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On the asymptotic expansion of the q-dilogarithm

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Abstract
In this work, we study some asymptotic expansion of the q-dilogarithm at $q = 1$ and $q = 0$ by using the Mellin transform and an adequate decomposition allowed by the Lerch functional equation.

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1 Introduction

Euler's dilogarithm is defined by [1]

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1. \tag{1.1}$$

In [2], Kirillov defines the following q-analog of the dilogarithm $Li_2(z)$:

$$Li_2(z;q) = \sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}, \quad |z| < 1, 0 < q < 1, \tag{1.2}$$

and he observes the following remarkable formula ([2], Section 2.5, Lemma 8):

$$\sum_{n=0}^{\infty} \frac{z^n}{(q,q)_n} = \exp\bigl(Li_2(z,q)\bigr), \quad |z| < 1, |q| < 1, \tag{1.3}$$

where

$$(q,q)_0 = 1, \qquad (q,q)_n = \prod_{k=0}^{n-1}\bigl(1-q^k\bigr), \quad n = 1,2,\ldots. \tag{1.4}$$

It seems a precise formulation of (1.3) going back to Ramanujan (see [3], Chapter 27, Entry 6) is given an asymptotic series for $Li_2(z;q)$ and Hardy and Littlewood [4] proved that for $|q| = 1$, the identity holds inside the radius of convergence of either series.

Let $\omega = e^{zx+2i\theta}$ with $\mathrm{Re}(z) > 1$, $x > 0$, and $0 < \theta < 1$. The main result of this work is the following complete asymptotic expansion of the q-dilogarithm function $Li_2(\omega;e^{-x})$ at

$x \to 0$:

$$Li_2(\omega, e^{-x}) \sim Ci_2(\theta)\frac{1}{x} + \left(\frac{1}{2} - z\right)Ci_1(\theta) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+1)!} \times B_{n+1}(z)B_{n+1}(1, e^{2i\pi\theta})x^n \quad \text{as } x \to 0 \tag{1.5}$$

and

$$Li_2(\omega, e^{-x}) \sim \frac{4\gamma}{\pi}B_2(\theta)\frac{\mathrm{i}}{x} + 4\sum_{n=1}^{\infty} \mathrm{i}^n \frac{\psi^{(n-1)}(z)B_{n+1}(\theta)}{(n+1)!}\left(\frac{2\pi}{x}\right)^n, \quad x \to \infty. \tag{1.6}$$

In Section 2.5, Corollary 10 of [2], Kirillov and Ueno and Nishizawa derived the asymptotic expansion (1.5) by using the Euler-Maclaurin summation formula; see also [5], an integral representation of Barnes type for the q-dilogarithm. Second, we use the Lerch functional equation to decompose the integrand and to apply the Cauchy theorem.

2 *q*-Dilogarithm

The polylogarithm is defined in the unit disk by the absolutely convergent series [1]

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad |z| < 1. \tag{2.1}$$

Several functional identities satisfied by the polylogarithm are available in the literature (see [6]). For $n = 2, \ldots$, the function $Li_n(z)$ can also be represented as

$$Li_n(z) = \int_0^z \frac{Li_{n-1}(t)}{t}\,dt, \quad n \in \mathbb{N}, \qquad Li_1(z) = -\log(1-z) = \int_0^z \frac{dt}{1-t}, \tag{2.2}$$

which is valid for all z in the cut plane $\mathbb{C} \setminus [1, \infty)$.

The notation $F(\theta, s)$ is used for the polylogarithm $Li_s(e^{2in\pi\theta})$ with θ real, called the periodic zeta function (see [7], Section 25.13) and is given by the Dirichlet series

$$F(\theta, s) = \sum_{n=1}^{\infty} \frac{e^{2in\pi\theta}}{n^s}, \quad \theta \in \mathbb{R}, \tag{2.3}$$

it converges for $\operatorname{Re} s > 1$ if $\theta \in \mathbb{Z}$, and for $\operatorname{Re} s > 0$ if $\theta \in \mathbb{R}/\mathbb{Z}$. This function may be expressed in terms of the Clausen functions $Ci_s(\theta)$ and $Si_s(\theta)$, and *vice versa* (see [1], Section 27.8):

$$Li_s(e^{\pm i\theta}) = Ci_s(\theta) \pm \mathrm{i}Si_s(\theta). \tag{2.4}$$

In [8], Koornwinder defines the q-analog of the logarithm function

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1,$$

as follows:

$$\log_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-q^n}, \quad |z| < 1, 0 < q < 1. \tag{2.5}$$

Recall that the q-analog of the ordinary integral (called Jackson's integral) is defined by

$$\int_0^z f(t)\,d_qt = (1-q)z\sum_{n=0}^{\infty} f\bigl(zq^n\bigr)q^n. \tag{2.6}$$

One can recover the ordinary Riemann integral as the limit of the Jackson integral for $q \uparrow 1$.

Lemma 2.1 *The function* $\log_q(z)$ *has the following* q*-integral representation*:

$$(1-q)\log_q(z) = \int_0^z \frac{1}{1-t}\,d_qt, \quad |z| < 1. \tag{2.7}$$

Moreover, it can be extended to any analytic function on $\mathbb{C} - \{q^{-n}, n \in \mathbb{N}_0\}$.

Proof Assume that $|z| < 1$, then from (2.5) we have

$$\begin{aligned}(1-q)\log_q(z) &= (1-q)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} z^n q^{nm}\\ &= (1-q)z\sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{nm}\\ &= (1-q)z\sum_{m=0}^{\infty} \frac{q^m}{1-zq^m}.\end{aligned}$$

The inversion of the order of summation is permitted, since the double series converges absolutely when $|z| < 1$.

Let K be a compact subset of $\mathbb{C} - \{q^{-n}, n \in \mathbb{N}_0\}$. There exists $N \in \mathbb{N}$ such that, for all $z \in K$, $|q^N z| < q$. Then for $n \geq N$ we have

$$\left|\frac{q^m}{1-zq^m}\right| \leq \frac{q^m}{1-q}. \tag{2.8}$$

Hence, the series $\sum_{m=N}^{\infty} \frac{q^m}{1-zq^m}$ converges uniformly in K. □

The q-dilogarithm (1.2) is related to Koornwinder's q-logarithm (2.5) by

$$Li_2(z,q) = \int_0^z \frac{\log_q(t)}{t}\,dt. \tag{2.9}$$

It follows that, for $n \geq 2$, we can also define

$$Li_n(z,q) = \int_0^z \frac{Li_{n-1}(t,q)}{t}\,dt. \tag{2.10}$$

This integral formula proves by induction that $Li_n(z,q)$ has an analytic continuation on $\mathbb{C} - [1,\infty)$. Moreover, for $|z| < 1$, we have

$$Li_n(z,q) = \sum_{k=1}^{\infty} \frac{z^k}{k^n(1-q^k)}.$$

This converges absolutely for $|z| < 1$ and defines a germ of a holomorphic function in the neighborhood of the origin. Note that

$$\lim_{q\uparrow 1}(1-q)Li_2\bigl((1-q)z,q\bigr) = Li_2(z),$$
$$\lim_{q\downarrow 0}(1-q)Li_2(z,q) = -\operatorname{Log}(1-z), \quad |z| < 1.$$

Let $\omega = e^{-zx+2\mathrm{i}\pi\theta}$, $\theta \in \mathbb{R}$, and $\operatorname{Re} z > 0$, we define

$$Ci_2(\omega,q) = \sum_{n=1}^{\infty} \frac{e^{-zx}\cos(2\pi n\theta)}{n(1-q^n)}, \tag{2.11}$$

$$Si_2(\omega,q) = \sum_{n=1}^{\infty} \frac{e^{-zx}\sin(2\pi n\theta)}{n(1-q^n)}. \tag{2.12}$$

Note that these functions can be considered as q-analogs of the Clausen functions (2.4) and are related to the q-dilogarithm by

$$Li_2(\omega,q) = Ci_2(\omega,q) + \mathrm{i}Si_2(\omega,q). \tag{2.13}$$

Now, we will use the Mellin transform method to obtain the integral representation

$$Li_2\bigl(\omega,e^{-x}\bigr) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta(s,z)F(\theta,s)\Gamma(s)x^{-s}\,ds, \quad c > 1, \tag{2.14}$$

where

$$\omega = e^{-zx+2\mathrm{i}\pi\theta}, \qquad x > 0, \qquad \operatorname{Re} z > 1, \qquad 0 < \theta < 1.$$

Recall that the Mellin transform for a locally integrable function $f(x)$ on $(0,\infty)$ is defined by

$$M(f,s) = \int_0^{\infty} f(x)x^{s-1}\,dx, \tag{2.15}$$

which converges absolutely and defines an analytic function in the strip

$$a < \operatorname{Re} s < b,$$

where a and b are real constants (with $a < b$) such that, for $\varepsilon > 0$,

$$f(x) = \begin{cases} \mathcal{O}(x^{-a-\varepsilon}) & \text{as } x \to 0^+, \\ \mathcal{O}(x^{-b-\varepsilon}) & \text{as } x \to +\infty. \end{cases} \tag{2.16}$$

The inversion formula reads

$$f(x) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} M(f,s)x^{-s}ds, \tag{2.17}$$

where c satisfies $a < c < b$. Equation (2.17) is valid at all points $x \geq 0$ where $f(x)$ is continuous.

We first compute the Mellin transform $M(\psi_n(x), s)$, where

$$\psi_n(x) = \frac{e^{-nzx}}{n(1 - e^{-nx})}, \quad x > 0, \operatorname{Re} z > 1, n \in \mathbb{N}. \tag{2.18}$$

Since

$$\psi_n(x) \sim \frac{1}{nx}, \quad x \to 0^+, \tag{2.19}$$

$$\psi_n(x) \sim \frac{1}{n} e^{-nx(z-1)}, \quad x \to +\infty. \tag{2.20}$$

We concluded that $M(\psi_n(x), s)$ is defined in the half-plane $\operatorname{Re} s > 0$. That is, the constants a and b satisfy $a = 1$ and $b = +\infty$, which values can be used for all $n \geq 1$ and $\operatorname{Re} z > 1$. The Mellin transform of $\psi_n(x)$ can be obtained from the following integral representation of the Hurwitz zeta function $\zeta(s, z)$:

$$\zeta(s, z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} x^{s-1}\, dx \quad \bigl(\operatorname{Re} s > 0, \bigl|\arg(1 - z)\bigr| < \pi; \operatorname{Re} s > 1, z = 1\bigr). \tag{2.21}$$

Note that $\zeta(s, z)$ is expressed also by the series

$$\zeta(s, z) = \sum_{k=1}^{\infty} \frac{1}{(z + k)^s}, \quad \operatorname{Re} s > 1, z \neq -1, -2, \ldots. \tag{2.22}$$

For the other values of z, $\zeta(s, z)$ is defined by analytic continuation. It has a meromorphic continuation in the s-plane, its only singularity in $\mathbb{C}$ being a simple pole at $s = 1$,

$$\zeta(s, z) = \frac{1}{s - 1} - \psi(z) + \mathcal{O}(s - 1). \tag{2.23}$$

Applying the Mellin inversion theorem to the integral (2.21), we then find

$$\psi_n(x) = \frac{1}{2\mathrm{i}\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta(s, z)\Gamma(s)(nx)^{-s}\, ds. \tag{2.24}$$

We use the Stirling formula, which shows that, for finite σ,

$$\Gamma(\sigma + \mathrm{i}t) = \mathcal{O}\bigl(|t|^{\sigma-1} e^{-\frac{1}{2}\pi|t|}\bigr) \quad \bigl(|t| \to +\infty\bigr) \tag{2.25}$$

and the well-known behavior of $\zeta(s, z)$ (see [9])

$$\zeta(s, z) = \mathcal{O}\bigl(|t|^{\tau(\sigma)} \log|t|\bigr), \tag{2.26}$$

where

$$\tau(\sigma) = \begin{cases} \frac{1}{2} - \sigma, & \sigma \leq 0, \\ \frac{1}{2}, & 0 \leq \sigma \leq \frac{1}{2}, \\ 1 - \sigma, & \frac{1}{2} \leq \sigma \leq 1, \\ 0, & \sigma \geq 1. \end{cases}$$

Then we obtain the following majorization of the modulus of the integrand in (2.24):

$$\mathcal{O}\big(|t|^{\tau(\sigma)+\sigma-1}\log|t|\big). \tag{2.27}$$

Consequently, the integral (2.24) converges absolutely in the whole vertical strip of the half-plane $\operatorname{Re} s > 0$. Then we replace x by nx, where n is a positive integer, and sum over n, and we then obtain

$$Li_2\big(\omega, e^{-x}\big) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \zeta(s,z)F(\theta,s+1)\Gamma(s)x^{-s}\,ds, \quad c > 1, \tag{2.28}$$

where

$$\omega = e^{-zx+2i\pi\theta}, \qquad x > 0, \qquad \operatorname{Re} z > 1, \qquad 0 < \theta < 1.$$

3 Asymptotic at $q = 1$

The integral (2.28) will be used to derive asymptotic expansions of the q-dilogarithm. The contour of integration is moved at first to the left to obtain an asymptotic expansion at $q = 1$ and then to the right to get an asymptotic expansion at $q = 0$.

Let us consider the function

$$g(s) = \zeta(s,z)F(\theta,s+1)\Gamma(s). \tag{3.1}$$

The periodic function zeta function $F(\theta,s)$ has an extension to an entire function in the s-plane (see [10]). Hence, the function $g(s)$ has a meromorphic continuation in the s-plane, its only singularity in $\mathbb{C}$ coincides with the pole of $\Gamma(s)$ and $\zeta(s,z)$ being a simple pole at $s = 1, 0, -1, -2, \ldots$.

Now we compute the residues of the poles. The special values at $s = -1, -2\ldots$ of the periodic zeta function are reduced to the Apostol-Bernoulli polynomials (see [10]),

$$F(\theta,-n) = -\frac{B_{n+1}(1, e^{2i\pi\theta})}{n+1}. \tag{3.2}$$

We need also the following asymptotic expansions of $\Gamma(s)$ and $\zeta(s)$ at $s = 0$:

$$\Gamma(s) = \frac{1}{s} - \gamma + \mathcal{O}\big(s^2\big), \tag{3.3}$$

$$\zeta(s) = \frac{1}{2} - z + s\log\frac{\Gamma(z)}{2\pi} + \mathcal{O}\big(s^2\big). \tag{3.4}$$

Hence,

$$\lim_{s\to 1}(s-1)g(s) = Li_2\big(e^{2i\pi\theta}\big),$$

$$\lim_{s\to -n}(s+n)g(s) = \frac{(-1)^n}{(n+1)(n+1)!}B_{n+1}(z)B_{n+1}\big(1, e^{2i\pi\theta}\big).$$

Here $B_n(z)$ is the Bernoulli polynomial (see [1]).

Let N be an integer and d real number such that $-N-1 < d < -N$. We consider the integral taken round the rectangular contour with vertices at $d \pm iA$ and $c \pm iA$, so that the side in $\mathrm{Re}(s) < 0$ parallel to the imaginary axis passes midway between the poles $s = 1, 0 - 1, -2, \ldots, -N$. The contribution from the upper and lower sides $s = \sigma \pm \mathrm{i}A$ vanishes as $|A| \to +\infty$, since the modulus of the integrand is controlled by

$$\mathcal{O}\bigl(|A|^{\tau(\sigma)+\sigma-1/2} \log|A| e^{-\frac{1}{2}\pi|A|}\bigr). \tag{3.5}$$

This follows from Stirling's formula (2.25), the behavior $\zeta(s, z)$ being given by (2.26), and the following estimation:

$$\bigl|F(\theta, s+1)\bigr| \leq \zeta(\sigma+1) = \mathcal{O}(1), \quad |A| \to +\infty.$$

Displacement of the contour (2.28) to the left then yields

$$\begin{aligned} Li_2\bigl(\omega, e^{-x}\bigr) = {} & Ci_2(\theta)\frac{1}{x} + \left(\frac{1}{2} - z\right) Ci_1(\theta) \\ & + \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(n+1)(n+1)!} B_{n+1}(z) B_{n+1}\bigl(1, e^{2\mathrm{i}\pi\theta}\bigr) x^n + R_N(x), \end{aligned} \tag{3.6}$$

where the remainder integral $R_N(z)$ is given by

$$R_N(x) = \frac{1}{2\mathrm{i}\pi} \int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \zeta(s, z) F(\theta, s+1) \Gamma(s) x^{-s} \, ds, \quad x > 0, \mathrm{Re}\, z > 1. \tag{3.7}$$

From (3.5), we find

$$\bigl|R_N(x)\bigr| = \mathcal{O}\left(\frac{1}{x^{N+1}}\right).$$

4 Asymptotic at $q = 0$

Recall that the periodic zeta function satisfies the functional equation (see [10])

$$\begin{aligned} F(\theta, s) = {} & \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \bigl\{ e^{\frac{\pi \mathrm{i}(1-s)}{2}} \zeta(1-s, \theta) + e^{\frac{\pi \mathrm{i}(s-1)}{2}} \zeta(1-s, 1-\theta) \bigr\} \\ & (\mathrm{Re}\, s > 1, 0 < \theta < 1), \end{aligned} \tag{4.1}$$

first given by Lerch, whose proof follows the lines of the first Riemann proof of the functional equation for $\zeta(x)$.

It is well known that the asymptotic expansion near infinity via the Mellin transform is obtained by displacement of the contour of integration in the Mellin inversion formulas (2.16) to the right-hand side (see [11]). However, the integrand (2.28) has no poles in the half-plane $\mathrm{Re}\, s > 1$. The periodic zeta function $F(\theta, s)$ has an analytic continuation to the whole s-space for $0 < \theta < 1$. Moreover, the poles of $\Gamma(1-s)$ in equation (4.1) at $s = -1, -2 \ldots$ are canceled by the zeros of the function

$$e^{\frac{\pi \mathrm{i}(1-s)}{2}} \zeta(1-s, \theta) + e^{\frac{\pi \mathrm{i}(s-1)}{2}} \zeta(1-s, 1-\theta).$$

On the other hand from (4.1) we easily obtain

$$\Gamma(s)\bigl\{F(\theta, s+1) + F(1-\theta, s+1)\bigr\} = -\frac{(2\pi)^{s+1}}{2s \sin\frac{\pi s}{2}} \bigl\{\zeta(-s, \theta) + \zeta(-s, 1-\theta)\bigr\}, \tag{4.2}$$

where we are able to simplify (4.2) by the well-known reflection formulas

$$\frac{\pi}{\sin \pi s} = \Gamma(s)\Gamma(1-s), \qquad \frac{\sin \pi s}{\pi} = \frac{2}{\pi} \sin \frac{\pi s}{2} \sin \frac{\pi(1-s)}{2}.$$

Proceeding similar to above we also obtain

$$\Gamma(s)\bigl\{F(\theta,s) - F(1-\theta,s)\bigr\} = \frac{(2\pi)^{s+1}}{2s\cos\frac{\pi(s)}{2}}\bigl\{\zeta(-s,\theta) - \zeta(-s,1-\theta)\bigr\}. \tag{4.3}$$

Moreover, the integral representation (2.28) is valid for all $0 < \theta < 1$. So we can replace θ by $1-\theta$ in its integrand. Using the above decomposition (4.2) and (4.3), we then obtain

$$Ci_2\bigl(\omega, e^{-x}\bigr) = -\frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(2\pi)^{s+1}\zeta(s,z)}{2s\sin\frac{\pi s}{2}}\bigl\{\zeta(-s,\theta) + \zeta(-s,1-\theta)\bigr\}\frac{ds}{x^s} \tag{4.4}$$

and

$$Si_2\bigl(\omega, e^{-x}\bigr) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(2\pi)^{s+1}\zeta(s,z)}{2s\cos\frac{\pi s}{2}}\bigl\{\zeta(-s,\theta) - \zeta(-s,1-\theta)\bigr\}\frac{ds}{x^s}, \tag{4.5}$$

where $\omega = e^{-zx+2\mathrm{i}\pi\theta}$, $0 < x$, $0 < \theta < 1$, $0 < \operatorname{Re} z$, and $1 < c < 2$.

Note that the special values $\zeta(n,z)$ ($n \in \mathbb{N}_0$) are expressed in terms of the polygamma function $\psi(z)$,

$$\zeta(n+1,z) = \frac{(-1)^{n+1}}{n!}\psi^{(n)}(z), \quad z \neq 0,-1,-2,\ldots, \tag{4.6}$$

and $\zeta(-n,z)$ ($n \in \mathbb{N}$) is reduced to the Bernoulli polynomial

$$\zeta(-n,z) = -\frac{B_{n+1}(z)}{n+1}. \tag{4.7}$$

Applying the identities for the Bernoulli polynomial

$$B_n(1-\theta) = (-1)^n B_n(\theta),$$

we obtain

$$\zeta(-n,\theta) + \zeta(-n,1-\theta) = \bigl((-1)^{n+1} - 1\bigr)\frac{B_{n+1}(\theta)}{n+1}, \tag{4.8}$$

$$\zeta(-n,\theta) - \zeta(-n,1-\theta) = \bigl((-1)^{n} - 1\bigr)\frac{B_{n+1}(\theta)}{n+1}. \tag{4.9}$$

The integrand in (4.4) has a meromorphic continuation in the s-plane, its only singularity in the half-plane $\operatorname{Re} s > 0$ coincides with the pole of $1/\sin\frac{\pi s}{2}$ being a simple pole at $s = 2,4,\ldots$. Then by the Cauchy integral, we can shift the contour in (4.4) to the right, picking up the residues at $s = 2,\ldots,2N$, with the result

$$Ci_2\bigl(\omega, e^{-x}\bigr) = 4\sum_{n=1}^{N}(-1)^n \frac{\psi^{(2n-1)}(z)B_{2n+1}(\theta)}{(2n+1)!}\left(\frac{2\pi}{x}\right)^{2n} + Q_N(x), \tag{4.10}$$

where

$$Q_N(x) = -\frac{1}{2i\pi}\int_{c+2N-i\infty}^{c+2N+i\infty} \frac{(2\pi)^{s+1}\zeta(s,z)}{2s\sin\frac{\pi s}{2}}\bigl\{\zeta(-s,\theta)+\zeta(-s,1-\theta)\bigr\}\frac{ds}{x^s}. \tag{4.11}$$

Using the following estimations in a vertical strip $s = \sigma + it$, $\sigma \neq 0, \pm 1, \pm 2, \dots,$

$$\frac{1}{\sin\frac{\pi s}{2}} = \mathcal{O}\bigl(|t|^{-1}e^{-\frac{\pi}{2}|t|}\bigr), \tag{4.12}$$

we obtain

$$\bigl|Q_N(x)\bigr| = \mathcal{O}\biggl(\frac{1}{x^{2N+1}}\biggr). \tag{4.13}$$

Similarly,

$$Si_2\bigl(\omega, e^{-x}\bigr) = \frac{4\gamma}{\pi}B_2(\theta)\frac{1}{x} + 4\sum_{n=1}^{N}(-1)^n\frac{\psi^{(2n)}(z)B_{2n+2}(\theta)}{(2n+2)!}\biggl(\frac{2\pi}{x}\biggr)^{2n+1} + \mathcal{O}\biggl(\frac{1}{x^{2N+2}}\biggr). \tag{4.14}$$

Proposition 4.1 *Let* $\omega = e^{-zx+2i\pi\theta}$, $x > 0$, $\operatorname{Re} z > 1$ *and* $0 < \theta < 1$. *Then*

$$Li_2\bigl(\omega, e^{-x}\bigr) \sim \frac{4\gamma}{\pi}B_2(\theta)\frac{i}{x} + 4\sum_{n=1}^{\infty} i^n\frac{\psi^{(n-1)}(z)B_{n+1}(\theta)}{(n+1)!}\biggl(\frac{2\pi}{x}\biggr)^n, \quad x \to \infty. \tag{4.15}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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