Chapter 5:
Generating Random Numbers from Distributions

See Reading Assignment

Review

1. Inverse Transform

Generate a number $\mathbf{u}_i$ between 0 and 1 (one U-axis) and then find the corresponding $\mathbf{x}_i$ coordinate by using $\mathbf{F}^{-1}(\cdot)$.

2. The Convolution Method

The distribution of the sum of two or more random variables is called the *convolution*.

3. Acceptance/Rejection Method

Replace $f(x)$ by a simple PDF, $w(x)$, which can be sampled from more easily. $w(x)$ is based on the development of a majorizing function for $f(x)$.

5. Mixed, Truncated and Shifted Dist.

- We consider three random variate generation methods
  - Mixture Distributions
  - Truncated Distributions
  - Shifted Distributions

- The new methods depend on previous methods.

- These methods give flexibility in modeling the randomness.

Simulation from Mixed Dist.

The distribution of a random variable X is a mixture distribution if the CDF of X has the form

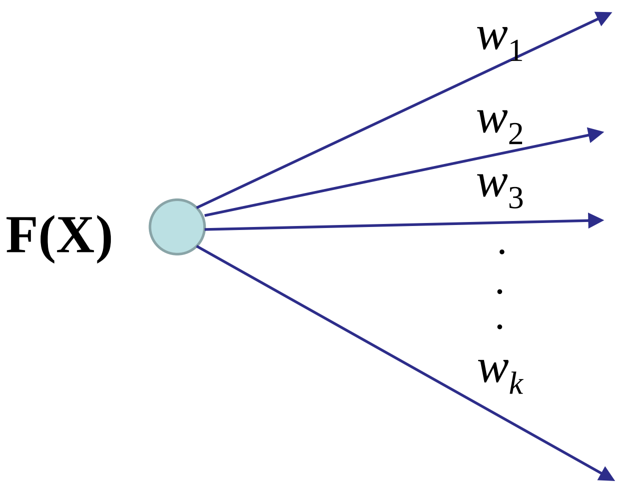
$$F_X(x) = \sum_{i=1}^{k} \omega_i F_{X_i}(x)$$

where $0 < \omega_i < 1, \sum_{i=1}^{k} \omega_i = 1, k \geq 2$, *and* $F_{X_i}(x)$ *is the CDF of a continuous or discrete random variable* $X_i, i = 1, \ldots, k$.

Simulation from Mixed Dist.

Independent

Cont. Prob. | Disc. Prob.



$F_1(X)$ → P{Choose 1st CDF = } = w_1

$F_2(X)$ → P{Choose 2nd CDF = } = w_2

$F_3(X)$ → P{Choose 3rd CDF = } = w_3

.
.
.

$F_k(X)$ → P{Choose kth CDF = } = w_k

Simulation from Mixed Dist.

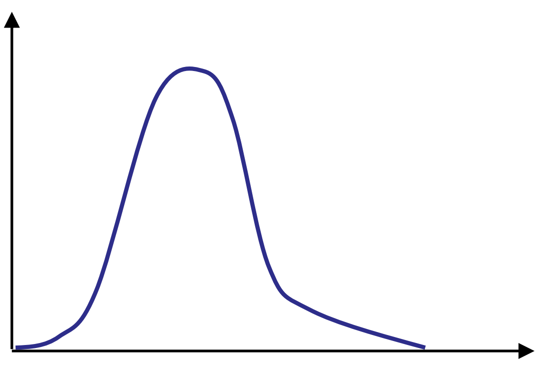
- Mixture distributions combine the characteristics of two or more distributions,
- More flexibility in modeling many processes.
- Example, standard distributions, such as the normal, Weibull, and lognormal, have a single mode. Mixture distributions are often utilized for the modeling of data sets that have more than one mode.

Simulation from Mixed Dist.

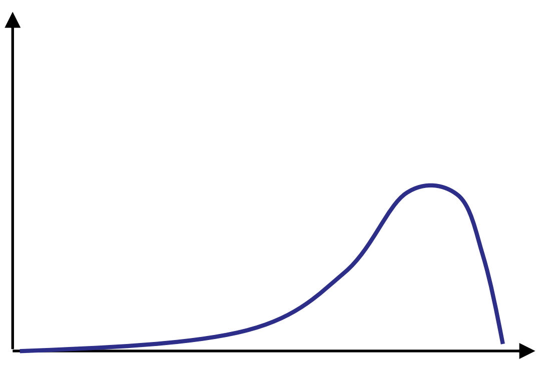
Example

Process: event follow some distribution in three days of the week and the event change to another distribution from four days of the week

1st three days of the week dist.



2nd four days of the week dist.



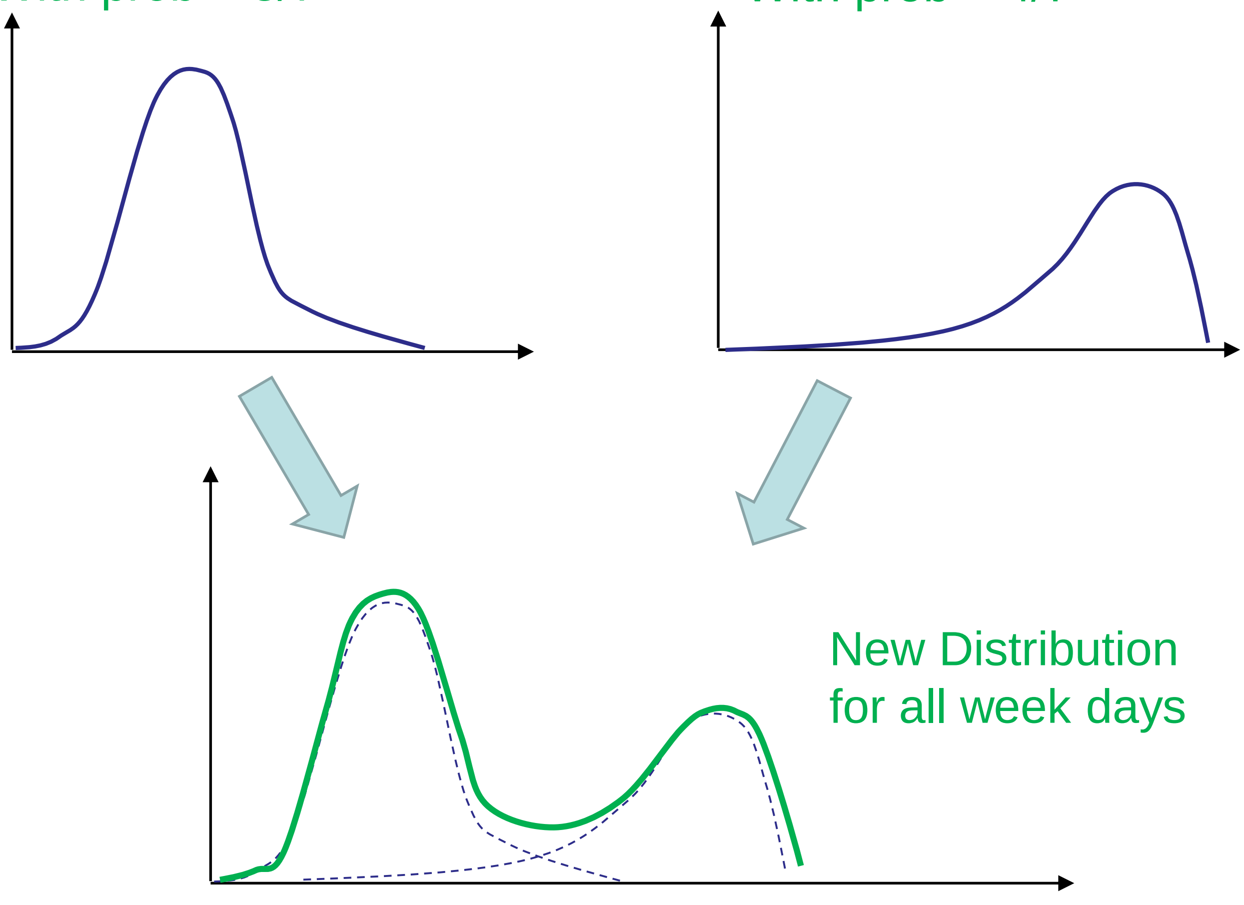
5. Mixed, Truncated and Shifted Dist.

5.1 Mixture Distribution

Example

1st three days of the week dist.
With prob = 3/7

2nd four days of the week dist.
With prob = 4/7



New Distribution
for all week days

Simulation from Mixed Dist.

- **<u>Example</u>**

  Suppose the time that it takes to pay with a credit card, X_1, is exponentially distributed with a mean of 1.5 min and the time that it takes to pay with cash, X_2, is exponentially distributed with a mean of 1.1min. In addition, suppose that the chance that a person pays with credit is 70%. Then, the overall distribution representing the payment service time, X, has an hyperexponential distribution with parameters $\omega_1 = 0.7$, $\omega_2 = 0.3$, $\lambda_1 = 1/(1.5)$, and $\lambda_2 = 1/(1.1)$.

Simulation from Mixed Dist.

- **Example**

  Then, distribution of the payment service time, X, has an hyperexponential distribution with parameters

  $\omega_1 = 0.7$, *Exponential* $\lambda_1 = 1/1.5$

  and $\omega_2 = 0.3$, *Exponential* $\lambda_2 = 1/1.1$

$$F_X(x) = \omega_1 F_{X_1}(x) + \omega_2 F_{X_2}(x)$$

$$F_{X_1}(x) = 1 - \exp\ (-\lambda_1 x)$$

$$F_{X_2}(x) = 1 - \exp\ (-\lambda_2 x)$$

Simulation from Mixed Dist.

- **Example**

$$F_X(x) = \omega_1 F_{X_1}(x) + \omega_2 F_{X_2}(x)$$

$$F_{X_1}(x) = 1 - \exp\ (-\lambda_1 x)$$

$$F_{X_2}(x) = 1 - \exp\ (-\lambda_2 x)$$

- The algorithm for this distribution is

```
1: Generate u ~ U(0, 1)
2: Generate v ~ U(0, 1)
3: IF u ≤ 0.7 THEN
4:     X = F_{X_1}^{-1}(v) = -1.5 ln(1 - v)
5: ELSE
6:     X = F_{X_2}^{-1}(v) = -1.1 ln(1 - v)
7: END IF
8: RETURN X
```

Simulation from Mixed Dist.

- **Example**

$$F_X(x) = \omega_1 F_{X_1}(x) + \omega_2 F_{X_2}(x)$$

$$F_{X_1}(x) = 1 - \exp\ (-\lambda_1 x)$$

$$F_{X_2}(x) = 1 - \exp\ (-\lambda_2 x)$$

n	U	Choose F	V	Get X
1	0.592	F1(X)	0.641	1.537
2	0.818	F2(X)	0.984	4.520
3	0.375	F1(X)	0.495	1.026
4	0.371	F1(X)	0.902	3.483
5	0.812	F2(X)	0.815	1.859
6	0.961	F2(X)	0.026	0.029
7	0.168	F1(X)	0.188	0.312
8	0.274	F1(X)	0.082	0.129
9	0.438	F1(X)	0.387	0.733
10	0.925	F2(X)	0.243	0.306

Simulation from Mixed Dist.

- ***Notes***
- In Example generating *X* use the inverse transform method for generating from the two exponential distribution.

- General mixture distribution might be any distribution.
  Ex.: mixture of a gamma and a lognormal dist.

- To give flexibility in modeling and generation, use any generation technique.
  Ex.: one $F_1(x)$ *use inverse transform, other* $F_2(x)$ *use acceptance/ rejection, and* $F_3(x)$ *use convolution.*