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PROBABILISTIC LIMIT GROUPS UNDER A T-NORM

G. JÄGER AND T. M. G. AHSANULLAH

Abstract. We introduce probabilistic limit groups under a t-norm and study their basic properties. We show that for the classes of strict t-norms, all categories of probabilistic limit groups under such t-norms are isomorphic. The same is true for nilpotent t-norms. We further show that for each probabilistic limit group under a t-norm there is a natural probabilistic uniform limit structure which has the same underlying probabilistic Cauchy structure as the probabilistic limit group.

1. Introduction

Probabilistic limit spaces were introduced by Liviu C. Florescu [4]. He used net convergence to describe such spaces. A formulation in terms of filter convergence was given by G. D. Richardson and D. C. Kent [16]. Probabilistic convergence spaces extend the theory of probabilistic metric spaces (see [12] and [17]) and probabilistic topological spaces (see [5]) by assigning to a filter a probability to converge to a point. This was subsequently generalized by Harald Nusser [14], who used t-norms in various axiom systems of probabilistic spaces.

This paper looks at a special class of probabilistic limit spaces, where the underlying set additionally carries a group structure. The compatibility of the group operations with the convergence structure is usually expressed by continuity of the group operations. However, in some applications (for example, when looking at normed vector spaces), this compatibility—in the probabilistic case—has to be defined differently. We give a suitable definition and study the resulting category of probabilistic

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limit groups under a t-norm. If the t-norm is given by the minimum t-norm, then the compatibility of the group operations coincides with their continuity. A similar approach in the realm of convergence approach spaces can be found in [11]. We further show that each probabilistic limit group allows a probabilistic uniformization and that for certain classes of t-norms, the categories of probabilistic limit groups under t-norms of these classes are isomorphic.

A *t-norm* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a binary operation on $[0,1]$ which is associative, commutative, non-decreasing in each argument, and has 1 as the unit. A t-norm is called *continuous* if it is continuous as a mapping from $[0,1]\times[0,1] \longrightarrow [0,1]$. A special class of t-norms is given by *continuous Archimedean t-norms*. These are determined by continuous, strictly decreasing additive generators $S : [0,1] \longrightarrow [0,\infty]$ with $S(1) = 0$ such that for all $\alpha, \beta \in [0,1]$

$$\alpha * \beta = S^{(-1)}(S(\alpha) + S(\beta))$$

with the *pseudo-inverse*

$$S^{(-1)}(u) = \bigvee\{x \in [0,1] \ : \ S(x) > u\} = \begin{cases} v & \text{if } S(v) = u \\ 0 & \text{if } u > S(0) \end{cases} .$$

Note that $\bigvee \emptyset = \bigwedge[0,1] = 0$ here.

We also note that the pseudo-inverse $S^{(-1)} : [0,\infty] \longrightarrow [0,1]$ is continuous, surjective, and strictly decreasing on $[0, S(0)]$ and that $S(S^{(-1)}(u)) = u$ if $u \le S(0)$ and that $S^{(-1)}(S(u)) = u$ for all $u \in [0,1]$. Continuous Archimedean t-norms can be separated into two classes.

- $S(0) = \infty$. These are the *strict t-norms*. In this case, $S^{(-1)} = S^{-1}$. A typical example is the product t-norm $\alpha * \beta = \alpha\beta$ with additive generator $S(x) = -\ln(x)$ (and $S(0) = \infty$).
- $S(0) < \infty$. These are the *nilpotent t-norms*. Noting that for an additive generator S for a continuous Archimedean t-norm and for all $a > 0$, $\overline{S}(x) = aS(x)$ defines an additive generator for the same t-norm, we can always assume for a nilpotent t-norm that $S(0) = 1$. A typical example for a nilpotent t-norm is the Lukasiewicz t-norm $\alpha*\beta = (\alpha+\beta-1)\vee 0$ with additive generator $S(x) = 1 - x$.

An example of a non-Archimedean t-norm is the *minimum t-norm* $\alpha*\beta = \alpha \wedge \beta$. We note that $\alpha * \beta \le \alpha \wedge \beta$ for any t-norm. For further results on t-norms, we refer the reader to [17] and [9].

We finally fix some notation. For a set X, we denote $P(X)$ its power set. We denote the set of all filters $\mathbb{F}, \mathbb{G}, \mathbb{H}, ...$ on the set X by $\mathbb{F}(X)$. We order this set by set inclusion and we denote for $x \in X$ the point filter by $[x] = \{F \subseteq X \ : \ x \in F\}$. For a group $(X, \cdot)$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, we define $\mathbb{F} \odot \mathbb{G}$ as the filter generated by the sets $F \cdot G = \{xy \ : \ x \in F, y \in G\}$

where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. The filter $\mathbb{F}^{-1}$ is generated by the sets $F^{-1} = \{x^{-1} : x \in F\}$ for $F \in \mathbb{F}$. The following properties are then not difficult to prove.

Lemma 1.1. *Let e denote the neutral element in the group $(X, \cdot)$. For $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathbb{F}(X)$ and $f : X \longrightarrow Y$ a group homomorphism (where Y is also a group), we have*

(i) $\mathbb{F} \odot \mathbb{F}^{-1} \leq [e]$ *and* $\mathbb{F}^{-1} \odot \mathbb{F} \leq [e]$;
(ii) $[x] \odot [x]^{-1} = [x]^{-1} \odot [x] = [e]$;
(iii) $[x^{-1}] = [x]^{-1}$;
(iv) $(\mathbb{F} \odot \mathbb{G}) \odot \mathbb{H} = \mathbb{F} \odot (\mathbb{G} \odot \mathbb{H})$;
(v) $(\mathbb{F}^{-1})^{-1} = \mathbb{F}$;
(vi) $(\mathbb{F} \odot \mathbb{G})^{-1} = \mathbb{G}^{-1} \odot \mathbb{F}^{-1}$;
(vii) $[e] \odot \mathbb{F} = \mathbb{F} \odot [e] = \mathbb{F}$;
(viii) $(\mathbb{F} \wedge \mathbb{G})^{-1} = \mathbb{F}^{-1} \wedge \mathbb{G}^{-1}$;
(ix) $(\mathbb{F} \wedge \mathbb{G}) \odot \mathbb{H} = (\mathbb{F} \odot \mathbb{H}) \wedge (\mathbb{G} \odot \mathbb{H})$;
(x) $f(\mathbb{F} \odot \mathbb{G}) = f(\mathbb{F}) \odot f(\mathbb{G})$;
(xi) $f(\mathbb{F}^{-1}) = (f(\mathbb{F}))^{-1}$.

For a subset A of an ordered set X we write, in case of existence, $\bigvee A$ for its supremum and $\bigwedge A$ for its infimum. If $A = \{\alpha, \beta\}$, then we write $\alpha \wedge \beta = \bigwedge A$ and $\alpha \vee \beta = \bigvee A$. For notions from category theory, we refer the reader to [1].

2. Probabilistic Limit Spaces, Probabilistic Cauchy Spaces, and Probabilistic Uniform Limit Spaces

A *probabilistic limit space* [16] is a pair $(X, \overline{q})$ of a set X and a non-empty family of mappings $\overline{q} = (q_\lambda : \mathbb{F}(X) \longrightarrow P(X))_{\lambda \in [0,1]}$ that satisfies the following axioms.

(PL1) $x \in q_\alpha([x])$ for all $\alpha \in [0,1]$, $x \in X$;
(PL2) $q_\alpha(\mathbb{F}) \subseteq q_\alpha(\mathbb{G})$ whenever $\mathbb{F} \leq \mathbb{G}$;
(PL3) $q_\beta(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$ whenever $\alpha \leq \beta$;
(PL4) $q_0(\mathbb{F}) = X$;
(PL5) $x \in q_{\alpha \wedge \beta}(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $x \in q_\beta(\mathbb{G})$ for all $\alpha, \beta \in [0,1], \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$.

If, instead of (PL5), the weaker axiom

(wPL5) $x \in q_{\alpha * \beta}(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $x \in q_\beta(\mathbb{G})$

is satisfied, then we speak of a *probabilistic limit space under a t-norm* $*$ [14]. A probabilistic limit space is, therefore, a probabilistic limit space under the minimum t-norm. It is not difficult to show that (PL5) is equivalent to

(PL5′) $x \in q_\alpha(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $x \in q_\alpha(\mathbb{G})$.

Also, it is clear that (PL5) implies (wPL5). In [14], probabilistic limit spaces are called *componentwise probabilistic limit spaces.*

Example 2.1. The *discrete probabilistic limit space* $(X, \overline{q^d})$ is defined by $x \in q^d_\alpha(\mathbb{F}) \iff \mathbb{F} = [x]$.

Example 2.2. The *indiscrete probabilistic limit space* $(X\overline{q^i})$ is defined by $q^i_\alpha(\mathbb{F}) = X$ for all $\mathbb{F} \in \mathbb{F}(X)$ and all $\alpha \in [0, 1]$.

Further examples are described in [16] and are also mentioned later in section 3.

A mapping $f : X \longrightarrow X'$ between the probabilistic limit spaces under the t-norm $*$, $(X, \overline{q})$ and $(X', \overline{q}')$, is *continuous* if, for all $\alpha \in [0, 1]$ and all $\mathbb{F} \in \mathbb{F}(X)$, we have $f(q_\alpha(\mathbb{F})) \subseteq q'_\alpha(f(\mathbb{F}))$. The category of all probabilistic limit spaces under the t-norm $*$ with the continuous mappings as morphisms is denoted by $PLIM^*$. It is shown in [14] that $PLIM^*$ is a topological and extensional construct and for $* = \wedge$, $PLIM^\wedge$ is Cartesian closed.

A *probabilistic Cauchy space under the t-norm* $*$ [14] is a pair $(X, \overline{C})$ of a set X and a non-empty family of subsets of $\mathbb{F}(X)$, $\overline{C} = (C_\alpha)_{\alpha \in [0,1]}$, that satisfies the following axioms.

(PC1) $[x] \in C_\alpha$ for all $x \in X$ and all $\alpha \in [0, 1]$;
(PC2) $\mathbb{G} \in C_\alpha$ whenever $\mathbb{F} \in C_\alpha$ and $\mathbb{F} \leq \mathbb{G}$;
(PC3) $C_\beta \subseteq C_\alpha$ whenever $\alpha \leq \beta$;
(PC4) $C_0 = \mathbb{F}(X)$;
(PC5) $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha * \beta}$ whenever $\mathbb{F} \in C_\alpha$, $\mathbb{G} \in C_\beta$, and $\mathbb{F} \vee \mathbb{G}$ exists.

A mapping $f : X \longrightarrow X'$ between two probabilistic Cauchy spaces under the t-norm $*$, $(X, \overline{C})$ and $(X, \overline{C}')$, is called *Cauchy-continuous* if, for all $\alpha \in [0, 1]$, we have $f(C_\alpha) \subseteq C'_\alpha$. The category of probabilistic Cauchy spaces under the t-norm $*$ and Cauchy continuous mappings is denoted by $PChy^*$.

Lemma 2.3. *Let $(X, \overline{C})$ be a probabilistic Cauchy space under the t-norm $\wedge$. Then (PC5) is equivalent to the axiom*

(PC5$^\wedge$) $\mathbb{F} \wedge \mathbb{G} \in C_\alpha$ whenever $\mathbb{F} \in C_\alpha$, $\mathbb{G} \in C_\alpha$, and $\mathbb{F} \vee \mathbb{G}$ exists.

Proof. If (PC5) is true, then we simply choose $\alpha = \beta$. If (PC5$^\wedge$) is true, then, for $\mathbb{F} \in C_\alpha$ and $\mathbb{G} \in C_\beta$, we conclude with (PC3) that $\mathbb{F} \in C_{\alpha\wedge\beta}$ and $\mathbb{G} \in C_{\alpha\wedge\beta}$. Therefore, if $\mathbb{F} \vee \mathbb{G}$ exists by (PC5$^\wedge$), then $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha\wedge\beta}$. $\square$

Therefore, probabilistic Cauchy spaces under the t-norm $\wedge$ are *componentwise probabilistic Cauchy spaces* [14, Definitions 2.9(2)]. The category $PChy^*$ is topological but not hereditary and quotients are not productive, not even for $* = \wedge$. However, $PChy^\wedge$ is Cartesian closed; see [14].

Let $(X, \overline{C}) \in |PChy^*|$. For $\alpha \in [0,1]$, $x \in X$, and $\mathcal{F} \in \mathcal{F}(X)$, we define $x \in q^C_\alpha(\mathbb{F})$ if $\mathbb{F} \wedge [x] \in C_\alpha$. It is easy to see that $(X, \overline{q}^C)$ is then a probabilistic limit space under the t-norm $*$. We only show (PL5). If $x \in q^C_\alpha(\mathbb{F}) \cap q^C_\beta(\mathbb{G})$, then $\mathbb{F} \wedge [x] \in C_\alpha$ and $\mathbb{G} \wedge [x] \in C_\beta$. Because $(\mathbb{F} \wedge [x]) \vee (\mathbb{G} \wedge [x])$ exists, we conclude from (PC5) that $(\mathbb{F} \wedge \mathbb{G}) \wedge [x] = (\mathbb{F} \wedge [x]) \wedge (\mathbb{G} \wedge [x]) \in C_{\alpha * \beta}$, and hence $x \in q^C_{\alpha*\beta}(\mathbb{F} \wedge \mathbb{G})$. It is also easy to see that for a Cauchy-continuous mapping $f : (X, \overline{C}) \longrightarrow (X', \overline{C}')$, the mapping $f : (X, \overline{q^C}) \longrightarrow (X', \overline{q^{C'}})$ is continuous.

A *probabilistic uniform limit space under the t-norm* $*$ [14] is a pair $(X, \overline{L})$ of a set X and a non-void family of subsets of $\mathbb{F}(X \times X)$, $\overline{L} = (L_\alpha)_{\alpha \in [0,1]}$, that satisfies the following axioms.

(PUL1) $[x] \times [x] \in L_\alpha$ for all $x \in X$ and all $\alpha \in [0,1]$;
(PUL2) $\Psi \in L_\alpha$ whenever $\Phi \le \Psi$ and $\Phi \in L_\alpha$;
(PUL3) $L_\alpha \subseteq L_\beta$ whenever $\beta \le \alpha$;
(PUL4) $L_0 = \mathbb{F}(X \times X)$;
(PUL5) $\Phi \wedge \Psi \in L_\alpha$ whenever $\Phi, \Psi \in L_\alpha$;
(PUL6) $\Phi^{-1} \in L_\alpha$ whenever $\Phi \in L_\alpha$;
(PUL7) $\Phi \circ \Psi \in L_{\alpha*\beta}$ whenever $\Phi \in L_\alpha$, $\Psi \in L_\beta$, and $\Phi \circ \Psi$ exists.

If instead of the axiom (PUL5), the weaker axiom

(wPUL5) $\Phi \wedge \Psi \in L_{\alpha*\beta}$ whenever $\Phi \in L_\alpha, \Psi \in L_\beta$

is satisfied, then we call $(X, \overline{L})$ a *weak probabilistic uniform limit space under* $*$.

A mapping $f : X \longrightarrow X'$ between two (weak) probabilistic uniform limit spaces $(X, \overline{L})$ and $(X', \overline{L}')$ is called *uniformly continuous* if $(f \times f)(L_\alpha) \subseteq L'_\alpha$ for all $\alpha \in [0,1]$. The category of all probabilistic uniform limit spaces under the t-norm $*$ with uniformly continuous mappings as morphisms is denoted by $PULIM^*$. The category of weak probabilistic uniform limit spaces under the t-norm $*$ is denoted by $WPULIM^*$.

Lemma 2.4. *Let $(X, \overline{L})$ be a probabilistic uniform limit space under the t-norm $\wedge$. Then (PUL7) is equivalent to the axiom*

(PUL7$^\wedge$) $\Phi \circ \Psi \in L_\alpha$ whenever $\Phi \in L_\alpha$, $\Psi \in L_\alpha$, and $\Phi \circ \Psi$ exists.

Proof. Similar to the proof of Lemma 2.3. □

Therefore, probabilistic uniform limit spaces under the t-norm $\wedge$ are *componentwise probabilistic uniform limit spaces* [14, Definitions 2.4(2)]. The category $PULIM^*$ is topological and not hereditary and products of quotients are quotiens. $PULIM^\wedge$ is Cartesian closed [14].

For $(X, \overline{L})$ and for $\alpha \in [0,1]$ and $\mathbb{F} \in \mathbb{F}(X)$, we define $\mathbb{F} \in C^L_\alpha$ if $\mathbb{F} \times \mathbb{F} \in L_\alpha$. Then $(X, \overline{C^L}) \in |PChy^*|$. The axioms (PC1)–(PC4) are easy. We prove (PC5). If $\mathbb{F} \in C^L_\alpha$, $\mathbb{G} \in C^L_\beta$, and $\mathbb{F} \vee \mathbb{G}$ exists, then $\mathbb{F} \times \mathbb{F} \in L_\alpha$

and $\mathbb{G} \times \mathbb{G} \in L_\beta$. By (PUL7), then $\mathbb{F} \times \mathbb{G} = (\mathbb{F} \times \mathbb{F}) \circ (\mathbb{G} \times \mathbb{G}) \in L_{\alpha*\beta}$ and also $\mathbb{G} \times \mathbb{F} \in L_{\alpha*\beta}$. Because $\alpha * \beta \leq \alpha, \beta$, we conclude with (PUL3) that $\mathbb{F} \times \mathbb{F} \in L_{\alpha*\beta}$ and $\mathbb{G} \times \mathbb{G} \in L_{\alpha*\beta}$, and hence, by (PUL5), also

$$(\mathbb{F} \wedge \mathbb{G}) \times (\mathbb{F} \wedge \mathbb{G}) = (\mathbb{F} \times \mathbb{F}) \wedge (\mathbb{F} \times \mathbb{G}) \wedge (\mathbb{G} \times \mathbb{F}) \wedge (\mathbb{G} \times \mathbb{G}) \in L_{\alpha*\beta}.$$

But this means $\mathbb{F} \wedge \mathbb{G} \in C^L_{\alpha*\beta}$.

It is also not difficult to see that for a uniformly continuous mapping $f : (X, \overline{L}) \longrightarrow (X', \overline{L}')$, the mapping $f : (X, \overline{C^L}) \longrightarrow (X', \overline{C^{L'}})$ is Cauchy-continuous.

We further define for $(X, \overline{L}) \in |PULIM^*|$ the underlying probabilistic limit space $(X, \overline{q^L})$ by $x \in q^L_\alpha(\mathbb{F})$ if $\mathbb{F} \times [x] \in L_\alpha$. It is then not difficult to show that for $\mathbb{F} \in \mathbb{F}(X)$, $q^{C^L}_\alpha(\mathbb{F}) \subseteq q^L_\alpha(\mathbb{F}) \subseteq q^{C^L}_{\alpha*\alpha}(\mathbb{F})$. Hence, if for all $\alpha \in [0,1]$ we have $\alpha * \alpha = \alpha$, then $q^L_\alpha = q^{C^L}_\alpha$. It is well known that this is only the case if $* = \wedge$.

3. Probabilistic Limit Groups and Weak Probabilistic Limit Groups

We consider now a group $(X, \cdot)$. A triple $(X, \cdot, \overline{q})$ is called a *probabilistic limit group under the t-norm* $*$, if

(PL) $(X, \overline{q})$ is a probabilistic limit space (under the t-norm $\wedge$);
(PLM) $xy \in q_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $y \in q_\beta(\mathbb{G})$;
(PLI) $x^{-1} \in q_\alpha(\mathbb{F}^{-1})$ whenever $x \in q_\alpha(\mathbb{F})$.

The category of probabilistic limit groups under the t-norm $*$ with continuous group homomorphisms as morphisms is denoted by PLG^*.

We note that in case $* = \wedge$, the axiom (PLM) is equivalent to the axiom

(PLM′) $xy \in q_\alpha(\mathbb{F} \odot \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $y \in q_\alpha(\mathbb{G})$.

A triple $(X, \cdot, \overline{q})$ is called a *weak probabilistic limit group under the t-norm* $*$ if

(PL*) $(X, \overline{q})$ is a probabilistic limit space under the t-norm $*$;
(PLM) $xy \in q_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $y \in q_\beta(\mathbb{G})$;
(PLI) $x^{-1} \in q_\alpha(\mathbb{F}^{-1})$ whenever $x \in q_\alpha(\mathbb{F})$.

The category of weak probabilistic limit groups under the t-norm $*$ with continuous group homomorphisms as morphisms is denoted by $WPLG^*$.

Example 3.1. For a group $(X, \cdot)$, the discrete space $(X, \cdot, \overline{q^d})$ and the indiscrete space $(X, \cdot, \overline{q^i})$ are probabilistic limit groups under any t-norm.

Example 3.2. Typical examples of probabilistic limit groups under a t-norm are given by normed vector spaces $(X, \|\cdot\|)$. We define, for $x \in X$

and $\epsilon > 0$, $B(x,\epsilon) = \{y \in X \;:\; \|x-y\| < \epsilon\}$, and with this, we define for $0 \leq \alpha \leq \infty$,

$$\mathbb{U}^x_\alpha = [\{B(x,\epsilon) \;:\; \epsilon \geq \alpha\}] .$$

Further, let $*$ be a strict t-norm with additive generator $S : [0,1] \longrightarrow [0,\infty]$, i.e., $\alpha * \beta = S^{-1}(S(\alpha)+S(\beta))$. Define for $0 \leq \alpha < 1$

$$x \in q_\alpha(\mathbb{F}) \iff \mathbb{F} \geq \mathbb{U}^x_{S(\alpha)},$$

and $q_1(\mathbb{F}) = \bigcap_{\alpha<1} q_\alpha(\mathbb{F})$. Then $(X,\overline{q})$ is a probabilistic limit space, as can easily be verified. Furthermore, the axiom (PLM) is satisfied. By the triangular inequality for the norm, we deduce $B(x,S(\alpha)) + B(y,S(\beta)) \subseteq B(x+y,S(\alpha)+S(\beta)) = B(x+y,S(\alpha*\beta))$. From this it follows that $\mathbb{U}^x_{S(\alpha)} \oplus \mathbb{U}^y_{S(\beta)} \geq \mathbb{U}^{x+y}_{S(\alpha*\beta)}$. Hence, if $x \in q_\alpha(\mathbb{F})$ and $y \in q_\beta(\mathbb{G})$, then $\mathbb{F} \geq \mathbb{U}^x_\alpha$ and $\mathbb{G} \geq \mathbb{U}^y_\beta$, and therefore $\mathbb{F} \oplus \mathbb{G} \geq \mathbb{U}^{x+y}_{S(\alpha*\beta)}$. But this means $x+y \in q_{\alpha*\beta}(\mathbb{F} \oplus \mathbb{G})$. The axiom (PLI) finally follows because $z \in B(x,S(\alpha))$ implies $-z \in B(-x,S(\alpha))$. Hence, $(X,+,\overline{q})$ is a probabilistic limit group under the t-norm $*$.

Example 3.3. Let λ be the Lebesgue measure on $[0,1]$ and let τ be the usual topology on $\mathbb{R}$. The set $X = \{f : [0,1] \longrightarrow \mathbb{R} : f \text{ is measurable}\}$ can be considered as a commutative group by defining $(f+g)(x) = f(x)+g(x)$ and $(-f)(x) = -f(x)$ for $x \in [0,1]$. We define, for $\alpha \in [0,1]$, $f \in X$ and $\mathbb{F} \in \mathbb{F}(X)$, $f \in q_\alpha(\mathbb{F})$ if there is $A \subseteq [0,1]$ with $\lambda(A) \leq 1-\alpha$ and $\mathbb{F}(x) \xrightarrow{\tau} f(x)$ for all $x \notin A$. Then $(X,\overline{q})$ is a probabilistic limit space under the Lukasiewicz t-norm $\alpha * \beta = (\alpha+\beta-1) \vee 0$. Richardson and Kent [16] mention that $(X,\overline{q})$ satisfies the axioms (PL1)–(PL4) and that q_1 describes convergence almost everywhere. So we need to show the axiom (PL5). Let $f \in q_\alpha(\mathbb{F})$ and $g \in q_\beta(\mathbb{G})$. Then there are $A, B \subseteq [0,1]$ such that $\lambda(A) \leq 1-\alpha$ and $\lambda(B) \leq 1-\beta$ such that for all $x \notin A$, $\mathbb{F}(x) \xrightarrow{\tau} f(x)$ and for all $x \notin B$, $\mathbb{G}(x) \xrightarrow{\tau} f(x)$. Hence, for all $x \notin A \cup B$, we have $\mathbb{F}(x) \wedge \mathbb{G}(x) \xrightarrow{\tau} f(x)$ and because $\lambda(A\cup B) \leq \lambda(A)+\lambda(B) \leq (1-\alpha)+(1-\beta) = 1-(\alpha+\beta-1)$, we see that $\mathbb{F} \wedge \mathbb{G} \in q_{\alpha*\beta}(f)$.

We finally show that it is also a weak probabilistic limit group under the Lukasiewicz t-norm. Let $f \in q_\alpha(\mathbb{F})$ and $g \in q_\beta(\mathbb{G})$. Then there are $A, B \subseteq [0,1]$ with $\lambda(A) \leq 1-\alpha$ and $\lambda(B) \leq 1-\beta$ such that for all $x \notin A$, $\mathbb{F}(x) \xrightarrow{\tau} f(x)$ and for all $x \notin B$, $\mathbb{G}(x) \xrightarrow{\tau} g(x)$. But then for all $x \notin A\cup B$, $\mathbb{F}\oplus\mathbb{G}(x) \xrightarrow{\tau} (f+g)(x)$ and because $\lambda(A\cup B) \leq \lambda(A)+\lambda(B) \leq 1-(\alpha*\beta)$, we see that $f+g \in q_{\alpha*\beta}(\mathbb{F}\oplus\mathbb{G})$. Hence, the axiom (PLM) is true. The axiom (PLI) is easy and left for the reader.

Example 3.4. Let again $X = \{f : [0,1] \longrightarrow \mathbb{R} : f \text{ is measurable}\}$ and define the equivalence relation $f \sim g$ if $f = g$ λ-almost everywhere. Richardson and Kent [16] define the following probabilistic limit structure on

$Y = X/_\sim$. For $\alpha > 0$, define $f \in q_\alpha(\mathbb{F})$ if, for all $a > 0$ and all $\epsilon < \alpha$, there is $F \in \mathbb{F}$ such that for all $g \in F$, we have $\lambda(\{x \in [0,1] : |f(x) - g(x)| < a\}) \geq \epsilon$. Further, define q_0 as the indiscrete topology on Y. Then $(Y, \overline{q})$ is a probabilistic limit space and Richardson and Kent [16] point out that q_1 is convergence in probability. We define the following group operations. For $f, g \in X$, we define $[f] + [g] = [f + g]$ and $-[f] = [-f]$. We show that $(Y, +, \overline{q})$ is a probabilistic limit group under the Lukasiewicz t-norm $\alpha * \beta = (\alpha + \beta - 1) \vee 0$. In order to show the axiom (PLM), let $f \in q_\alpha(\mathbb{F})$ and $g \in q_\beta(\mathbb{G})$. Further, let $a > 0$ and $0 < \epsilon < \alpha * \beta = \alpha + \beta - 1$. Then there are $\epsilon_1 < \alpha$ and $\epsilon_2 < \beta$ with $\epsilon_1 + \epsilon_2 - 1 = \epsilon$. There are $F \in \mathbb{F}$ and $G \in \mathbb{G}$ such that for all $h \in F$ and $k \in G$, $\lambda(\{x : |h(x) - f(x)| < \frac{a}{2}\}) \geq \epsilon_1$ and $\lambda(\{x : |k(x) - g(x)| < \frac{a}{2}\}) \geq \epsilon_2$. From $|h(x) - f(x)| < \frac{a}{2}$ and $|k(x) - g(x)| < \frac{a}{2}$, we conclude $|(h + k)(x) - (f + g)(x)| < a$ for all $h \in F$ and all $k \in G$. Hence, for all $p \in F \oplus G$, we have $|p(x) - (f + g)(x)| < a$. Now for all $h \in F$, $k \in G$, and $p \in F \oplus G$, we have

$$\{x : |h(x) - f(x)| < \frac{a}{2}\} \cap \{x : |k(x) - g(x)| < \frac{a}{2}\} \subseteq$$
$$\{x : |p(x) - (f + g)(x)| < a\}$$

and hence,

$$\begin{aligned} & \lambda(\{x : |p(x) - (f + g)(x)| < a\}) \\ \geq\ & \lambda(\{x : |h(x) - f(x)| < \frac{a}{2}\}) + \lambda(\{x : |k(x) - g(x)| < \frac{a}{2}\}) - 1 \\ \geq\ & \epsilon_1 + \epsilon_2 - 1 \,=\, \epsilon. \end{aligned}$$

Hence, $f + g \in q_{\alpha * \beta}(\mathbb{F} \oplus \mathbb{G})$. The axiom (PLI) is left for the reader.

Example 3.5. We finally present an example of a probabilistic limit group under the minimum t-norm. To this end, we consider for two probabilistic limit groups under the minimum t-norm, $(X, \cdot_X, \overline{q}_X)$ and $(Y, \cdot_Y, \overline{q}_Y)$, the set $C(X, Y)$ of continuous mappings from $(X, \overline{q}_X)$ to $(Y, \overline{q}_Y)$. This set carries a natural function space structure by defining for $\Phi \in \mathbb{F}(C(X, Y))$, $f \in C(X, Y)$, and $\alpha \in [0, 1]$, $f \in q^\alpha_{C(X,Y)}(\Phi)$ if, for all $\mu \leq \alpha$, we have that $f(x) \in q^\mu_Y(ev(\Phi \times \mathbb{F}))$ whenever $x \in q^\mu_X(\mathbb{F})$. Here, $ev : C(X, Y) \times X \longrightarrow Y$ is the evaluation mapping $ev(f, x) = f(x)$. In fact, this function space structure is called the *probabilistic limit structure of continuous convergence* and makes the category $PLIM^\wedge$ cartesian closed; see [14]. We define the following group operations on $C(X, Y)$. For $f, g \in C(X, Y)$, we define $fg(x) = f(x)g(x)$ and $f^{-1}(x) = (f(x))^{-1}$. It is not difficult to show that with these definitions, $fg \in C(X, Y)$ and $f^{-1} \in C(X, Y)$, and that $C(X, Y)$ then becomes a group. We show that with the probabilistic limit structure of continuous convergence, $C(X, Y)$ is a probabilistic limit group under the minimum t-norm. In order to show the axiom (PLM′), let $f \in q^{\alpha}_{C(X,Y)}(\Phi)$ and $g \in q^{\alpha}_{C(X,Y)}(\Psi)$. Further, let $x \in q^{\mu}_{X}(\mathbb{F})$ for $\mu \leq \alpha$. Then $f(x) \in q^{\mu}_{Y}(ev(\Phi \times \mathbb{F}))$ and $g(x) \in q^{\mu}_{Y}(ev(\Psi \times \mathbb{F}))$. Hence, by (PLM′) for the probabilistic limit group Y, we conclude

$$fg(x) = f(x)g(x) \in q^{\mu}_{Y}(ev(\Phi \times \mathbb{F}) \odot ev(\Psi \times \mathbb{F})).$$

It is not difficult to prove that $ev(\Phi\times\mathbb{F})\odot ev(\Psi\times\mathbb{F}) \leq ev((\Phi\odot\Psi)\times\mathbb{F}))$, and hence the axiom (PLM′) follows. For (PLI) we note that $ev(\Phi^{-1} \times \mathbb{F}) = (ev(\Phi \times \mathbb{F}))^{-1}$ and leave the details for the reader.

It is possibly worthwhile to note that even if we restrict this construction on the set $C_h(X, Y)$ of continuous group homomorphisms from X to Y, the category $PLG^{\wedge}$ does not become cartesian closed with the probabilistic limit structure of continuous convergence. This is due to the fact that the evaluation mapping is not a group homomorphism.

Let $(X, \overline{q})$ be a probabilistic limit space under the t-norm $*$. We define the following structure on the product $X \times X$. For $\Phi \in \mathbb{F}(X \times X)$, $\alpha \in [0, 1]$, and $(x, y) \in X \times X$, we define

$$(x, y) \in (q \otimes q)_{\alpha}(\Phi) \iff$$
$$\exists\, \alpha_1, \alpha_2 \in [0, 1] \text{ s.t. } \alpha_1 * \alpha_2 \geq \alpha,\ x \in q_{\alpha_1}(pr_1(\Phi)), y \in q_{\alpha_2}(pr_2(\Phi)).$$

It is easy to see that $(X \times X, \overline{q \otimes q})$ is a probabilistic limit space under the t-norm $*$. Furthermore, we have the following result.

Lemma 3.6. *Let $(X, \overline{q}) \in |PLIM^*|$. The following are equivalent.*

(1) *The axiom (PLM) is true.*
(2) *The multiplication $m : (X \times X, \overline{q \otimes q}) \longrightarrow (X, \overline{q})$ is continuous.*

Proof. First, let the axiom (PLM) be true and let $\Phi \in \mathbb{F}(X \times X)$, $\alpha \in [0, 1]$, and $(x, y) \in (q \otimes q)_{\alpha}(\Phi)$. Then there are $\alpha_1, \alpha_2 \in [0, 1]$ such that $\alpha_1 * \alpha_2 \geq \alpha$, $x \in q_{\alpha_1}(pr_1(\Phi))$, and $y \in q_{\alpha_2}(pr_2(\Phi))$. But then $m(x, y) = xy \in q_{\alpha_1*\alpha_2}(pr_1(\Phi) \otimes pr_2(\Phi)) = q_{\alpha_1*\alpha_2}(m(pr_1(\Phi) \times pr_2(\Phi))$. Because $\alpha_1 * \alpha_2 \geq \alpha$ and $pr_1(\Phi) \times pr_2(\Phi) \leq \Phi$, we obtain with (PL2) and (PL3) $xy \in q_{\alpha}(m(\Phi))$. Hence, the multiplication is continuous. Conversely, assume that the multiplication is continuous and let $x \in q_{\alpha}(\mathbb{F})$ and $y \in q_{\beta}(\mathbb{G})$. Then $(x, y) \in (q \otimes q)_{\alpha*\beta}(\mathbb{F} \times \mathbb{G})$, and hence, by continuity of m, $xy \in q_{\alpha*\beta}(m(\mathbb{F} \times \mathbb{G})) = q_{\alpha*\beta}(\mathbb{F} \otimes \mathbb{G})$. □

It is further clear that the axiom (PLI) is equivalent to the continuity of the mapping $inv : (X, q) \longrightarrow (X, q)$, $x \longmapsto x^{-1}$. Hence, we can characterize weak probabilistic limit groups under the t-norm $*$ as groups with a probabilistic limit structure under the t-norm $*$ where multiplication and taking inverses are continuous mappings. It is not clear how to define suitable product structures for (non-weak) probabilistic limit groups under the t-norm $*$. But, for the case $* = \wedge$, we note that $\overline{q \otimes q}$ is the same as the product structure $\overline{q \times q}$ on $X \times X$. Hence, probabilistic limit groups under the minimum t-norm can be characterized as groups with a probabilistic limit structure for which the group operations are continuous.

Theorem 3.7. *The categories PLG^* and $WPLG^*$ are topological over GRP.*

Proof. Let $(f_j : (X, \cdot) \longrightarrow (X_j, \cdot_j, \overline{q^j})_{j\in J}$ be a source where the mappings f_j are group homomorphisms. The initial probabilistic limit structure (under the t-norm $*$) $\overline{q}$ on X is given in [14]:

$$x \in q_\alpha(\mathbb{F}) \iff \forall\, j \in J\ : f_j(x) \in q^j_\alpha(f_j(\mathbb{F})).$$

If $x \in q_\alpha(\mathbb{F})$ and $y \in q_\beta(\mathbb{G})$, then $f_j(x) \in q^j_\alpha(f_j(\mathbb{F}))$ and $f_j(y) \in q^j_\beta(\mathbb{G})$ for all $j \in J$. Hence, by (PLM) for the spaces $(X_j, \cdot_j, \overline{q^j})$, $f_j(xy) = f_j(x)f_j(y) \in q^j_{\alpha*\beta}(f_j(\mathbb{F}) \odot f_j(\mathbb{G})) = q^j_{\alpha*\beta}(f_j(\mathbb{F} \odot \mathbb{G}))$ for all $j \in J$. Hence, $xy \in q_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$, and therefore the space $(X, \cdot, \overline{q})$ satisfies (PLM). For (PLI), we note that, for all $j \in J$, we have $f_j(x^{-1}) = (f_j(x))^{-1}$ and $f_j(\mathbb{F}^{-1}) = (f_j(\mathbb{F}))^{-1}$. Hence, if $x \in q_\alpha(\mathbb{F})$, then, for all $j \in J$, we obtain $f_j(x^{-1}) = (f_j(x))^{-1} \in q^j_\alpha((f_j(\mathbb{F}))^{-1}) = q^j_\alpha(f_j(\mathbb{F}^{-1}))$. Therefore, $x^{-1} \in q_\alpha(\mathbb{F}^{-1})$ and (PLI) is satisfied for $(X, \cdot, \overline{q})$. $\square$

Probabilistic limit groups are homogeneous in the following sense.

Lemma 3.8. *Let $(X, \cdot, \overline{q}) \in |WPLG^*|$, $\alpha \in [0, 1]$, $\mathbb{F} \in \mathbb{F}(X)$, and $x \in X$. Then $x \in q_\alpha(\mathbb{F})$ if and only if $e \in q_\alpha([x^{-1}] \odot \mathbb{F})$ if and only if $e \in q_\alpha(\mathbb{F} \odot [x^{-1}])$.*

Proof. Let $x \in q_\alpha(\mathbb{F})$. By (PL1), we have $x^{-1} \in q_1([x^{-1}])$, and hence, by (PLM), $e = x^{-1}x \in q_{\alpha*1}([x^{-1}] \odot \mathbb{F}) = q_\alpha([x^{-1}] \odot \mathbb{F})$. Conversely, if $e \in q_\alpha([x^{-1}] \odot \mathbb{F})$, then because $x \in q_1([x])$, we conclude with (PLM) $x = xe \in q_{1*\alpha}([x] \odot ([x^{-1}] \odot \mathbb{F})) = q_\alpha(\mathbb{F})$. $\square$

We can use homogeneity and characterize a probabilistic limit group solely in terms of convergence of filters to the unit element e.

Lemma 3.9. *Let $(X, \cdot)$ be a group. Then $(X, \cdot, \overline{q}) \in |PLG^*|$ if and only if the following axioms are satisfied.*

- (PLH) *$x \in q_\alpha(\mathbb{F})$ if and only if $e \in q_\alpha([x^{-1}] \odot \mathbb{F})$ if and only if $e \in q_\alpha(\mathbb{F} \odot [x^{-1}])$;*
- (PL1e) *$e \in q_\alpha([e])$ for all $\alpha \in [0, 1]$;*
- (PL2e) *$\mathbb{F} \le \mathbb{G}$ and $e \in q_\alpha(\mathbb{F})$ imply $e \in q_\alpha(\mathbb{G})$;*
- (PL3e) *$e \in q_\alpha(\mathbb{F}) \cap q_\alpha(\mathbb{G})$ implies $e \in q_\alpha(\mathbb{F} \wedge \mathbb{G})$;*
- (PLMe) *$e \in q_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$ whenever $e \in q_\alpha(\mathbb{F}) \cap q_\beta(\mathbb{G})$;*
- (PLIe) *$e \in q_\alpha(\mathbb{F}^{-1})$ whenever $e \in q_\alpha(\mathbb{F})$.*

Proof. The necessity of the stated axioms is obvious. We check the sufficiency. By using Lemma 1.1 and the given conditions, we can easily show that $(X, \overline{q})$ is a probabilistic limit space under the minimum t-norm $\wedge$. We only show (PLM). Let $x \in q_\alpha(\mathbb{F})$ and $y \in q_\beta(\mathbb{G})$. By homogeneity, then $e \in q_\alpha([x^{-1}] \odot \mathbb{F})$ and $e \in q_\beta(\mathbb{G} \odot [y^{-1}])$. Hence, by (PLMe), then also $e \in q_{\alpha*\beta}(([x^{-1}] \odot \mathbb{F}) \odot (\mathbb{G} \odot [y^{-1}]))$. Again, by homogeneity, then $x \in q_{\alpha*\beta}((\mathbb{F} \odot \mathbb{G}) \odot [y^{-1}])$ and, using homogeneity once more, $e \in q_{\alpha*\beta}((\mathbb{F} \odot \mathbb{G}) \odot [y^{-1}] \odot [x^{-1}]) = q_{\alpha*\beta}((\mathbb{F} \odot \mathbb{G}) \odot [(xy)^{-1}])$. But this means $xy \in q_{\alpha*\beta}((\mathbb{F} \odot \mathbb{G}))$. The axiom (PLI) can be shown similarly. $\square$

Lemma 3.10. *Let $(X, \cdot)$ be a group. Then $(X, \cdot, \overline{q}) \in |WPLG^*|$ if and only if the following axioms are satisfied.*

(PLH) $x \in q_\alpha(\mathbb{F})$ *if and only if* $e \in q_\alpha([x^{-1}] \odot \mathbb{F})$ *if and only if* $e \in q_\alpha(\mathbb{F} \odot [x^{-1}])$;

(PL1e) $e \in q_\alpha([e])$ *for all* $\alpha \in [0, 1]$;

(PL2e) $\mathbb{F} \le \mathbb{G}$ *and* $e \in q_\alpha(\mathbb{F})$ *imply* $e \in q_\alpha(\mathbb{G})$;

(PL3e) $e \in q_\alpha(\mathbb{F}) \cap q_\beta(\mathbb{G})$ *implies* $e \in q_{\alpha*\beta}(\mathbb{F} \wedge \mathbb{G})$;

(PLMe) $e \in q_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$ *whenever* $e \in q_\alpha(\mathbb{F}) \cap q_\beta(\mathbb{G})$;

(PLIe) $e \in q_\alpha(\mathbb{F}^{-1})$ *whenever* $e \in q_\alpha(\mathbb{F})$.

Proof. This is similar to that of Lemma 3.9. $\square$

4. Isomorphy of Categories of Probabilistic Limit Groups Under t-Norms

In order to show that for certain classes of t-norms, the categories of probabilistic limit groups under t-norms of such a class are all isomorphic, we adapt and generalize an idea of [2] and introduce a new category.

For $\omega \in [0, \infty]$, an *ω-limit tower group* $(X, \cdot, \overline{p})$ is a group $(X, \cdot)$ together with family of mappings $\overline{p} = (p_\epsilon : \mathbb{F}(X) \longrightarrow P(X))_{\epsilon \in [0,\infty]}$ such that the following axioms are satisfied.

(LT1) $x \in p_\epsilon([x])$ for all $\epsilon \in [0, \infty]$, $x \in X$;
(LT2) $p_\epsilon(\mathbb{F}) \subseteq p_\epsilon(\mathbb{G})$ whenever $\mathbb{F} \le \mathbb{G}$;
(LT3) $p_\delta(\mathbb{F}) \subseteq p_\epsilon(\mathbb{F})$ whenever $\delta \le \epsilon$;
(LT4) $p_\epsilon(\mathbb{F}) = X$ whenever $\epsilon \ge \omega$;
(LT5) $x \in p_\epsilon(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in p_\epsilon(\mathbb{F})$ and $x \in p_\epsilon(\mathbb{G})$;
(LTM) $xy \in p_{\epsilon+\delta}(\mathbb{F} \odot \mathbb{G})$ whenever $x \in p_\epsilon(\mathbb{F})$ and $y \in p_\delta(\mathbb{G})$;
(LTI) $x^{-1} \in p_\epsilon(\mathbb{F}^{-1})$ whenever $x \in p_\epsilon(\mathbb{F})$, for all $\epsilon, \delta \in [0, \infty]$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$.

If, instead of (LT5), the weaker condition

(LT5*) $x \in p_{\epsilon+\delta}(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in p_\epsilon(\mathbb{F}) \cap p_\delta(\mathbb{G})$

is satisfied, then $(X, \cdot, \overline{q})$ is called a *weak ω-limit tower group*.

If $\omega = \infty$, then spaces $(X, \overline{p})$, where $\overline{p}$ satisfies the axioms (LT1)–(LT5), are called *limit tower spaces* (see [2]). They require additionally a left-continuity condition.

A mapping $f : X \longrightarrow X'$ between two (weak) ω-limit tower groups $(X, \cdot, \overline{q})$ and $(X', \cdot', \overline{q'})$ is called *continuous* if $f(p_\epsilon(\mathbb{F})) \subseteq p'_\epsilon(f(\mathbb{F}))$ for all $\epsilon \in [0, \infty]$ and all $\mathbb{F} \in \mathbb{F}(X)$. The category whose objects are the ω-limit tower groups and continuous group homomorphisms as morphisms is denoted by LTG_ω; the category whose objects are the weak ω-limit tower groups and continuous group homomorphisms as morphisms is denoted by $WLTG_\omega$.

We consider now an Archimedean t-norm with continuous, strictly decreasing additive generator $S : [0,1] \longrightarrow [0, \infty]$ with $S(1) = 0$, such that $\alpha * \beta = S^{(-1)}(S(\alpha) + S(\beta))$ with the pseudo-inverse

$$S^{(-1)}(u) = \begin{cases} v & \text{if } S(v) = u \\ 0 & \text{if } u > S(0) \end{cases} .$$

We define for $(X, \cdot, \overline{q})$ and for the t-norm $*$ the $S(0)$-limit tower group $(X, \cdot, \Psi_S(\overline{q}))$ by $\Psi_S(\overline{q})_\epsilon(\mathbb{F}) = q_{S^{(-1)}(\epsilon)}(\mathbb{F})$.

Lemma 4.1. *For $(X, \cdot, \overline{q}) \in |PLG^*|$, we have $(X, \cdot, \Psi_S(\overline{q})) \in |LTG_{S(0)}|$.*

Proof. The axioms (LT1), (LT2), (LT3), and (LT5) are easy; see [2]. (LT4) follows because, from $\epsilon \geq S(0)$, we conclude $S^{(-1)}(\epsilon) = 0$. We prove (LTM). Let $x \in \Psi_S(\overline{q})_\epsilon(\mathbb{F})$ and $y \in \Psi_S(\overline{q})_\delta(\mathbb{G})$. Then $x \in q_{S^{(-1)}(\epsilon)}(\mathbb{F})$ and $y \in q_{S^{(-1)}(\delta)}(\mathbb{G})$, and hence, by (PLM), $xy \in q_{S^{(-1)}(\epsilon)*S^{(-1)}(\delta)}(\mathbb{F} \odot \mathbb{G})$. If $\epsilon+\delta \geq S(0)$, then $S^{(-1)}(\epsilon+\delta) = 0$, and therefore $xy \in q_{S^{(-1)}(\epsilon+\delta)}(\mathbb{F}\odot\mathbb{G}) = \Psi_S(\overline{q})_{\epsilon+\delta}(\mathbb{F}\odot\mathbb{G})$. If $\epsilon+\delta < S(0)$, then both $\epsilon, \delta < S(0)$ and then $S^{(-1)}(\epsilon) * S^{(-1)}(\delta) = S^{(-1)}(\epsilon + \delta)$, and therefore also $xy \in q_{S^{(-1)}(\epsilon+\delta)}(\mathbb{F} \odot \mathbb{G}) = \Psi_S(\overline{q})_{\epsilon+\delta}(\mathbb{F} \odot \mathbb{G})$. The axiom (LTI) is easy and left for the reader. $\square$

Lemma 4.2. *We have $(X, \cdot, \Psi_S(\overline{q})) \in |WLTG_{S(0)}|$ for $(X, \cdot, \overline{q}) \in |WPLG^*|$.*

Proof. We need to check the axiom (LT5*), but this is similar to the proof of (LTM) above. $\square$

It follows easily from this that $\Psi_S : PLG^* \longrightarrow LTG_{S(0)}$, $(X, \cdot, \overline{q}) \longmapsto (X, \cdot, \Psi_S(\overline{q}))$, $f \longmapsto f$ is a functor; see [2].

For an $S(0)$-limit tower group $(X, \cdot, \overline{p})$, we define now a probabilistic limit group under the t-norm $*$, $(X, \cdot, \Phi_S(\overline{p}))$ by $\Phi_S(\overline{p})_\alpha(\mathbb{F}) = p_{S(\alpha)}(\mathbb{F})$.

Lemma 4.3. *For $(X, \cdot, \overline{p}) \in |LTG_{S(0)}|$, we have $(X, \cdot, \Phi_S(\overline{p})) \in |PLG^*|$.*

Proof. We only prove the axiom (PLM) and leave the others to the reader. Let $x \in \Phi_S(\overline{p})_\alpha(\mathbb{F})$ and $y \in \Phi_S(\overline{p})_\beta(\mathbb{G})$. Then $x \in p_{S(\alpha)}(\mathbb{F})$ and $y \in p_{S(\beta)}(\mathbb{G})$ and by (LTM), then $xy \in p_{S(\alpha)+S(\beta)}(\mathbb{F} \odot \mathbb{G})$. By definition of the t-norm, we have $S(\alpha * \beta) = S(S^{(-1)}(S(\alpha) + S(\beta)))$. If $S(\alpha) + S(\beta) \leq S(0)$, then $S(\alpha * \beta) = S(\alpha) + S(\beta)$, and hence $xy \in p_{S(\alpha*\beta)}(\mathbb{F} \odot \mathbb{G}) = \Phi_S(\overline{p})_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$. If $S(\alpha) + S(\beta) > S(0)$, then $S(\alpha * \beta) = S(0)$ and $\Phi_S(\overline{p})_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G}) = p_{S(0)}(\mathbb{F} \odot \mathbb{G}) = X$, and hence, also in this case, $xy \in \Phi_S(\overline{p})_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$. □

Lemma 4.4. *For $(X, \cdot, \overline{p}) \in |WLTG_{S(0)}|$, we have $(X, \cdot, \Phi_S(\overline{p})) \in |WPLG^*|$.*

It is again not difficult to show that $\Phi_S : LTG_{S(0)} \longrightarrow PLG^*$, $(X, \cdot, \overline{p}) \longmapsto (X, \cdot, \Phi_S(\overline{p}))$, $f \longmapsto f$ is a functor.

Now we note that $(\Phi_S \circ \Psi_S \overline{q})_\alpha = q_{S^{(-1)}(S(\alpha))} = q_\alpha$. If $\epsilon \leq S(0)$, then $S(S^{(-1)}(\epsilon)) = \epsilon$, and hence $(\Psi_S \circ \Phi_S \overline{p})_\epsilon = p_{S(S^{(-1)}(\epsilon))} = p_\epsilon$. If $\epsilon > S(0)$, then trivially $(\Psi_S \circ \Phi_S \overline{p})_\epsilon = X = p_\epsilon$. Hence, both functors, Ψ_S and Φ_S, are isomorphism functors and we can state the following result.

Theorem 4.5. *The categories PLG^* and $LTG_{S(0)}$ are isomorphic and the categories $WPLG^*$ and $WLTG_{S(0)}$ are isomorphic.*

Now we note that, for any strict t-norm, there is an additive generator S and $S(0) = \infty$. For any nilpotent t-norm, there is an additive generator S and we may assume $S(0) = 1$. Hence, we can deduce the following theorem.

Theorem 4.6. (1) *For strict t-norms, all categories PLG^* are isomorphic.*

(2) *For strict t-norms, all categories $WPLG^*$ are isomorphic.*

(3) *For nilpotent t-norms, all categories PLG^* are isomorphic.*

(4) *For nilpotent t-norms, all categories $WPLG^*$ are isomorphic.*

Hence, it is sufficient to study probabilistic limit groups under Archimedean t-norms only for "prototype" t-norms, e.g., for the product t-norm in the strict case and for the Lukasiewicz t-norm in the nilpotent case. We would like to point out that the examples in section 3 dealt with the "prototype t-norms" minimum t-norm, Lukasiewicz t-norm, and strict t-norms (where we could have chosen the product t-norm).

5. Probabilistic Uniformization of Probabilistic Limit Groups

Let $(X, \cdot, \overline{q})$ be a probabilistic limit group under the t-norm $*$. We further define the mapping $\omega_l : X \times X \longrightarrow X$, $(x, y) \longmapsto x^{-1}y$. The following lemma is not difficult to prove.

Lemma 5.1. *Let $(X, \cdot)$ and $(X', \cdot')$ be groups with unit elements e and e', respectively. Then for $x \in X$; $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$; $\Phi, \Psi \in \mathbb{F}(X \times X)$; and a group homomorphism $f : X \longrightarrow X'$, we have*

(1) $\omega_l([x] \times [x]) = [e]$;
(2) $\omega_l(\mathbb{F} \times \mathbb{G}) = \mathbb{F}^{-1} \odot \mathbb{G}$;
(3) $\omega_l(\Phi^{-1}) = \omega_l(\Phi^{-1})$;
(4) $\omega_l(\Phi) \odot \omega_l(\Psi) \leq \omega_l(\Phi \circ \Psi)$;
(5) $f(\omega_l(\Phi)) = \omega_l((f \times f)(\Phi))$.

We define $\overline{L^q}$ by $\Phi \in L^q_\alpha$ if $e \in q_\alpha(\omega_l(\Phi))$.

Lemma 5.2. (1) *If* $(X, \cdot, \overline{q}) \in |PLG^*|$, *then* $(X, \overline{L^q}) \in |PULIM^*|$.
(2) *If* $(X, \cdot, \overline{q}) \in |WPLG^*|$, *then* $(X, \overline{L^q}) \in |WPULIM^*|$.

Proof. (1) We have by Lemma 5.1(1) that $e \in q_\alpha(\omega_l([x] \times [x]))$, and hence $[x] \times [x] \in L^q_\alpha$ and (PUL1) is true. (PUL2), (PUL3), and (PUL4) are easy and left for the reader. For (PUL5), let $\Phi, \Psi \in L^q_\alpha$. Then $e \in q_\alpha(\omega_l(\Phi)) \cap q_\alpha(\omega_l(\Psi))$, and hence $e \in q_\alpha(\omega_l(\Phi) \wedge \omega_l(\Psi)) = q_\alpha(\omega_l(\Phi \wedge \Psi))$. Therefore, we conclude that $\Phi \wedge \Psi \in L^q_\alpha$. (PUL6) follows directly from Lemma 5.1(3) and, for (PUL7), we use Lemma 5.1(4).

(2) We only need to prove the axiom (wPUL5). If $\Phi \in L^q_\alpha$ and $\Psi \in L^q_\beta$, then by (wPL5) $e \in q_{\alpha * \beta}(\omega_l(\Phi \wedge \Psi))$, and hence $\Phi \wedge \Psi \in L^q_{\alpha * \beta}$. □

Lemma 5.3. *Let* $(X, \cdot, \overline{q}), (X', \cdot', \overline{q'}) \in |WPLG^*|$ *and let* $f : X \longrightarrow X'$ *be a group homomorphism. The following are equivalent.*

(1) $f : (X, \overline{q}) \longrightarrow (X', \overline{q'})$ *is continuous.*
(2) $f : (X, \overline{L^q}) \longrightarrow (X', \overline{L^{q'}})$ *is uniformly continuous.*

Proof. First, let $f : (X, \overline{q}) \longrightarrow (X', \overline{q'})$ be continuous. If $\Phi \in L^q_\alpha$, then $e \in q_\alpha(\omega_l(\Phi))$, and hence $e' = f(e) \in q'_\alpha(f(\omega_l(\Phi))) = q'_\alpha(\omega_l((f \times f)(\Phi)))$. Hence, $(f \times f)(\Phi) \in L^{q'}_\alpha$, and therefore $f : (X, \overline{L^q}) \longrightarrow (X', \overline{L^{q'}})$ is uniformly continuous. Conversely, assume that $f : (X, \overline{L^q}) \longrightarrow (X', \overline{L^{q'}})$ is uniformly continuous. If $x \in q^L_\alpha(\mathbb{F})$, then $\mathbb{F} \times [x] \in L_\alpha$, and hence $f(\mathbb{F}) \times [f(x)] = f(\mathbb{F}) \times f([x]) = (f \times f)(\mathbb{F} \times \mathbb{F}) \in L'_\alpha$. But this means $f(x) \in q^{L'}_\alpha(f(\mathbb{F}))$, and hence f is continuous. □

Lemma 5.4. *Let* $(X, \cdot, \overline{q}) \in |WPLG^*|$. *Then* $\overline{q^{L^q}} = \overline{q}$.

Proof. We have $x \in q^{L^q}_\alpha(\mathbb{F})$ if and only if $\mathbb{F} \times [x] \in L^q_\alpha$ if and only if $[x] \times \mathbb{F} \in L^q_\alpha$ if and only if $e \in q_\alpha(\omega_l([x] \times \mathbb{F})) = q_\alpha([x^{-1}] \odot \mathbb{F})$ if and only if $x \in q_\alpha(\mathbb{F})$. □

For a probabilistic limit group $(X, \cdot, \overline{q})$, we further define the probabilistic Cauchy space $(X, \overline{C^q})$ by $\mathbb{F} \in C^q_\alpha$ if $e \in q_\alpha(\mathbb{F}^{-1} \odot \mathbb{F})$. This definition generalizes in an obvious way the corresponding definition in the category of limit groups; see [3], [7]. By Lemma 5.1(2), we immediately see that $\mathbb{F} \in C^q_\alpha$ if and only if $\mathbb{F} \times \mathbb{F} \in L^q_\alpha$ if and only if $\mathbb{F} \in C^{L^q}_\alpha(\mathbb{F})$. In this sense, the Cauchy filters of the probabilistic limit group and of its probabilistic uniformization are the same. We obtain from this the following result.

Lemma 5.5. *Let* $(X, \cdot, \overline{q}) \in |LPG^*|$. *Then, for all* $\alpha \in [0, 1]$, $C^q_\alpha = C^{L^q}_\alpha$, *and hence* $(X, \overline{C^q}) \in |PChy^*|$.

6. Conclusions

We defined probabilistic limit groups under t-norms, gave examples of such spaces, and studied their basic properties. We showed that a probabilistic limit group allows a natural probabilistic uniformization. We further showed that it is sufficient to study probabilistic limit groups under prototype t-norms in the strict and in the nilpotent cases. It is known that probabilistic limit spaces under Archimedean t-norms can be identified with certain limit tower spaces (see [2] and [15]). These latter spaces have a close connection to approach spaces (see [2] and [10]). Hence, the study of probabilistic limit groups under t-norms may lead to further insight into the theory of approach (limit) groups that was initiated in [11].

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