x_1 = 0$ & $0 < x_2 < 1 \Rightarrow -\infty < Z_1 < \infty$ & $-\infty < Z_2 < \infty$

$0 < x_2 < 1$, $x_1 = 1 \Rightarrow Z_1 = 0$ & $Z_2 = 0$.

$x_2 = 0$, $0 < x_1 < 1 \Rightarrow 0 < Z_1 < \infty$, $Z_2 = 0$

$x_2 = 1$, $0 < x_1 < 1 \Rightarrow -\infty < Z_1 < \infty$ $Z_2 = 0$.

$\Rightarrow$ Support of Z_1, Z_2 is,

$S_1 = \{(z_1, z_2) : -\infty < z_1 < \infty, -\infty < z_2 < \infty\}$.

5.3 SEVERAL RANDOM VARIABLES

Exercises

5.3-1. Let X_1 and X_2 be independent Poisson random variables with respective means $\lambda_1 = 2$ and $\lambda_2 = 3$. Find

(a) $P(X_1 = 3, X_2 = 5)$.

(b) $P(X_1 + X_2 = 1)$.
Hint. Note that this event can occur if and only if $\{X_1 = 1, X_2 = 0\}$ or $\{X_1 = 0, X_2 = 1\}$.

5.3-2. Let X_1 and X_2 be independent random variables with respective binomial distributions $b(3, 1/2)$ and $b(5, 1/2)$. Determine

(a) $P(X_1 = 2, X_2 = 4)$.

(b) $P(X_1 + X_2 = 7)$.

5.3-3. Let X_1 and X_2 be independent random variables with probability density functions $f_1(x_1) = 2x_1$, $0 < x_1 < 1$, and $f_2(x_2) = 4x_2^3$, $0 < x_2 < 1$, respectively. Compute

(a) $P(0.5 < X_1 < 1 \text{ and } 0.4 < X_2 < 0.8)$.

(b) $E\left(X_1^2 X_2^3\right)$.

5.3-4. Let X_1 and X_2 be a random sample of size $n = 2$ from the exponential distribution with pdf $f(x) = 2e^{-2x}$, $0 < x < \infty$. Find

(a) $P(0.5 < X_1 < 1.0, 0.7 < X_2 < 1.2)$.

(b) $E[X_1(X_2 - 0.5)^2]$.

5.3-5. Let X_1 and X_2 be observations of a random sample of size $n = 2$ from a distribution with pmf $f(x) = x/6$, $x = 1, 2, 3$. Then find the pmf of $Y = X_1 + X_2$. Determine the mean and the variance of the sum in two ways.

5.3-6. Let X_1 and X_2 be a random sample of size $n = 2$ from a distribution with pdf $f(x) = 6x(1 - x)$, $0 < x < 1$. Find the mean and the variance of $Y = X_1 + X_2$.

5.3-7. The distributions of incomes in two cities follow the two Pareto-type pdfs

$$f(x) = \frac{2}{x^3}, \quad 1 < x < \infty, \quad \text{and} \quad g(y) = \frac{3}{y^4}, \quad 1 < y < \infty,$$

respectively. Here one unit represents \$20,000. One person with income is selected at random from each city. Let X and Y be their respective incomes. Compute $P(X < Y)$.

5.3-8. Suppose two independent claims are made on two insured homes, where each claim has pdf

$$f(x) = \frac{4}{x^5}, \qquad 1 < x < \infty,$$

in which the unit is \$1000. Find the expected value of the larger claim. Hint: If X_1 and X_2 are the two independent claims and $Y = \max(X_1, X_2)$, then

$$G(y) = P(Y \le y) = P(X_1 \le y)P(X_2 \le y) = [P(X \le y)]^2.$$

Find $g(y) = G'(y)$ and $E(Y)$.

5.3-9. Let X_1, X_2 be a random sample of size $n = 2$ from a distribution with pdf $f(x) = 3x^2$, $0 < x < 1$. Determine

(a) $P(\max X_i < 3/4) = P(X_1 < 3/4, X_2 < 3/4)$.

(b) The mean and the variance of $Y = X_1 + X_2$.

5.3-10. Let X_1, X_2, X_3 denote a random sample of size $n = 3$ from a distribution with the geometric pmf

$$f(x) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}, \qquad x = 1, 2, 3, \ldots.$$

(a) Compute $P(X_1 = 1, X_2 = 3, X_3 = 1)$.

(b) Determine $P(X_1 + X_2 + X_3 = 5)$.

(c) If Y equals the maximum of X_1, X_2, X_3, find

$$P(Y \le 2) = P(X_1 \le 2)P(X_2 \le 2)P(X_3 \le 2).$$

5.3-11. Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively. Find

(a) $P(X_1 = 2, X_2 = 2, X_3 = 5)$.

(b) $E(X_1 X_2 X_3)$.

(c) The mean and the variance of $Y = X_1 + X_2 + X_3$.

5.3-12. Let X_1, X_2, X_3 be a random sample of size $n = 3$ from the exponential distribution with pdf $f(x) = e^{-x}$, $0 < x < \infty$. Find

$$P(1 < \min X_i) = P(1 < X_1, 1 < X_2, 1 < X_3).$$

5.3-13. A device contains three components, each of which has a lifetime in hours with the pdf

$$f(x) = \frac{2x}{10^2}\, e^{-(x/10)^2}, \qquad 0 < x < \infty.$$

The device fails with the failure of one of the components. Assuming independent lifetimes, what is the probability that the device fails in the first hour of its operation? HINT: $G(y) = P(Y \le y) = 1 - P(Y > y) = 1 - P$ (all three $> y$).

5.3-14. Let X_1, X_2, X_3 be independent random variables that represent lifetimes (in hours) of three key components of a device. Say their respective distributions are exponential with means 1000, 1500, and 2000. Let Y be the minimum of X_1, X_2, X_3 and compute $P(Y > 1000)$.

5.3-15. Three drugs are being tested for use as the treatment of a certain disease. Let p_1, p_2, and p_3 represent the probabilities of success for the respective drugs. As three patients come in, each is given one of the drugs in a random order. After $n = 10$ "triples" and assuming independence, compute the probability that the maximum number of successes with one of the drugs exceeds eight if, in fact, $p_1 = p_2 = p_3 = 0.7$.

5.3-16. Each of eight bearings in a bearing assembly has a diameter (in millimeters) that has the pdf

$$f(x) = 10x^9, \qquad 0 < x < 1.$$

Assuming independence, find the cdf and the pdf of the maximum diameter (say, Y) of the eight bearings and compute $P(0.9999 < Y < 1)$.

5.3-17. In considering medical insurance for a certain operation, let X equal the amount (in dollars) paid for the doctor and let Y equal the amount paid to the hospital. In the past, the variances have been $\text{Var}(X) = 8100$, $\text{Var}(Y) = 10{,}000$, and $\text{Var}(X + Y) = 20{,}000$. Due to increased expenses, it was decided to increase the doctor's fee by \$500 and increase the hospital charge Y by 8%. Calculate the variance of $X + 500 + (1.08)Y$, the new total claim.

5.3-18. The lifetime in months of a certain part has a gamma distribution with $\alpha = \theta = 2$. A company buys three such parts and uses one until it fails, replacing it with a second part. When the latter fails, it is replaced by the third part. What are the mean and the variance of the total lifetime (the sum of the lifetimes of the three parts) associated with this situation?

5.3-19. Two components operate in parallel in a device, so the device fails when and only when both components fail. The lifetimes, X_1 and X_2, of the respective components are independent and identically distributed with an exponential distribution with $\theta = 2$. The cost of operating the device is $Z = 2Y_1 + Y_2$, where $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. Compute $E(Z)$.

5.3-20. Let X and Y be independent random variables with nonzero variances. Find the correlation coefficient of $W = XY$ and $V = X$ in terms of the means and variances of X and Y.

5.3-21. Flip $n = 8$ fair coins and remove all that came up heads. Flip the remaining coins (that came up tails) and remove the heads again. Continue flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, \ldots, 8$. Then $Y = \max(X_1, X_2, \ldots, X_8)$.

(a) Show that $P(Y \le y) = [1 - (1/2)^y]^8$.

(b) Show that the pmf of Y is defined by $P(Y = y) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8, \quad y = 1, 2, \ldots.$

(c) Use a computer algebra system such as *Maple* or *Mathematica* to show that the mean of Y is $E(Y) = 13{,}315{,}424/3{,}011{,}805 = 4.421$.

(d) What happens to the expected value of Y as the number of coins is doubled?

① $X_1 \sim P(\lambda_1) = P(2)$ $p(X_1) = \frac{e^{-\lambda_1}(\lambda_1)^{x_1}}{x_1!}$

$X_2 \sim P(\lambda_2) = P(3)$

X_1, & X_2 are independent

(a) so $P(X_1=3, X_2=5) = P(X_1=3) \cdot P(X_2=5)$

$$= \frac{e^{-\lambda_1}(\lambda_1)^3}{3!} \times \frac{e^{-\lambda_2}(\lambda_2)^5}{5!}$$

$$= \frac{e^{-2}\, 2^3}{3!} \times \frac{e^{-3}\, 3^5}{5!} = \underline{0.018} \text{ Ans}$$

(b) $P(X_1 + X_2 = 1)$

We know that X_1 & X_2 are discrete random Variable with integer Value.

so $P(X_1+X_2=1) = P(X_1=0, X_2=1) + P(X_1=1, X_2=0)$

$$= \frac{e^{-2}(2)^0}{0!} \times \frac{e^{-3}(3)^1}{1!} + \frac{e^{-2}(2)^1}{1!} \times \frac{e^{-3}(3)^0}{0!}$$

$$= 3 \times e^{-5} + 2 \times e^{-5}$$

$$= 5e^{-5}$$

$$= 0.0336 \text{ Ans}$$

(2) $X_1 \sim B(3, \frac{1}{2}) \quad X_2 \sim B(5, \frac{1}{2})$

(a) $P(X_1 = 2, X_2 = 4)$

x_1 & X_2 are independent so

$$P(X_1 = 2, X_2 = 4) = P(X_1 = 2) . P(X_2 = 4)$$

$$= \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2} \times \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4}$$

$$= 3\left(\frac{1}{2}\right)^3 \times 5\left(\frac{1}{2}\right)^5$$

$$= \frac{15}{(2)^8} = \underline{\underline{0.0586}}$$

(b) $P(X_1 + X_2 = 7) = P(X_1 = 2, X_2 = 5) + P(X_1 = 3, X_2 = 4)$

$$= \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2} \times \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0$$

$$+ \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 \times \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4}$$

$$= 3\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^3 \times 5\left(\frac{1}{2}\right)^5$$

$$= \frac{3+5}{2^8} = \frac{8}{2^8}$$

$$= \underline{\underline{0.03125}}$$

(3) (a) X_1 & X_2 are Independent.

so $P(0.5 < X_1 < 1 \;\&\; 0.4 < X_2 < 0.8)$

$\Rightarrow P(0.5 < X_1 < 1) \cdot P(0.4 < X_2 < 0.8)$

$X_1 : f(x_1) = 2x_1 \quad x_1 \in (0,1)$

$X_2 : f(x_2) = 4x_2^3 \quad x_2 \in (0,1)$ Given

$$\Rightarrow \left(\int_{0.5}^{1} f(x_1)\,dx_1\right) \times \int_{0.4}^{0.8} f(x_2)\,dx_2$$

$$= \left(\int_{0.5}^{1} 2x_1\,dx_1\right) \times \int_{0.4}^{0.8} 4x_2^3\,dx_2$$

$$= 0.288$$

(3) (b) $E(X_1^2 X_2^3) = \iint x_1^2 x_2^3 f(x_1, x_2)\, dx_1 dx_2$

$$= \int_0^1 \int_0^1 x_1^2 f(x_1)\, dx_1 \; x_2^3 f(x_2)\, dx_2$$

$$= \int_0^1 x_1^2 \, 2x_1 \, dx_1 \; \int_0^1 x_2^3 \, 4x_2^3 \, dx_2$$

$$= \frac{2}{4} \times \frac{4}{7}$$

$$= 2/7$$

Chapter 5

5.1-1 $g(y) = 2y,\ 0 < y < 1.$

5.1-3 $g(y) = (1/8)y^5 e^{-y^2/2},\ 0 < y < \infty.$

5.1-5 Exponential distribution with mean 2.

5.1-7 **(a)** $F(x) = (\ln x - \ln c - 0.03)/0.04$ and $f(x) = 1/(0.04x),\ ce^{0.03} \le x \le ce^{0.07}$, where $c = 50{,}000$;

(b) The interest for each of n equal parts is R/n. The amount at the end of the year is $50{,}000(1+R/n)^n$; the limit as $n \to \infty$ is $50{,}000\,e^R$.

5.1-9 **(a)** $G(y) = P(Y \le y) = P(X \le \ln y) = 1 - e^{-y}$, $0 < y < \infty$;

(b) $G(y) = 1 - \exp[-e^{(\ln y - \theta_1)/\theta_2}],\ 0 < y < \infty$;
$g(y) = \exp[-e^{(\ln y - \theta_1)/\theta_2}][e^{(\ln y - \theta_1)/\theta_2}][1/\theta_2 y]$, $0 < y < \infty$;

(c) A Weibull distribution with $G(y) = 1 - e^{-(y/\beta)^\alpha},\ 0 < y < \infty$;
$g(y) = (\alpha y^{\alpha-1}/\beta^\alpha)e^{-(y/\beta)^\alpha},\ 0 < y < \infty$, where α is the shape parameter and β is the scale parameter;

(d) $\exp(-e^{-2}) = 0.873.$

5.1-11 **(a)** $\dfrac{1}{2} - \dfrac{\arctan 1}{\pi} = 0.25;$

(b) $\dfrac{1}{2} - \dfrac{\arctan 5}{\pi} = 0.0628;$

(c) $\dfrac{1}{2} - \dfrac{\arctan 10}{\pi} = 0.0317.$

5.1-13 **(b)** **(i)** $\exp(\mu + \sigma^2/2)$, **(ii)** $\exp(2\mu + 2\sigma^2)$,
(iii) $\exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)$.

5.1-15 **(a)** $g(y) = \dfrac{1}{\sqrt{2\pi y}} \exp(-y/2), \qquad 0 < y < \infty;$

(b) $g(y) = \dfrac{3}{2}\sqrt{y}, \qquad 0 < y < 1.$

5.2-1 $g(y_1, y_2) = (1/4)e^{-y_2/2},\ 0 < y_1 < y_2 < \infty;$
$g_1(y_1) = (1/2)e^{-y_1/2},\ 0 < y_1 < \infty;$
$g_2(y_2) = (y_2/4)e^{-y_2/2},\ 0 < y_2 < \infty$; no.

5.2-3 $\mu = \dfrac{r_2}{r_2 - 2}$, $r_2 > 2$;

$$\sigma^2 = \frac{2r_2^2(r_1 + r_2 - 2)}{r_1(r_2 - 2)^2(r_2 - 4)},\ r_2 > 4.$$

5.2-5 **(a)** 14.80; **(b)** 1/7.01 = 0.1427; **(c)** 0.95.

5.2-9 840.

5.2-11 **(a)** 0.1792; **(b)** 0.1792.

5.2-13 **(a)**

$$\begin{aligned} G(y_1, y_2) &= \int_0^{y_1}\int_u^{y_2} 2(1/1000^2)\exp[-(u+v)/1000]\,dv\,du \\ &= 2\exp[-(y_1+y_2)/1000] - \exp[-y_1/500] \\ &\quad -2\exp[-y_2/1000] + 1, \quad 0 < y_1 < y_2 < \infty; \end{aligned}$$

(b) $2e^{-6/5} - e^{-12/5} \approx 0.5117$.

5.3-1 **(a)** 0.0182; **(b)** 0.0337.

5.3-3 **(a)** 36/125; **(b)** 2/7.

5.3-5

$$g(y) = \begin{cases} 1/36, & y = 2, \\ 4/36, & y = 3, \\ 10/36, & y = 4, \\ 12/36, & y = 5, \\ 9/36, & y = 6; \end{cases}$$

$\mu = 14/3, \sigma^2 = 10/9$.

5.3-7 2/5.

5.3-9 **(a)** 729/4096; **(b)** $\mu = 3/2$; $\sigma^2 = 3/40$.

5.3-11 **(a)** 0.0035; **(b)** 8; **(c)** $\mu_Y = 6, \sigma_Y^2 = 4$.

5.3-13 $1 - e^{-3/100} \approx 0.03$.

5.3-15 0.0384.

5.3-17 \$21,816.

5.3-19 5.

5.3-21 **(c)** Using *Maple*, we obtain $\mu = 13{,}315{,}424/3{,}011{,}805 = 4.4211$.

(d) $E(Y) = 5.377$ with 16 coins, $E(Y) = 6.355$ with 32 coins.

5.4-1 **(a)**

$$g(y) = \begin{cases} 1/64, & y = 3, 12, \\ 3/64, & y = 4, 11, \\ 6/64, & y = 5, 10, \\ 10/64, & y = 6, 9, \\ 12/64, & y = 7, 8. \end{cases}$$

5.4-3 **(a)** $M(t) = e^{7(e^t-1)}$; **(b)** Poisson, $\lambda = 7$; **(c)** 0.800.

5.4-5 0.925.

5.4-7 **(a)** $M(t) = 1/(1-5t)^{21}$, $t < 1/5$;

(b) gamma distribution, $\alpha = 21, \theta = 5$.

5.4-11 **(a)** $g(w) = 1/12,\ w = 0, 1, 2, \ldots, 11$; **(b)** $h(w) = 1/36,\ w = 0, 1, 2, \ldots, 35$.

5.4-13 **(a)**

$$h_1(w_1) = \begin{cases} 1/36, & w_1 = 0, \\ 4/36, & w_1 = 1, \\ 10/36, & w_1 = 2, \\ 12/36, & w_1 = 3, \\ 9/36, & w_1 = 4; \end{cases}$$

(b) $h_2(w) = h_1(w)$;

(c)

$$h(w) = \begin{cases} 1/1296, & w = 0 \\ 8/1296, & w = 1, \\ 36/1296, & w = 2, \\ 104/1296, & w = 3, \\ 214/1296, & w = 4, \\ 312/1296, & w = 5, \\ 324/1296, & w = 6, \\ 216/1296, & w = 7, \\ 81/1296, & w = 8; \end{cases}$$

(d) With denominators equal to $6^8 = 1{,}679{,}616$, the respective numerators of $0, 1, \ldots, 16$ are 1, 16, 136, 784, 3,388, 11,536, 31,864, 72,592, 137,638, 217,776, 286,776, 311,472, 274,428, 190,512, 99,144, 34,992, 6,561;

(e) They are becoming more symmetrical as the value of n increases.

5.4-15 **(b)** $\mu_Y = 25/3, \sigma_Y^2 = 130/9$;

(c)

$$P(Y = y) = \begin{cases} 96/1024, & y = 4, \\ 144/1024, & y = 5, \\ 150/1024, & y = 6, \\ 135/1024, & y = 7. \end{cases}$$

5.4-17 $Y - X + 25$ is $b(50,\ 1/2)$;

$$P(Y - X \ge 2) = \sum_{k=27}^{50} \binom{50}{k}\left(\frac{1}{2}\right)^{50} = 0.3359.$$

5.4-19 $1 - 17/2e^3 = 0.5768$.

5.4-21 0.4207.

5.5-1 **(a)** 0.4772; **(b)** 0.8561.

5.5-3 **(a)** 46.58, 2.56; **(b)** 0.8447.

5.5-5 **(b)** 0.05466; 0.3102.

5.5-7 0.9830.

5.5-9 **(a)** 0.3085; **(b)** 0.2267.

5.5-11 $0.8413 > 0.7734$, select X.

5.5-13 **(a)** $t(2)$; **(c)** $\mu_V = 0$; **(d)** $\sigma_V = 1$;
(e) In part (b), numerator and denominator are not independent.

5.5-15 **(a)** 2.567; **(b)** -1.740; **(c)** 0.90.

5.6-1 0.4772.

5.6-3 0.8185.

5.6-5 **(a)** $\chi^2(18)$; **(b)** 0.0756, 0.9974.

5.6-7 0.6247.

5.6-9 $P(1.7 \le Y \le 3.2) = 0.6749$; the normal approximation is 0.6796.

5.6-11 \$444,338.13.

5.6-13 0.9522.

5.6-15 **(a)** $\displaystyle\int_0^{25} \frac{1}{\Gamma(13)2^{13}}\, y^{13-1}e^{-y/2}\, dy = 0.4810$;
(b) 0.4448 using normal approximation.

5.7-1 **(a)** 0.2878, 0.2881; **(b)** 0.4428, 0.4435; **(c)** 0.1550, 0.1554.

5.7-3 0.9258 using normal approximation, 0.9258 using binomial.

5.7-5 0.3085.

5.7-7 0.6247 using normal approximation, 0.6148 using Poisson.

5.7-9 **(a)** 0.5548; **(b)** 0.3823.

5.7-11 0.6813 using normal approximation, 0.6788 using binomial.

5.7-13 **(a)** 0.3802; **(b)** 0.7571.

5.7-15 0.4734 using normal approximation; 0.4749 using Poisson approximation with $\lambda = 50$; 0.4769 using $b(5000, 0.01)$.

5.7-17 0.6455 using normal approximation, 0.6449 using Poisson.

5.8-1 **(a)** 0.84; **(b)** 0.082.

5.8-3 $k = 1.464$; 8/15.

5.8-5 **(a)** 0.25; **(b)** 0.85; **(c)** 0.925.

5.8-7 **(a)** $E(W) = 0$; the variance does not exist.

5.9-1 **(a)** 0.9984; **(b)** 0.998.

5.9-3 $M(t) = \left[1 - \dfrac{2t\sigma^2}{n-1}\right]^{-(n-1)/2} \to e^{\sigma^2 t}$.