MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE - 3

Derivative of a complex function - Cauchy Riemann conditions - Analytic functions

Derivative of a complex function-a

- A complex function $f(z)$ is said to have a derivative continuous at a point z_0 if and only if the limit ;

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

I) Exists

II) Is finite

III) Does not depend on the direction of approaching z

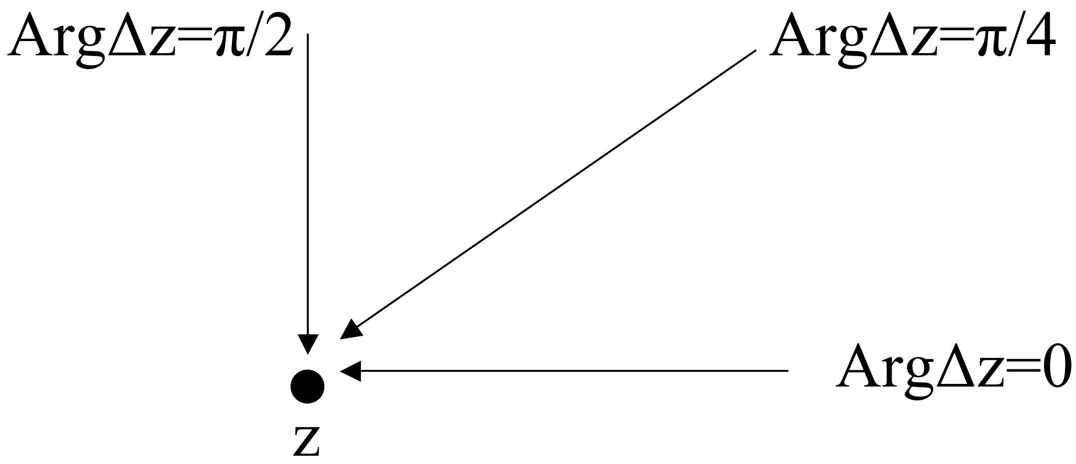
Derivative of a complex function-b

- We use the following notation

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \left.\frac{df}{dz}\right|_{z=z_0} = f^{'}(z_0)$$

- As we have mentioned we can approach a complex number with different ways. The only condition is the length of Δz to tend to zero. This means that Arg(Δz) is not defined. The limit must be independent of the Arg(Δz).

Derivative of a complex function-c



- Example: Check if the following function is differentiable

$$f(z) = z + \frac{z - z^*}{2}$$

Derivative of a complex function-d

- When the derivative of two functions *f* and *g* exist at a point the following rules apply.

$$\frac{d}{dz}\left(f(z)+g(z)\right)=f'(z)+g'(z)$$

$$\frac{d}{dz}\left(f(z)\cdot g(z)\right)=f'(z)\cdot g(z)+f(z)\cdot g'(z)$$

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right)=\frac{f'(z)\cdot g(z)-f(z)\cdot g'(z)}{g^2(z)}$$

$$\frac{d}{dz}f(g(z))=\frac{df}{dg}\cdot\frac{dg}{dz}$$

Cauchy-Riemann conditions-1

- Let a complex function $f(z) = u(x,y)+iv(x,y)$. If the derivative of this function exists at a certain point z.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- These are the famous Cauchy-Riemann conditions. The conditions imply also that

$$f^{'}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Cauchy-Riemann conditions-2

- Cauchy-Riemann conditions are necessary but not sufficient. In this case we have the following theorem:
- If the function $f(z) = u(x,y)+iv(x,y)$ has the following properties in the vicinity of a point $z_0=(x_0,y_0)$:

1. $u(x,y)$ and $v(x,y)$ have continuous partial derivatives at z_0.
2. $$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}, \qquad \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = -\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}$$

- Then the derivative of f at $z_0=(x_0,y_0)$ exists.

Cauchy-Riemann conditions in polar coordinates

- The previous theorem can be also expressed through the polar form when $z_0 \neq 0$
- If the function $f(z) = u(r,\theta)+iv(r,\theta)$ has the following properties:

1. $u(r,\theta)$ and $v(r,\theta)$ have continuous partial derivatives at (r_0,θ_0).

2. $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

- Then the derivative of f at z=(r_0,θ_0) exists and can be written as:

$$f^{'}(z_0) = e^{-i\theta_0}\left(\left.\frac{\partial u}{\partial r}\right|_{(r_0,\theta_0)} + i\left.\frac{\partial v}{\partial r}\right|_{(r_0,\theta_0)}\right)$$

Analytic functions-a

- A function $f(z)$ is called *analytic* in a region of the complex plane if it is single-valued and differentiable at any point in this region.
- If this region is the entire complex value then the function is called *entire.*
- If the function is analytic around a point z_0 then the point is a regular one. Otherwise is called *singular point*.
- Example of analytic function are the polynomial functions.

Analytic functions-b

- If two functions are analytic in a region *D* then their sum and product are analytic functions in this region. Also their ratio is analytic provided that the denominator is different than zero in the entire region *D*.
- The composition of two analytic functions is also an analytic function.
- It can be proved also that if a function is analytic in a region *D* and its derivative is zero in the entire region then the function is constant in this region.

Harmonic functions-1

- If a function $f(z) = u(x,y)+iv(x,y)$ is analytic, then:

a) Cauchy-Riemann conditions imply that

$$\vec{\nabla}^2 u = \vec{\nabla}^2 v = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

This is Laplace equation. One of the most important equations in physics (electrostatics, ideal fluid flow etc). Functions that obey this equation are called *harmonic.*

Harmonic functions-2

b) The curves $u(x,y)$=const. and $v(x,y)$=const. are orthogonal, that means:

Where $$\vec{\nabla} u \cdot \vec{\nabla} v = 0$$

$$\vec{\nabla} u = \frac{\partial u}{\partial x}\mathbf{x} + \frac{\partial u}{\partial y}\mathbf{y}, \qquad \vec{\nabla} v = \frac{\partial v}{\partial x}\mathbf{x} + \frac{\partial v}{\partial y}\mathbf{y}$$

Where **x** and **y** are the unit vectors along x and y axes

Harmonic functions-3

- If two functions u and v are harmonic in a region D and their partial derivatives satisfy the Cauchy-Riemann equations then v is called **the harmonic conjugate** of u.
- The following theorems do hold:
- A) If the function $f(z) = u(x,y)+iv(x,y)$ is analytic in a region D, the functions u and v are harmonic in D.
- B) A function $f(z) = u(x,y)+iv(x,y)$ is analytic in a region D, if and only if, v is the harmonic conjugate of u.