

PHYS 501 - 3rd Homework Solutions

$$\textcircled{1} \quad \vec{F} = -\vec{\nabla}V = -\vec{\nabla}_c T = -c \vec{\nabla} T = -c A \vec{\nabla} \left\{ \left(\frac{\rho}{w_0} \right)^{2\ell} \exp \left(-\frac{2\rho^2}{w_0^2} \right) \delta_{\rho}^{(k)} \left(\frac{2\rho^2}{w_0^2} \right) \right\}$$

If $\rho = 0$ then $\delta_{\rho}^{(k)} \left(\frac{2\rho^2}{w_0^2} \right) = 1$, thus

$$\begin{aligned} F &= -c A \vec{\nabla} \left\{ \left(\frac{\rho}{w_0} \right)^{2\ell} \exp \left(-\frac{2\rho^2}{w_0^2} \right) \right\} = -c A \hat{\rho}_0 \frac{\partial}{\partial \rho} \left\{ \left(\frac{\rho}{w_0} \right)^{2\ell} \exp \left(-\frac{2\rho^2}{w_0^2} \right) \right\} \\ &= -c A \hat{\rho}_0 \left\{ \exp \left(-\frac{2\rho^2}{w_0^2} \right) \frac{\partial}{\partial \rho} \left(\frac{\rho}{w_0} \right)^{2\ell} + \left(\frac{\rho}{w_0} \right)^{2\ell} \frac{\partial}{\partial \rho} \exp \left(-\frac{2\rho^2}{w_0^2} \right) \right\} \\ &= -c A \hat{\rho}_0 \left\{ \frac{2\ell}{w_0^{2\ell}} \exp \left(-\frac{2\rho^2}{w_0^2} \right) \rho^{2\ell-1} - \frac{4\rho}{w_0^2} \left(\frac{\rho}{w_0} \right)^{2\ell} \exp \left(-\frac{2\rho^2}{w_0^2} \right) \right\} \\ &= -c A \exp \left(-\frac{2\rho^2}{w_0^2} \right) \left\{ \frac{2\ell}{w_0} \left(\frac{\rho}{w_0} \right)^{2\ell-1} - \frac{4\rho}{w_0^2} \left(\frac{\rho}{w_0} \right)^{2\ell} \right\} \hat{\rho}_0 \end{aligned}$$

(2)

$$\vec{\omega} = \frac{d\vec{\omega}}{dt} = \dot{\vec{\omega}} = (\dot{\rho} \hat{\rho}_o(t) + \rho \dot{\phi} \hat{\phi}_o(t) + \dot{v}_z \hat{k}) \Rightarrow$$

$$\vec{\alpha} = (\dot{\rho} \hat{\rho}_o(t)) + (\rho \dot{\phi} \hat{\phi}_o(t)) + \dot{v}_z \hat{k} \Rightarrow$$

$$\vec{\alpha} = \ddot{\rho} \hat{\rho}_o(t) + \dot{\rho} \ddot{\hat{\rho}}_o(t) + (\rho \ddot{\phi}) \hat{\phi}_o(t) + (\rho \dot{\phi}) \ddot{\hat{\phi}}_o(t) + \dot{v}_z \hat{k}$$

$$\begin{aligned}\vec{\alpha} &= \ddot{\rho} \hat{\rho}_o(t) + \dot{\rho} \ddot{\hat{\rho}}_o(t) + (\dot{\rho} \dot{\phi} + \rho \ddot{\phi}) \hat{\phi}_o(t) + (\rho \dot{\phi}) \ddot{\hat{\phi}}_o(t) + \dot{v}_z \hat{k} \\ &= \ddot{\rho} \hat{\rho}_o(t) + \dot{\rho} \left(\dot{\hat{\rho}}_o(t) + (\dot{\rho} \dot{\phi} + \rho \ddot{\phi}) \hat{\phi}_o(t) + (\rho \dot{\phi}) \ddot{\hat{\phi}}_o(t) \right) + \dot{v}_z \hat{k} \\ &\quad + (\dot{\rho} \dot{\phi}) (-\dot{i} \sin \phi(t) + \dot{j} \cos \phi(t)) + \dot{v}_z \hat{k} \Rightarrow\end{aligned}$$

$$\begin{aligned}\vec{\alpha} &= \ddot{\rho} \hat{\rho}_o(t) + \dot{\rho} (-\dot{\phi} \dot{i} \sin \phi(t) + \dot{\phi} \dot{j} \cos \phi(t)) + (\dot{\rho} \dot{\phi} + \rho \ddot{\phi}) \hat{\phi}_o(t) \\ &\quad + (\rho \dot{\phi}) (-\dot{\phi} \cos \phi(t) \dot{i} + \dot{\phi} \sin \phi(t) \dot{j}) + \dot{v}_z \hat{k} \Rightarrow\end{aligned}$$

$$\vec{\alpha} = \ddot{\rho} \hat{\rho}_o(t) + \dot{\rho} \dot{\phi} \hat{\phi}_o(t) + (\dot{\rho} \dot{\phi} + \rho \ddot{\phi}) \hat{\phi}_o(t) +$$

$$\bullet \quad \dot{\rho} \dot{\phi}^2 \hat{\rho}_o(t) + \dot{v}_z \hat{k} = [\ddot{\rho} - \rho \dot{\phi}^2] \hat{\rho}_o(t) + \dot{\phi}_o [2 \dot{\rho} \dot{\phi} + \dot{\phi} \ddot{\phi}] + \alpha_z \hat{k}$$

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From question 5.21 we have

$$\hat{r} = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k} \quad (\alpha)$$

$$\hat{\theta}_o = \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k} \quad (\beta)$$

$$\hat{\varphi}_o = -\sin\phi \hat{i} + \cos\phi \hat{j} \quad (\gamma)$$

$$\frac{\partial \hat{r}}{\partial r} = 0, \quad \frac{\partial \hat{r}}{\partial \theta} = \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k} = \hat{\theta}_o$$

$$\frac{\partial \hat{r}}{\partial \phi} = -\sin\theta \sin\phi \hat{i} + \sin\theta \cos\phi \hat{j} = \sin\theta (-\sin\phi \hat{i} + \cos\phi \hat{j}) = \sin\theta \hat{\varphi}_o$$

$$\frac{\partial \hat{\theta}_o}{\partial r} = 0, \quad \frac{\partial \hat{\theta}_o}{\partial \theta} = -\sin\theta \cos\phi \hat{i} - \sin\theta \sin\phi \hat{j} - \cos\theta \hat{k} = -\hat{r}$$

$$\frac{\partial \hat{\theta}_o}{\partial \phi} = -\cos\theta \sin\phi \hat{i} + \cos\theta \cos\phi \hat{j} = \cos\theta (-\sin\phi \hat{i} + \cos\phi \hat{j}) = \cos\theta \hat{\varphi}_o$$

- $\frac{\partial \hat{\varphi}_o}{\partial r} = 0$

$$\frac{\partial \hat{\varphi}_o}{\partial \theta} = 0, \quad \frac{\partial \hat{\varphi}_o}{\partial \varphi} = -\cos \hat{\varphi} \hat{i} + \sin \hat{\varphi} \hat{j} = -(\cos \hat{\varphi} \hat{i} + \sin \hat{\varphi} \hat{j}) \quad (\text{d})$$

But remember that

$$\begin{aligned} \hat{r} &= \sin \theta \cos \hat{\varphi} \hat{i} + \sin \theta \sin \hat{\varphi} \hat{j} + \cos \theta \hat{k} \\ \hat{\theta}_o &= \cos \theta \cos \hat{\varphi} \hat{i} + \cos \theta \sin \hat{\varphi} \hat{j} - \sin \theta \hat{k} \end{aligned} \quad \left. \times \sin \right\}$$

we have $\sin \theta \hat{r} = \sin^2 \theta \cos^2 \hat{\varphi} \hat{i} + \sin^2 \theta \sin^2 \hat{\varphi} \hat{j} + \cos^2 \theta \hat{k}$
 $\cos \theta \hat{\theta}_o = \cos^2 \theta \cos^2 \hat{\varphi} \hat{i} + \cos^2 \theta \sin^2 \hat{\varphi} \hat{j} - \sin^2 \theta \hat{k}$ }^④

$$\sin \theta \hat{r} + \cos \theta \hat{\theta}_o = \cos \hat{\varphi} \hat{i} + \sin \hat{\varphi} \hat{j} \quad (\text{e})$$

From (e) we see that (d) becomes

$$\frac{\partial \hat{\varphi}_o}{\partial \varphi} = -(\sin \hat{\varphi} \hat{i} + \cos \hat{\varphi} \hat{j})$$

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$$\begin{aligned}
 -i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) &= -i \left(r \sin \theta \cos \phi \frac{\partial}{\partial y} - r \sin \theta \sin \phi \frac{\partial}{\partial x} \right) \\
 &= -i \left[r \sin \theta \cos \phi \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \cos \theta \sin \phi \frac{\partial}{\partial \phi} + \cos \theta \frac{\partial}{\partial \theta} \right) \right. \\
 &\quad \left. - r \sin \theta \cos \phi \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \cos \theta \cos \phi \frac{\partial}{\partial \phi} - \frac{\sin \phi}{r} \frac{\partial}{\partial \theta} \right) \right] \\
 &= -i \left[r \sin^2 \theta \cos^2 \phi \frac{\partial}{\partial r} + \sin \theta \cos \theta \cos \phi \sin \phi \frac{\partial}{\partial \theta} + r \sin \theta \cos^2 \phi \frac{\partial}{\partial \phi} \right. \\
 &\quad \left. - r \sin \theta \sin \theta \sin \phi \cos \phi \frac{\partial}{\partial r} - r \sin \theta \sin \theta \cos \phi \sin \phi \frac{\partial}{\partial \theta} - r \sin \theta \sin^2 \phi \frac{\partial}{\partial \phi} \right] \\
 &= -i \left[\cos^2 \phi \frac{\partial}{\partial r} + \sin^2 \phi \frac{\partial}{\partial \phi} \right] = -i \frac{\partial \phi}{\partial r}
 \end{aligned}$$