Chapter 5:
Generating Random Numbers from Distributions

Refer to Reading Matrials:

Inverse Transform (Continuous Dist.)

- First, generate a number $\mathbf{u}_i$ between 0 and 1 (one U-axis) and then find the corresponding $\mathbf{x}_i$ coordinate by using $\mathbf{F^{-1}(\cdot)}$.
- For various values of $\mathbf{u}_i$, the $\mathbf{x}_i$ will be properly "distributed" along the x-axis.
- This method is that there is a one-to-one mapping between $\mathbf{u}_i$ and $\mathbf{x}_i$ because of the monotone property of the CDF.

2. Inverse Transform (Discrete Dist.)

- The inverse CDF method also works for discrete distributions.
- A discrete random variable, X, with values $x_1, x_2, \dots, x_n$ the probability mass function (PMF) and denoted

$$f(x_i) = P(X = x_i)$$

$$\sum_{i=1}^{n} f(x_i) = 1$$

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

2. Inverse Transform (Discrete Dist.)

- Given a PMF for a discrete variable X

X	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
P{X}	p_1	p_2	p_3	p_4	p_5

With $0 \leq p_i \leq 1$

$$p_1+p_2+p_3+p_4+p_5 = 1$$

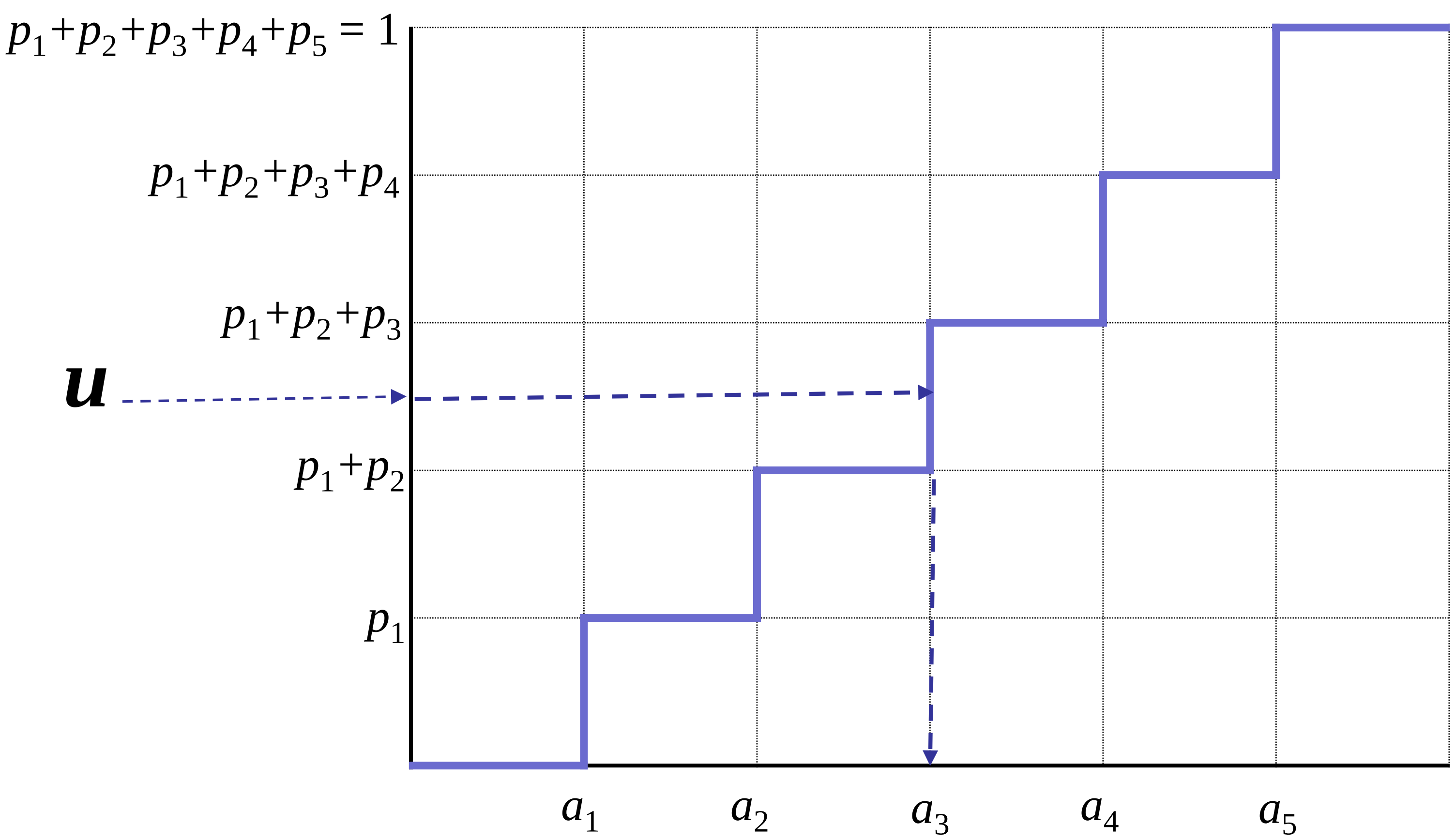
- The CDF of X is:

X	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$
P{X≤ a_i}	p_1	p_1+p_2	$p_1+p_2+p_3$	$p_1+p_2+p_3+p_4$	$p_1+p_2+p_3+p_4+p_5$

- We need to have for any u~ U[0,1] a value from X.

2. Inverse Transform (Discrete Dist.)

- The CDF graph



2. Inverse Transform (Discrete Dist.)

- From the CDF graph: for u~ U[0,1]

If $0 \le u \le p_1$

Return X = a_1

If $p_1 \le u \le p_1 + p_2$

Return X = a_2

If $p_1 + p_2 \le u \le p_1 + p_2 + p_3$

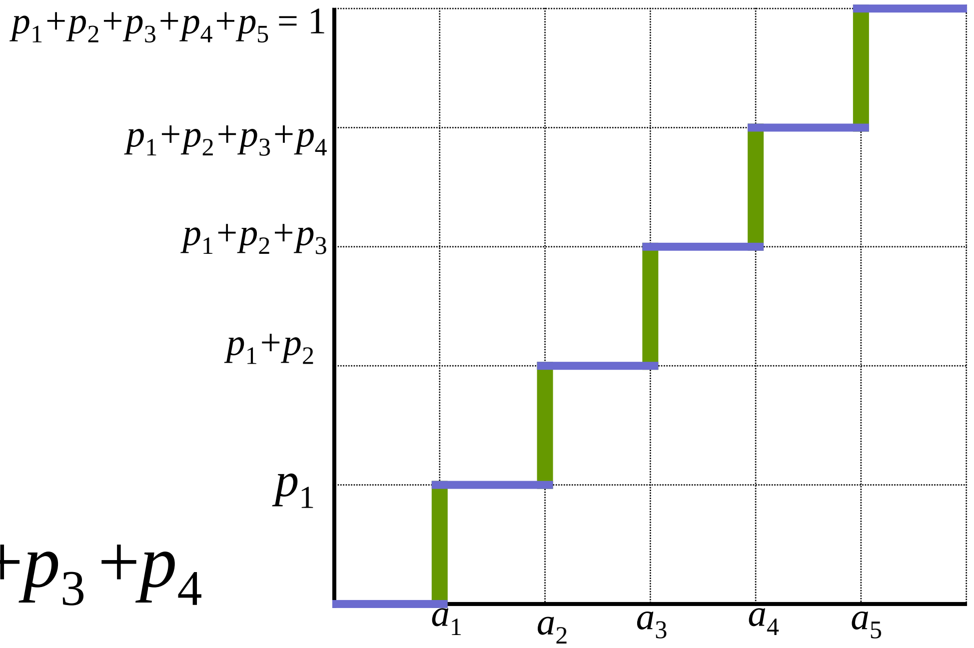
Return X = a_3

If $p_1 + p_2 + p_3 \le u \le p_1 + p_2 + p_3 + p_4$

Return X = a_4

If $p_1 + p_2 + p_3 + p_4 \le u \le 1$

Return X = a_5



2. Inverse Transform (Discrete Dist.)

Example:

The functional form of the PMF and CDF are given as follows:

$$P\{X = x\} = \begin{cases} 0.4 & x = 1 \\ 0.3 & x = 2 \\ 0.2 & x = 3 \\ 0.1 & x = 4 \end{cases} \qquad F(x) = \begin{cases} 0.0 & \text{if } x < 1 \\ 0.4 & \text{if } 1 \le x < 2 \\ 0.7 & \text{if } 2 \le x < 3 \\ 0.9 & \text{if } 3 \le x < 4 \\ 1.0 & \text{if } x \ge 4 \end{cases}$$

2. Inverse Transform (Discrete Dist.)

Example:

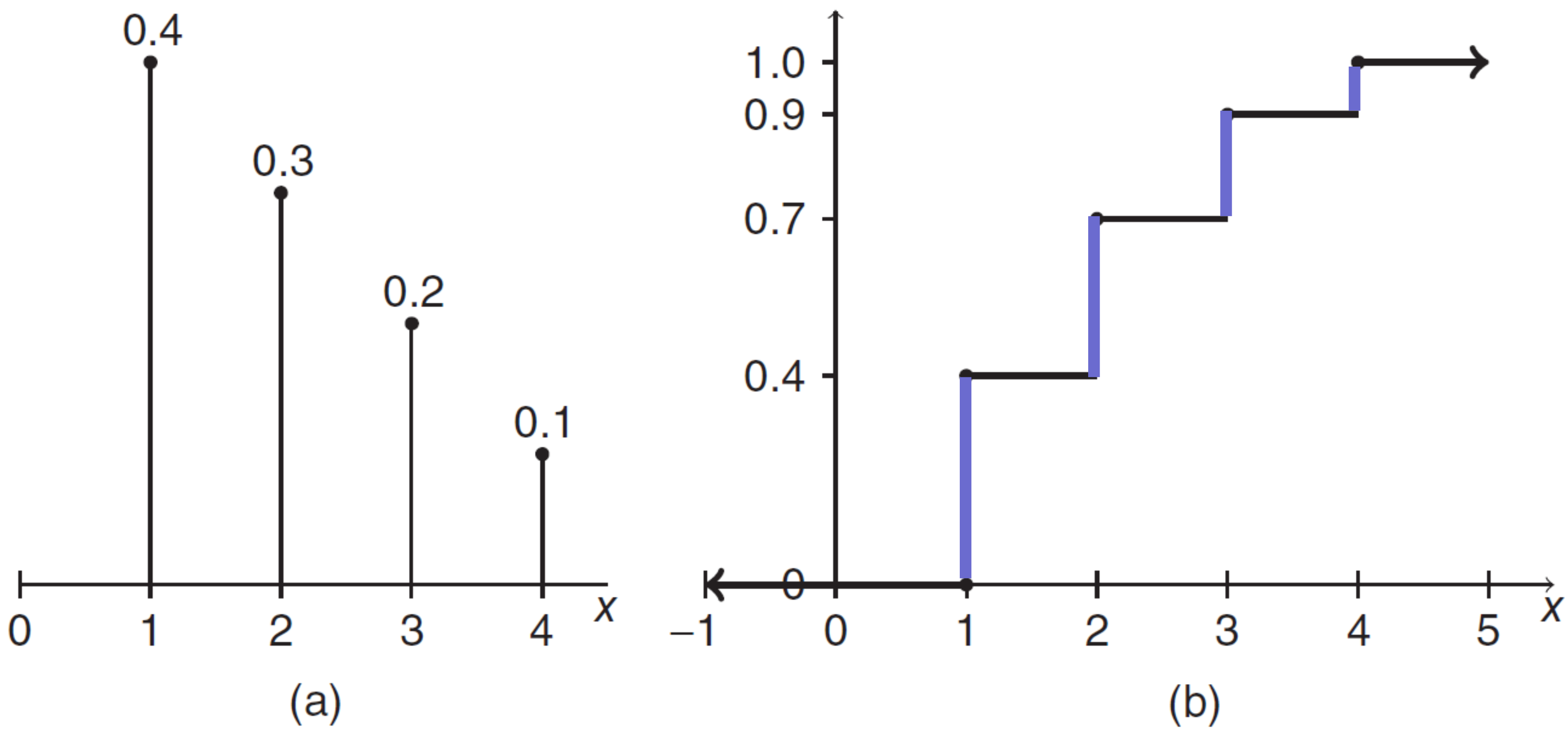


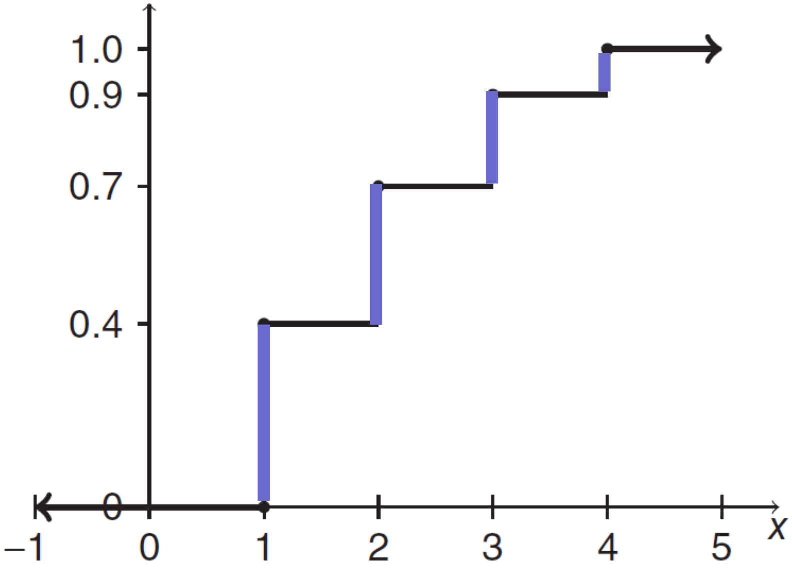
Figure 2.4 Example of (a) PMF and (b) CDF.

2. Inverse Transform (Discrete Dist.)

Example:

The inverse transform function is: u~ U[0,1]

$$F^{-1}(u) = \begin{cases} 1 & \text{if } 0.0 \le u \le 0.4 \\ 2 & \text{if } 0.4 < u \le 0.7 \\ 3 & \text{if } 0.7 < u \le 0.9 \\ 4 & \text{if } 0.9 < u \le 1.0 \end{cases}$$



Let

$u = 0.3021 \quad \rightarrow \quad X = 1$

$u = 0.9267 \quad \rightarrow \quad X = 4$

$u = 0.5694 \quad \rightarrow \quad X = 2$

2. Inverse Transform (Discrete Dist.)

Bernoulli (*p*)

X ~ Bernoulli(p)

Pr{X =1} = p and Pr{X = 0} = 1– p

For u~ U[0,1]

$$F(u)^{-1} = \begin{cases} 1 & ;\ 0 \le u \le p \\ 0 & ;\ p < u \le 1 \end{cases}$$

2. Inverse Transform (Discrete Dist.)

Example

X ~ Bernoulli(p=0.75)

Pr{X =1} = 0.75 and Pr{X = 0} = 0.25

For u~ U[0,1]

$$F(u)^{-1} = \begin{cases} 1 & ;\ 0 \le u \le 0.75 \\ 0 & ;\ 0.75 < u \le 1 \end{cases}$$

Let

$u = 0.3021 \rightarrow \quad X = 1$

$u = 0.9267 \rightarrow \quad X = 0$

$u = 0.5694 \rightarrow \quad X = 1$

2. Inverse Transform (Discrete Dist.)

Discrete Uniform(*a, b*)

X ~ DU[a, b]

Pr{X = x} = 1/n $\quad\quad a \leq x \leq b$

n : number of integer values between a and b

For u~ U[0,1]

$$F(u)^{-1} = a + [(b - a + 1)u]$$

[y] = the integer part of y

2. Inverse Transform (Discrete Dist.)

Example

X ~ DU[-1, 3]

Pr{X = x} = 1/5 $\qquad$ x = -1,0,1,2,3

For u~ U[0,1]

$$F(u)^{-1} = -1 + [(4+1)u]$$

Let

$u = 0.0321 \quad \rightarrow \quad X = -1 + [5(0.0321)] = -1$

$u = 0.9267 \quad \rightarrow \quad X = -1 + [5(0.9267)] = 3$

$u = 0.5694 \quad \rightarrow \quad X = -1 + [5(0.5694)] = 1$

2. Inverse Transform (Discrete Dist.)

<u>Binomial (*n,p*)</u>

X ~ Bin(n,p) → P{X=k} = $\binom{n}{k} p^k (1-p)^{n-k}$

- Compute CDF

$$P\{X \le k\} = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$$

- Do the inverse function as descried in the example.

2. Inverse Transform (Discrete Dist.)

Example

Let X ~ Bin(n=5, p=1/3)

$$P\{X=k\} = \binom{5}{k}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{5-k}$$

- Compute CDF

$$P\{X \le k\} = \sum_{i=0}^{k}\binom{5}{i}\left(\frac{1}{3}\right)^{i}\left(\frac{2}{3}\right)^{5-i}$$

k	0	1	2	3	4	5
P{X=*k*}	0.132	0.329	0.329	0.165	0.041	0.004
P{X<=*k*}	0.132	0.461	0.790	0.955	0.996	1.000

2. Inverse Transform (Discrete Dist.)

Example

Let X ~ Bin(n=5, p=1/3)

k	0	1	2	3	4	5
P{X=*k*}	0.132	0.329	0.329	0.165	0.041	0.004
P{X<=*k*}	0.132	0.461	0.790	0.955	0.996	1.000

- Compute the inverse function: For u~ U[0,1]

if $0 \le \boldsymbol{u} \le 0.132$	Then	$k = 0$
if $0.132 < \boldsymbol{u} \le 0.461$	Then	$k = 1$
if $0.461 < \boldsymbol{u} \le 0.790$	Then	$k = 2$
if $0.790 < \boldsymbol{u} \le 0.955$	Then	$k = 3$
if $0.955 < \boldsymbol{u} \le 0.996$	Then	$k = 4$
if $0.996 < \boldsymbol{u} \le 1$	Then	$k = 5$

2. Inverse Transform (Discrete Dist.)

Geometric (p)

X ~ Geo(p) → P{X=k} = $(1-p)^{k-1}p$

- Compute CDF

$$P\{X \leq k\} = \sum_{i=0}^{k} (1-p)^{k-1} p$$

- The inverse function is given by

$$X = F(u)^{-1} = \left[\frac{\ln(1-u)}{\ln(1-p)}\right]$$

[y] is the integer part of y

2. Inverse Transform (Discrete Dist.)

Example

Let X ~ Geo(p = 0.35)

$$P\{X=k\} = (0.65)^{k-1}(0.35)$$

- The inverse function is given by

$$X = F(u)^{-1} = \left\lfloor \frac{\ln(1-u)}{\ln(0.65)} \right\rfloor$$

Let u~ U[0,1]

$$u = 0.0321 \rightarrow X = \left\lfloor \frac{\ln(0.9679)}{\ln(0.65)} \right\rfloor = \left\lfloor \frac{-2.61232}{-0.4308} \right\rfloor = \lfloor 0.075 \rfloor = 0$$

$$u = 0.9267 \rightarrow X = \left\lfloor \frac{\ln(0.0733)}{\ln(0.65)} \right\rfloor = \left\lfloor \frac{-0.0323}{-0.4308} \right\rfloor = \lfloor 6.066 \rfloor = 6$$

$$u = 0.5694 \rightarrow X = \left\lfloor \frac{\ln(0.4306)}{\ln(0.65)} \right\rfloor = \left\lfloor \frac{-0.8426}{-0.4308} \right\rfloor = \lfloor 1.955 \rfloor = 1$$