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Results in Mathematics

Almost Contact Lagrangian Submanifolds of Nearly Kaehler 6-Sphere

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Abstract. For a Lagrangian submanifold M of S^6 with nearly Kaehler structure, we provide conditions for a canonically induced almost contact metric structure on M by a unit vector field, to be Sasakian. Assuming M contact metric, we show that it is Sasakian if and only if the second fundamental form annihilates the Reeb vector field ξ, furthermore, if the Sasakian submanifold M is parallel along ξ, then it is the totally geodesic 3-sphere. We conclude with a condition that reduces the normal canonical almost contact metric structure on M to Sasakian or cosymplectic structure.

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1. Introduction

In [8], Ejiri Showed that a Lagrangian submanifold of the nearly Kaehler 6-dimensional unit sphere S^6 is orientable and minimal. Lagrangian submanifolds of the nearly Kaehler unit 6-sphere were studied by Dillen and Vrancken [6], Dillen et al. [7] and others. Deshmukh and Hadi [5] proved the following result.

Theorem (D–H). *Let M be a compact 3-dimensional Lagrangian submanifold of S^6 with nearly Kaehler structure (g, J). Then there exists a global unit vector field ξ on M, and if $J\xi$ is parallel in the normal bundle, then M is a Sasakian manifold.*

Their proof is based on the fact (see Martinet [9]) that a compact orientable 3-dimensional manifold does carry a contact structure, and the construction of a canonical almost contact metric structure defined by

$\varphi X = G(X, J\xi)$ where J is the almost complex structure and G is the covariant derivative of J and X an arbitrary vector field tangent to M. Intrigued by this result, Vrancken [12] showed that the second fundamental form of a Sasakian Lagrangian submanifold M of the nearly Kaehler unit 6-sphere annihilates the Reeb vector field, and provided a complete classification of such submanifolds. In this context, as the second fundamental form annihilates ξ, Chen's basic equality (see [3]) is satisfied (see [6]).

In this paper, we examine the more general situation when M (not necessarily compact) admits a global unit vector field ξ, and show that this induces a canonical almost contact metric structure on M with the metric induced by embedding, and an underlying (1,1)-tensor field F on M. We will consider two cases when the canonical structure is (i) contact metric, and (ii) normal almost contact metric; and show that the structure reduces to Sasakian in case (i) and Sasakian or Cosymplectic in case (ii), under the assumption that F is divergence-free.

Let us briefly review the Lagrangian submanifolds of the nearly Kaehler 6-sphere. Let J be the almost complex structure defined on S^6 inherited from the Cayley division algebra [8]. Then (S^6, J, g) is a nearly Kaehler manifold, where g is the standard metric on S^6 of constant curvature 1. Define a tensor field G of type (1,2) on S^6 by $G(X,Y) = (\bar{\nabla}_X J)(Y)$, where X, Y are arbitrary vector fields on S^6, and $\bar{\nabla}$ the Riemannian connection on S^6 with respect to the Riemannian metric g on S^6. G satisfies the following properties (see [7,8]):

$$G(X,Y) = -G(Y,X) \tag{1}$$

$$G(X,JY) = -JG(X,Y) \tag{2}$$

$$g(G(X,Y),Z) = -g(G(X,Z),Y) \tag{3}$$

$$\begin{aligned} g(G(X,Y),G(Z,W)) &= g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &\quad + g(JX,Z)g(Y,JW) - g(JX,W)g(Y,JZ) \end{aligned} \tag{4}$$

$$(\bar{\nabla}_X G)(Y,Z) = g(Y,JZ)X + g(X,Z)JY - g(X,Y)JZ \tag{5}$$

where X, Y, Z are arbitrary vector fields on S^6.

We denote the metrics of S^6 and its submanifold M by the same letter g, and the normal bundle of M by ν. If $JTM = \nu$, where TM is the tangent bundle of M, then M is said to be a Lagrangian submanifold of S^6. If ∇ and $\nabla^\perp$ denote the Riemannian connection induced on M, and the connection in the normal bundle ν respectively, then we have (see [8])

$$\nabla^\perp_X JY = J\nabla_X Y + G(X,Y) \tag{6}$$

$$\sigma(X,Y) = JA_{JY}X \tag{7}$$

$$JG(X, JG(Y,Z)) = g(X,Z)Y - g(X,Y)Z \tag{8}$$

$$-\sigma(X, JG(Y,Z)) + JG(\sigma(X,Y),Z) + JG(Y,\sigma(Z,X)) = 0 \tag{9}$$

where $X, Y \in \mathfrak{X}(M)$, σ is the second fundamental form and A_{JY} is the Weingarten map with respect to the normal vector field JY. Vrancken has pointed out (private communication) that the minus sign in the first term of Eq. (9) is missing in [8] and also on p. 403 in [3]. The correct form appears in the Lemma 3.2 of the paper [13] of Schafer–Smoczyk.

Let us also review almost contact metric structures. A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if carries a global 1-form η, a vector field ξ, a (1,1)-tensor field φ, and a Riemannian metric g satisfying

$$\begin{aligned}&\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1,\\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\end{aligned} \tag{10}$$

Obviously, $\varphi\xi = 0$, $\eta\circ\varphi = 0$, $g(X, \xi) = \eta(X)$, and ξ is unit. The almost contact metric structure (η, ξ, g) on M is called a contact metric structure if

$$(d\eta)(X, Y) = g(X, \varphi Y).$$

For a contact metric manifold, following Blair [1], we define a (1,1) tensor $h = \frac{1}{2}L_\xi\varphi$ which is known to be self-adjoint, trace-free, anti-commutes with φ, and annihilates ξ. We have the following formulas for a contact metric manifold:

$$\nabla_X\xi = -\varphi X - \varphi h X \tag{11}$$

$$Ric(\xi, \xi) = 2n - |h|^2. \tag{12}$$

The special case when $h = 0$ corresponds to K-contact metrics for which ξ is g-Killing. An almost contact metric is called Sasakian if

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{13}$$

where ∇ is the Riemannian connection of g. A contact metric is K-contact if and only if

$$Ric(X, \xi) = 2n\eta(X). \tag{14}$$

In dimension 3, K-contact condition is equivalent to Sasakian condition.
An almost contact metric structure on M is said to be normal if the almost complex structure $\mathcal{J}$ on $M\times\mathcal{R}$ defined by $\mathcal{J}(X, f\frac{d}{dt}) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$ is integrable. For a 3-dimensional almost contact metric manifold, we have the following formula (Olszak [10])

$$(\nabla_X\varphi)Y = g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi. \tag{15}$$

A 3-dimensional normal almost contact structure satisfies [10]

$$(\nabla_X\varphi)Y = a(g(X, Y)\xi - \eta(Y)X) + b(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{16}$$

$$\nabla_X\xi = -a\varphi X + b(X - \eta(X)\xi) \tag{17}$$

where a, b are smooth functions on M. Using Eq. (17) and that φ is anti self-adjoint with respect to g, we find

$$(L_\xi g)(X, Y) = 2b(g(X, Y) - \eta(X)\eta(Y)).$$

Next, Lie-differentiating the relation $\eta(X) = g(X, \xi)$ along ξ and using the foregoing equation gives $L_\xi \eta = 0$. Also, the use of Eqs. (16) and (17) shows that $L_\xi \varphi = 0$. From Eq. (17) we have $(d\eta)(X, Y) = -2ag(X, Y)$. Taking its Lie-derivative along ξ, noting that Lie-derivation commutes with exterior derivation, and using the values of $L_\xi g$, $L_\xi \eta$ and $L_\xi \varphi$ computed earlier, and also that $\eta(\varphi X) = 0$, we obtain $(\xi a + 2ab) g(\varphi X, Y) = 0$. As φ vanishes nowhere on M, in view of the first equation in (10), it follows that

$$\xi a = -2ab \tag{18}$$

We note here that almost contact metric structure satisfying the condition (16) is known as a trans-Sasakian structure (see Oubina [11]).
A cosymplectic manifold is a normal almost contact metric manifold such that η and Φ (the 2-form defined by $\Phi(X, Y) = g(X, \varphi Y)$) are both closed. This definition is equivalent to $\nabla \varphi = 0$ on an almost contact metric manifold. See [1].

Henceforth we will assume that (M, g) is a Lagrangian submanifold of the nearly Kaehler 6-sphere.

2. Canonical Almost Contact Metric Structure on M

First, we state and prove the following lemma.

Lemma 1. *A unit vector field ξ on a Lagrangian submanifold (M, g) of the nearly Kaehler 6-sphere, induces a canonical almost contact metric structure (φ, g, ξ) with structure tensor φ defined by $\varphi X = G(X, J\xi)$.*

Proof. We begin with the hypothesis that ξ is a global unit vector field on M with respect to the induced metric g on M, and define a 1-form η by $\eta(X) = g(X, \xi)$. We also note from Eq. (2) that, for $X \in \mathfrak{X}(M)$, the vector field $G(X, J\xi) = -JG(X, \xi)$ is tangential to M, because we know from lemma 4.1 of [8] that $G(X, Y)$ is normal to M for all vector fields X, Y tangent to M, and hence $JG(X, Y)$ is tangent to M, as M is Lagrangian. Hence we define a (1,1)-tensor φ on M by

$$\varphi X = G(X, J\xi)$$

which shows, in view of properties (1) and (2), that $\varphi(\xi) = 0$. We also have that

$$\begin{aligned} g(\varphi X, Y) &= g\left(G(X, J\xi), Y\right) = -g\left(G(J\xi, X), Y\right) = g\left(X, G(J\xi, Y)\right) \\ &= -g\left(X, G(Y, J\xi)\right) = -g(X, \varphi Y). \end{aligned}$$

Further, we have

$$\begin{aligned} \varphi^2 X &= G(G(X, J\xi), J\xi) = -JG(G(X, J\xi), \xi) \\ &= -JG(-JG(X, \xi), \xi) = -JG(\xi, JG(X, \xi)). \end{aligned}$$

Using Eq. (8) in the above equation shows that

$$\varphi^2 X = -X + \eta(X)\xi$$

for any $X \in \mathfrak{X}(M)$. Furthermore,

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(G(X, J\xi), G(Y, J\xi)) = g(G(X, \xi), G(Y, \xi)) \\ &= g(X, Y) - \eta(X)\eta(Y). \end{aligned}$$

□

Thus (φ, ξ, η, g) is an almost contact metric structure on M.

Definition 1. The structure (φ, ξ, η, g) defined by a unit vector field ξ, as defined in the above Lemma, will be called a canonical almost contact metric structure on M.

As M is Lagrangian, we can set $\nabla^{\perp}_X J\xi = JFX$, where F is a (1,1)-tensor field on M, and prove

Proposition 1. *Let M be a Lagrangian submanifold of S^6 with nearly Kaehler structure (g, J) and ξ be a global unit vector field on M, with the canonically induced almost contact metric structure (φ, g, ξ) on M. Then the structure is Sasakian if and only if $F = 0$.*

Proof. Using formulas (5), (7) and (9), we compute the covariant derivative of φ as follows.

$$\begin{aligned} (\nabla_X \varphi)(Y) &= \nabla_X G(Y, J\xi) - G(\nabla_X Y, J\xi) \\ &= \overline{\nabla}_X G(Y, J\xi) - \sigma(X, G(Y, J\xi)) - G(\nabla_X Y, J\xi) \\ &= (\bar{\nabla}_X G)(Y, J\xi) + G(\sigma(X, Y), J\xi) + G(Y, \nabla^{\perp}_X J\xi) \\ &\quad - G(Y, A_{J\xi} X) - \sigma(X, G(Y, J\xi)) \\ &= g(X, Y)\xi - \eta(Y)X - JG(\sigma(X, Y), \xi) - \sigma(X, G(Y, J\xi)) \\ &\quad - G(Y, A_{J\xi} X) + G(Y, \nabla^{\perp}_X J\xi) \\ &= g(X, Y)\xi - \eta(Y)X - JG(\sigma(X, Y), \xi) + \sigma(X, JG(Y, \xi)) \\ &\quad - JG(Y, \sigma(X, \xi)) + G(Y, \nabla^{\perp}_X J\xi) \\ &= g(X, Y)\xi - \eta(Y)X + G(Y, \nabla^{\perp}_X J\xi). \end{aligned}$$

As per our setting $\nabla^{\perp}_X J\xi = JFX$, the above equation becomes

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X + G(Y, JFX). \tag{19}$$

If $F = 0$, then obviously the structure is Sasakian. Conversely, if the structure is Sasakian, then (19) reduces to $G(Y, JFX) = 0$. Substituting ξ for Y, and using (1) and (2) gives $\varphi FX = 0$. Operating it by φ provides $F = 0$, because $\eta(FX) = g(FX, \xi) = g(JFX, J\xi) = g(\nabla^{\perp}_X J\xi, J\xi) = 0$. This completes the proof. □

Thus the question arises as to whether we may be able to weaken the condition on F for the canonical structure to reduce to Sasakian. In the next section, we provide an answer assuming the canonical structure to be contact metric.

3. Canonical Contact Metric Structure on M

Theorem 1. *Let M be a Lagrangian submanifold of S^6 with nearly Kaehler structure (g, J) and ξ be a global unit vector field such that the canonically induced almost contact metric structure (φ, g, ξ) is contact metric structure on M. Then the structure is Sasakian if and only if F is divergence-free.*

Proof. Substituting ξ for Y in (19) and using the property $\varphi\xi = 0$ gives $-\varphi\nabla_X\xi = \eta(X)\xi - X + G(\xi, JFX)$. Using Eqs. (10) and (11) in this we get

$$\varphi^2 hX = G(\xi, JFX). \tag{20}$$

Operating it by φ and noting that $\varphi^3 = -\varphi$ [which follows from Eq. (10)], we get $-\varphi hX = \varphi G(\xi, JFX)$. As $h\varphi = -\varphi h$ for a contact metric structure, using the definition of the canonical metric structure: $\varphi X = G(X, J\xi)$, and also the Eqs. (1), (2) and (8) we get

$$\begin{aligned} h\varphi X &= G(G(\xi, JFX), J\xi) = -JG(G(\xi, JFX), \xi) = -JG(-JG(\xi, FX), \xi) \\ &= -JG(\xi, JG(\xi, FX)) = -[g(\xi, FX)\xi - g(\xi, \xi)FX] = FX \end{aligned}$$

where we used $\eta(FX) = 0$ which was shown in the proof of Proposition 1. Thus we have

$$FX = h\varphi X. \tag{21}$$

We take the divergence on both sides of this equation and use the well-known formula (see Blair and Sharma [2]): $(div.h\varphi)(X) = Ric(\xi, X) - 2\eta(X)$ for a contact metric, in order to obtain

$$(div.F)(X) = Ric(\xi, X) - 2\eta(X). \tag{22}$$

Thus the vanishing of $div.F$ implies $Ric(\xi, X) = 2\eta(X)$. Hence, from Eq. (14) we conclude that the contact metric structure is K-contact, and since the dimension of M is 3, it is Sasakian. The converse is obvious. This completes the proof. □

Remark 1. The right hand side of Eq. (21) is metrically equivalent to half of the strain tensor (also known as the torsion tensor, see Chern and Hamilton [4]) $L_\xi g$, i.e. $(L_\xi g)(X, Y) = 2g(h\varphi X, Y)$ which follows from Eq. (11).

At this point, we present a generalization of a result of Vrancken stated in the beginning of Section 1, by considering M as a contact metric submanifold and proving the following result.

Theorem 2. *Suppose that the Lagrangian submanifold (M, g) of the nearly Kaehler 6-sphere S^6 (g, J) is a contact metric manifold. Then*

(i) *M is Sasakian if and only if its second fundamental form annihilates the Reeb vector field ξ,*

(ii) *for Sasakian M, structure tensor φ is given by $\varphi X = G(X, J\xi)$,*

(iii) *if the Sasakian submanifold M is parallel along ξ, i.e. the second fundamental form σ is parallel along ξ, then it is the totally geodesic 3-sphere.*

Remark 2. We recall from (p. 40 of [3]) that an isometrically embedded submanifold M of a Riemannian manifold $\bar{M}$ is called a parallel submanifold if the second fundamental form σ is parallel with respect to the van der Waerden–Bortolotti connection $\tilde{\nabla}$ as defined by the Eq. (25). Part (iii) of the above theorem considers weakening this parallelism of σ to parallelism along the Reeb vector field ξ (i.e $\tilde{\nabla}_\xi \sigma = 0$) of the Sasakian submanifold of the nearly Kaehler S^6, and shows that σ vanishes.

Proof of Theorem 2. Contracting the Gauss equation

$$g(R(X,Y)Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - g(\sigma(X,Z),\sigma(Y,W)) + g(\sigma(X,W),\sigma(Y,Z))$$

at X and W, with respect to a local orthonormal frame (e_i), $i = 1, 2, 3$ on M, and using the minimality of M, we obtain

$$Ric(Y,Z) = 2g(Y,Z) - \sum_i g(\sigma(e_i,Y),\sigma(e_i,Z)). \tag{23}$$

Substituting ξ for Y and Z in the above, and using the formula (12) yields the relation

$$|h|^2 = \sum_i g(\sigma(e_i,\xi),\sigma(e_i,\xi)).$$

Now, for a 3-dimensional contact metric manifold we know (see p. 94 of [1]) that

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$$

If $h = 0$, then the above equation reduces to the Sasakian condition (13). Conversely, if M is Sasakian, then comparing (13) with the above formula and subsequently substituting $Y = \xi$ gives $hX = g(hX,\xi)\xi = g(X,h\xi)\xi = 0$, because h is self-adjoint and annihilates ξ for a contact metric structure on M. Hence the contact metric M is Sasakian if and only if $h = 0$. Thus we conclude from the unnumbered equation following Eq. (23) whose right hand side is the sum of the squared norms of $\sigma(e_i,\xi)$, that M is Sasakian, i.e. $h = 0$ if and only if $\sigma(X,\xi) = 0$. Notice that for a contact metric, h is self-adjoint. This proves part (i).

For part (ii), we first note from Eq. (7) and our foregoing conclusion $\sigma(X,\xi) = 0$ that $A_{J\xi} = 0$. Now let (g, ϕ, ξ) be the Sasakian structure on the Lagrangian submanifold M and e be a local unit vector field. As ϕ is anti self-adjoint, $g(\phi e, e) = -g(e, \phi e)$ and hence $g(\phi e, e) = 0$, i.e. $\phi e \perp e$. From the last equation of (10), ϕe is also unit. Further, $g(\phi e, \xi) = -g(e, \phi\xi) = 0$, because $\phi\xi = 0$. Hence $\phi e \perp \xi$. Thus, $(e, \phi e, \xi)$ is a local orthonormal basis. It is known (see [8]) that

$$G(e, \phi e) = -J\xi, \quad G(\phi e, \xi) = -Je, \quad G(\xi, e) = -J\phi e.$$

Using the above equations, formulas (1), (6), and (11) with $h = 0$ for a Sasakian metric, we obtain the relations

$$\nabla^\perp_e J\xi = -J\phi e + G(e, \xi) = 0, \quad \nabla^\perp_{\phi e} J\xi = -J\phi^2 e + G(\phi e, \xi) = 0, \quad \nabla^\perp_\xi J\xi = 0$$

which show that $\nabla^\perp_X J\xi = 0$, i.e. $J\xi$ is parallel in the normal bundle. As shown in Lemma 1, the (1,1)-tensor φ defined by $\varphi X = G(X, J\xi)$ defines an almost contact metric structure (φ, ξ, η, g) on M. Using the results $A_{J\xi} = 0$ and $\nabla^\perp_X J\xi = 0$ and the fact that M has a Sasakian structure (η, ξ, ϕ, g) we find

$$\begin{aligned} \phi e &= -\nabla_e \xi = -\overline{\nabla}_e \xi = \overline{\nabla}_e JJ\xi = \left(\overline{\nabla}_e J\right)(J\xi) + J\overline{\nabla}_e J\xi \\ &= G(e, J\xi) = \varphi e. \end{aligned}$$

Similarly, we show that $\phi(\phi e) = \varphi(\phi e)$. As we already know $\phi\xi = \varphi\xi = 0$, it turns out that $\varphi = \phi$, proving part (ii). □

Finally, for part (iii), we find from the Codazzi equation that

$$(\tilde{\nabla}_X \sigma)(Y, Z) = (\tilde{\nabla}_Y \sigma)(X, Z) \tag{24}$$

where $\tilde{\nabla}$ is the van der Waerden–Bortolotti connection defined by

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{25}$$

Substituting $Y = \xi$ in (24), using the hypothesis $\tilde{\nabla}_\xi \sigma = 0$, definition (25) and the result $\sigma(X, \xi) = 0$, we immediately obtain $\sigma(\nabla_X \xi, Z) = 0$. But, as M is Sasakian, $\nabla_X \xi = -\varphi X$. Thus $\sigma(\varphi X, Z) = 0$. As X is an arbitrary tangent vector field on M, replacing X by φX in the foregoing equation, using the first equation of (10) and part (i) of this theorem, we obtain $\sigma = 0$, completing the proof.

4. Canonical Normal Almost Contact Metric Structure on M

Motivated by the condition: $div.F = 0$, assumed in Theorem 1, we suppose the canonical almost contact metric structure on M to be normal, and imposing $div.F = 0$, prove the following classification result.

Theorem 3. *Let the canonical almost contact metric structure on the Lagrangian submanifold M of the nearly Kaehler 6-sphere be normal. If F is divergence-free, then M is either Sasakian or Cosymplectic.*

Proof. By hypothesis, the canonical structure is a normal contact metric structure, and hence from Eq. (19) we have that

$$\nabla_X \xi = -\varphi X + FX. \tag{26}$$

Comparing it with Eq. (17) gives

$$FX = (1-a)\varphi X + b(X - \eta(X)\xi). \tag{27}$$

Differentiating (27) along an arbitrary vector field Y on M, we have

$$\begin{aligned}(\nabla_Y F)X = &-(Ya)\varphi X + (1-a)(\nabla_Y \varphi)X + (Yb)[X - \eta(X)\xi] \\ &-b((\nabla_Y \eta)X)\xi - b\eta(X)\nabla_Y \xi\end{aligned}$$

Let (e_i) $(i = 1, 2, 3)$ be a local orthonormal frame on M. Substituting $Y = e_i$ in the above equation, taking inner product with e_i and summing over $i = 1, 2, 3$, and using the hypothesis $div.F = 0$, we obtain

$$Xb - \eta(X)\xi b - (\varphi X)a = 2[a(1-a) + b^2]\eta(X). \tag{28}$$

Substituting ξ for X immediately provides

$$a(1-a) + b^2 = 0. \tag{29}$$

Hence (28) reduces to

$$Xb - (\varphi X)a - (\xi b)\eta(X) = 0. \tag{30}$$

Only two cases can occur: either (i) $b = 0$ on M and hence from (29) $a = 1$ or 0, or (ii) $b \neq 0$ on some open part $\mathcal{U}$ of M and hence $a \neq 0$, $a \neq 1$ on $\mathcal{U}$. Let us work on $\mathcal{U}$ and rule out case (ii). Differentiating Eq. (29) along an arbitrary vector field X on $\mathcal{U}$, and then substituting $X = \xi$ and also using (18) gives

$$Xb = \frac{2a-1}{2b}Xa, \quad \xi b = a(1-2a). \tag{31}$$

Using the above two equations in (30) provides

$$\frac{2a-1}{2b}Xa - (\varphi X)a + a(2a-1)\eta(X) = 0.$$

As X is arbitrary, substituting φX for X, using the first equation of (10), Eq. (18) and the property $\eta(\varphi X) = 0$, we obtain

$$\frac{2a-1}{2b}(\varphi X)a + Xa + 2ab\eta(X) = 0. \tag{32}$$

Eliminating $(\varphi X)a$ between the above two equations, and subsequently replacing X with φX we get $(\varphi X)a = 0$. Thus, (30) becomes $db = (\xi b)\eta$. Applying d on it and using Poincaré lemma: $d^2 = 0$, we have $d(\xi b)\wedge\eta + (\xi b)d\eta = 0$. Operating both sides of the resulting equation on the pair $(X, \varphi X)$, where X is an arbitrary vector field $\perp \xi$ on $\mathcal{U}$, we obtain $(\xi b)(d\eta)(X, \varphi X) = 0$. This can be written as $(\xi b)[g(\nabla_X \xi, \varphi X) - g(\nabla_{\varphi X}\xi, X)] = 0$. The use of Eq. (17) and first equation of (10) in the preceding equation provides $a(\xi b)g(\varphi X, \varphi X) = 0$. The use of the last equation of (10) turns the preceding equation into $a(\xi b)g(X, X) = 0$. As $a \neq 0$ on $\mathcal{U}$ and X is arbitrary, we obtain $\xi b = 0$ on $\mathcal{U}$. Hence $db = 0$, i.e. b is constant on $\mathcal{U}$. So, Eq. (31) implies $(1-2a)Xa = 0$. As $a \neq \frac{1}{2}$ anywhere on $\mathcal{U}$, otherwise (29) would be violated, we conclude that a is constant on $\mathcal{U}$. Finally, appealing to Eq. (32) provides $ab = 0$ which contradicts the assumption for case (ii). □

Hence we conclude that $a = 1, b = 0$ in which case Eq. (16) reduces to (13) and hence M is Sasakian, or $a = b = 0$ in which case (16) reduces to $\nabla\varphi = 0$, i.e. as defined in the introduction, M is cosymplectic, completing the proof.

5. Concluding Remark

For the canonical contact metric structure on the Lagrangian submanifold M of the nearly Kaehler S^6, we note that Eq. (26) holds. Also, Proposition 1 asserts that $F = 0$ if and only if the canonical structure M is Sasakian. Thus the tensor F measures the deviation of M from becoming Sasakian. More generally, Theorem 1 tells us that the condition $F = 0$ for the canonical structure to be Sasakian, can be weakened to $div.F = 0$, when the canonical almost contact metric on M is a contact metric.

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