problem 5-2:

Evaluate the limits:

1) $\lim_{n\to\infty} \frac{\sqrt[n]{e}+\sqrt[n]{e^2}+\dots+\sqrt[n]{e^{2n}}}{n}$

2) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\dots+\frac{1}{(2n)^2}\right)$

3) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+n^2}\right)$

2) $0 < \frac{1}{n^2}+\frac{1}{(n+1)^2}+\dots+\frac{1}{(n+n)^2} \le \overbrace{\frac{1}{n^2}+\frac{1}{n^2}+\dots+\frac{1}{n^2}}^{\#\to(n+1)} = \frac{n+1}{n^2}$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty}\frac{n+1}{n^2}=0$

$=$

$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\dots+\frac{1}{(2n)^2}\right)=0$, by Sandwich Theorem.

3) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+n^2}\right)$

We know that: $\int_0^1 \frac{dx}{1+x^2}=\frac{\pi}{4}$

$$\int_0^1 \frac{dx}{1+x^2} = \lim_{n\to\infty} R_p = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

$f(x)=\frac{1}{1+x^2}$, $b=1$, $a=0$, $\Delta x=\frac{b-a}{n}=\frac{1}{n}$

$x_k = a+k\cdot\Delta x = k\cdot\frac{1}{n}$

$$\int_0^1 \frac{dx}{1+x^2} = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}$$

$$= \lim \sum_{k=1}^{n}\frac{1}{1+\frac{k^2}{n^2}}\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\frac{k^2+n^2}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{n^2}{k^2+n^2} = \lim_{n\to\infty} n\cdot\sum_{k=1}^{n}\frac{1}{n^2+k^2}$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{1}{n^2+1}+\frac{1}{n^2+2^2}+\dots+\frac{1}{n^2+n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+n^2}\right)$$

$$= \frac{\pi}{4}$$