PHYSICS 404
3rd HOMEWORK – FALL 2019
Prof. V. Lempesis

Hand in: Thursday 31st of October 2019

1. (i) Find the general solution of the following differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{9}{x^2}\right)u=0\ .$$

(ii)Find the general solution of the following differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{1}{25x^2}\right)u=0\ .$$

(ii) Find the general solution of the following differential equation:

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(4x^2-16\right)u=0$$

Hint: See the discussion we did in Q3 of Handout 6.

(3 marks)

Solution:

(i) $\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{9}{x^2}\right)u=0\Rightarrow\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{3^2}{x^2}\right)u=0$

But 3 is an integer thus the general solution is given as

$$u(x)=AJ_3(x)+BN_3(x)$$

(ii)

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{1}{25x^2}\right)u=0\Rightarrow\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{(1/5)^2}{x^2}\right)u=0$$

$$u(x)=AJ_{1/5}(x)+BJ_{-1/5}(x)$$

(iii) As we have discussed the general solution is

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(4x^2-16\right)u=0\Rightarrow x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(2^2x^2-4^2\right)u=0$$

$$u(x)=AJ_4(2x)+BN_4(2x)$$

2. Prove that $N_{\nu-1} - N_{\nu+1} = 2N'_{\nu}$. (Hint: Q8 of Handout 6 could be very helpful)

(7 marks)

Solution

In Q. 8 we have shown that

$$N_{\nu+1}(x) = \frac{J_{\nu+1}(x)cos[(\nu+1)\pi] - J_{-(\nu+1)}(x)}{sin[(\nu+1)\pi]}$$

$$N_{\nu-1}(x) = \frac{J_{\nu-1}(x)cos[(\nu-1)\pi] - J_{-(\nu-1)}(x)}{sin[(\nu-1)\pi]}$$

Thus we have:

$$N_{\nu-1}(x) - N_{\nu+1}(x) =$$

$$\frac{J_{\nu-1}(x)cos[(\nu-1)\pi] - J_{-(\nu-1)}(x)}{sin[(\nu-1)\pi]} - \frac{J_{\nu+1}(x)cos[(\nu+1)\pi] - J_{-(\nu+1)}(x)}{sin[(\nu+1)\pi]} =$$

$$-\frac{-J_{\nu-1}(x)cos[\nu\pi] - J_{-(\nu-1)}(x)}{sin[\nu\pi]} + \frac{-J_{\nu+1}(x)cos[\nu\pi] - J_{-(\nu+1)}(x)}{sin[\nu\pi]} =$$

$$\frac{-J_{\nu+1}(x)cos[\nu\pi] - J_{-(\nu+1)}(x) + J_{\nu-1}(x)cos[\nu\pi] + J_{-(\nu-1)}(x)}{sin[\nu\pi]} =$$

$$\frac{[J_{\nu-1}(x) - J_{\nu+1}(x)]cos[\nu\pi] - [J_{-(\nu+1)}(x) - J_{-(\nu-1)}(x)]}{sin[\nu\pi]} =$$

$$\frac{2J'_{\nu}(x)cos[\nu\pi] - 2J'_{-\nu}(x)}{sin[\nu\pi]} = 2N'_{\nu}(x)$$

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Comment [1]: Some of you tried to prove the corresponding relation for $K\nu$ functions which involves the modified Bessel functions $I\nu$. I did not asked this.

3. We have seen that a **function** $f(x)$ is expressed as a Bessel series: $f(x) = \sum_{n=1} b_n J_m(\alpha_{mn} x)$ with α_{mn} the nth root of J_m and $0 < x < 1$ where S $b_n = \frac{2}{J^2_{m+1}(\alpha_{mn})}\int_0^1 f(x) J_m(\alpha_{mn} x)dx$.

Assume that we have the function $f(x)$=x(1-x) and m =1. Find the first three coefficients b_1 , b_2 , b_3 of the expansion series $f(x)=\sum_{n=1} b_n J_1\left(\alpha_{1n}x\right)$

Hint:

a) To evaluate integrals use the Wolfram online integrator at:
http://www.wolframalpha.com/calculators/integral-calculator/

b) To find the zeros of the Bessel functions you can use any internet source. For example at
http://mathworld.wolfram.com/BesselFunctionZeros.html

c) To find values of Bessel functions you can use the follow link
http://keisan.casio.com/exec/system/1180573474

(10 marks)

Solution

The quantities which we are asked to find are:

$$b_1=\frac{2}{J_2^2\left(\alpha_{11}\right)}\int_0^1 x\left(1-x\right)J_1\left(\alpha_{11}x\right)dx$$

$$b_2=\frac{2}{J_2^2\left(\alpha_{12}\right)}\int_0^1 x\left(1-x\right)J_1\left(\alpha_{12}x\right)dx$$

$$b_3=\frac{2}{J_2^2\left(\alpha_{13}\right)}\int_0^1 x\left(1-x\right)J_1\left(\alpha_{13}x\right)dx$$

The first three zeros of the Bessel function J_1(x) are taken from the link b): α_{11}=3.8317, α_{12}=7.0156, α_{13}=10.1735.

The values of the Bessel function J_2(x) at the zeroes of the Bessel function J_1(x) are taken from the link c) :

$J_2(\alpha_{11}) = 0.403$, $J_2(\alpha_{12}) = -0.3$, $J_2(\alpha_{13}) = 0.25$.

Using the link a) we get:

$$b_1=\frac{2}{J_2^2\left(\alpha_{11}\right)}\int_0^1 x\left(1-x\right)J_1\left(\alpha_{11}x\right)dx=\frac{2}{0.403^2}\times 0.0744=0.916$$

$$b_2 = \frac{2}{J_2^2\left(\alpha_{12}\right)}\int_0^1 x\left(1-x\right)J_1\left(\alpha_{12}x\right)dx = \frac{2}{(-0.3)^2}\times 0.0195 = 0.433$$

$$b_3 = \frac{2}{J_2^2\left(\alpha_{13}\right)}\int_0^1 x\left(1-x\right)J_1\left(\alpha_{13}x\right)dx = \frac{2}{0.25^2}\times 0.00989 = 0.316$$