Chapter 1:
Revie of Calculus and Probability

Refer to Text Book:

- “Operations Research: Applications and Algorithms” By Wayne L. Winston ,Ch. 12
- “Operations Research: An Introduction” By Hamdi Taha, Ch. 12

1. Review of Integral Calculus

- In our study of random variables, we often require a knowledge of the basics of integral calculus

- Consider two functions: $f(x)$ and F(x). Then F(x) is the **indefinite integral** of $f(x)$, if $F'(x) = f$(x)

$$F(x) = \int f(x)\, dx$$

No limits for the integration

1. Review of Integral Calculus

- Some rules for **indefinite integral** of f (*C is a constant*)

$$\int (1)\, dx = x + C$$

$$\int af(x)\, dx = a \int f(x)\, dx \qquad (a \text{ is any constant})$$

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int x^{-1}\, dx = \ln x + C$$

1. Review of Integral Calculus

- Some rules for **indefinite integral** of f (*C is a constant*)

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$$

$$\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int f(x)^{-1} f'(x) \, dx = \ln f(x) + C$$

1. Review of Integral Calculus

- For two functions $u(x)$ and $v(x)$,

the integration by parts is

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

$$\int e^{f(x)}f'(x)\,dx = e^{f(x)} + C$$

$$\int a^{f(x)}f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + C$$

where $a>0$ and $a\neq 1$

1. Review of Integral Calculus

The idea of integration is ***area under the curve***

- Divide [a,b] into n equal intervals Δ
- Draw bars with height equals to $f(x_n)$

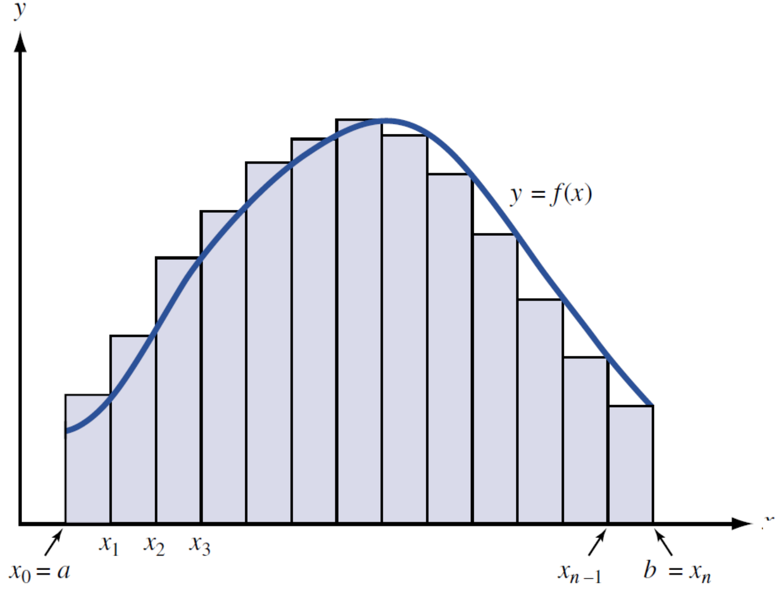


FIGURE 1
Relation of Area and Definite Integral

$$\lim_{\Delta \to 0} \sum_{i=1}^{i=n} f(x_i)\Delta$$

$$\int_a^b f(x)\, dx$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

1. Review of Integral Calculus

Example

The present is time is = 0. At a time *t* years from now, you earn income at a rate e^{2t}. How much money do you earn during the next 5 years?

Total money you earn during the next 5 years is ***area under the curve*** e^{2t}

$$\int_0^5 e^{2t}\,dt = \frac{1}{2}\int_0^5 2e^{2t}\,dt = \frac{1}{2}[e^{2t}]_0^5 = \frac{1}{2}[e^{10} - 1]$$

$$= 11{,}012.73 \text{ SR}$$

2. Review of Basic Probabilities

- **Definition:**

  Any situation where the outcome is uncertain is called an **experiment**.

Examples:

1. The weather condition of tomorrow
2. The number of students getting **A** in the course.
3. The outcome of tossing a fair coin twice.
4. The number of companies that closed **UP** in the stock market.

2. Review of Basic Probabilities

- **Definition:**

  For any experiment, the **sample space S** of the experiment consists of all possible outcomes for the experiment.

<u>Examples:</u>

The weather condition of tomorrow

S={sunny, cloudy, rainy}

The number of students getting A in the course

S = {0,1, 2 , 3 , …, all students}

The outcome of tossing a fair coin twice.

S={HH,HT,TH,TT}

2. Review of Basic Probabilities

Notes on Sample Space :

- The Sample Space is not always obvious
- Elements of sample space may change based on the focus of study.
- Who choose the definition of sample space?

S={sunny, cloudy, rainy}
S = {0,1, 2 , 3 , …, all students}
S={HH,HT,TH,TT}

What about temperature at a specific point in time?

2. Review of Basic Probabilities

- **Definition:**

  An **event** E consists of any collection of points (set of outcomes) in the sample space.

  A collection of events $E_1, E_2, \ldots, E_n$ is said to be a **mutually exclusive** collection of events if for any two events, E_i and E_j no points in common.

<u>Examples:</u>

The number of students getting **A** is odd number $=E_1$

The number of students getting **B** is odd number $=E_2$

2. Review of Basic Probabilities

- **Definition:**

  The probability P{E} of the event E happens *m* times If in an experiment repeated *n* times, is

$$P\{E\} = \lim_{n \to \infty} \frac{m}{n}$$

  This means that as number of repetition of the experiment increases the probability becomes more accurate.

2. Review of Basic Probabilities

- **Example:**

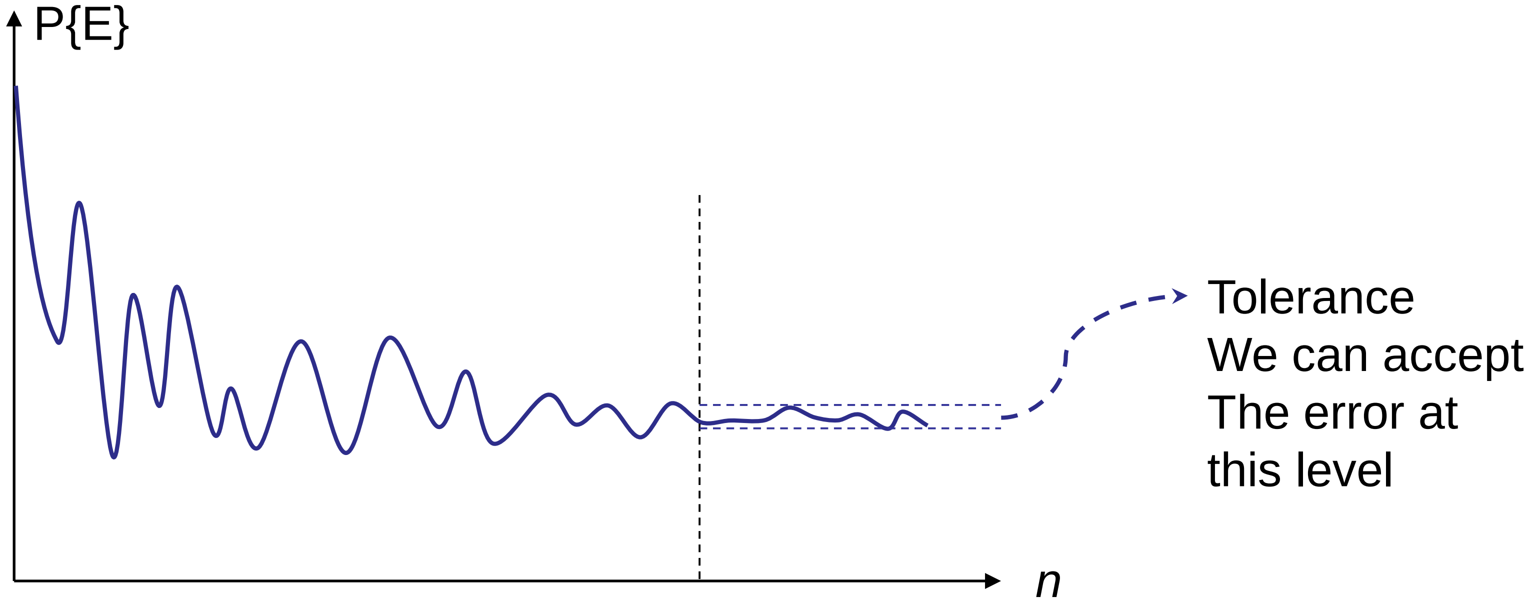
  Let E= Head appears in the an unbalances coin.

  You do not know the probability of P{E}

No. Exp.	**1**	**2**	**3**	**4**	**5**	**…**	**10**	**20**	**30**	**40**	**400**
No. Heads	1	2	2	3	3	…	6	13	21	29	268
P{E}	**1**	**1**	**0.667**	**0.75**	**0.6**	**…**	**0.6**	**0.65**	**0.7**	**0.725**	**0.67**

$P\{E\} \approx 0.67$

2. Review of Basic Probabilities



No matter how long the experiment repeats,
the probability remains within the tolerance limits

From that point of view the sampling techniques starts to emerge

This means that :

If the sample is chosen to satisfy the tolerance,
then no need to test the entire population

2. Review of Basic Probabilities

- **Definition:**

  The conditional probability of E_2 given E_1 is defined as :

$$P\{E_2|E_1\} = \frac{P\{E_1 and E_2\}}{P\{E_1\}}$$

  If E_2 given E_1 are independent then:

$$P\{E_2|E_1\} = P\{E_2\}$$
$$P\{E_1|E_2\} = P\{E_1\}$$
$$P\{E_1 and E_2\} = P\{E_1\}P\{E_2\}$$

2. Review of Basic Probabilities

- **Example:**

  You apply for a job with 99 other applicants from different colleges. Someone told that the accepted applicant is an applicant with statistics major. There are 32 statistics applicant applying with you for the same job.

P{You get accepted before the information}

$$= \frac{1}{100} = 0.01$$

P{You get accepted after the information}

= P{You get accepted **given that** the selected is from Statistics}$= \frac{1}{33} = 0.0303$

2. Review of Basic Probabilities

Baye's Rule:

1. Suppose that the population is divided into 4 sectors S_1, S_2, S_3, S_4 .
2. Sectors are independent and mutually exclusive.
3. From data the percentage of each sector is:

$$P\{S_1\}\ ,\ P\{S_2\},\ P\{S_3\},\ P\{S_4\}\ \text{and}$$

$$P\{S_1\}+P\{S_2\}+P\{S_3\}+P\{S_4\}=1$$

4. The experiment O is done on the population and the following results is obtained from each sector

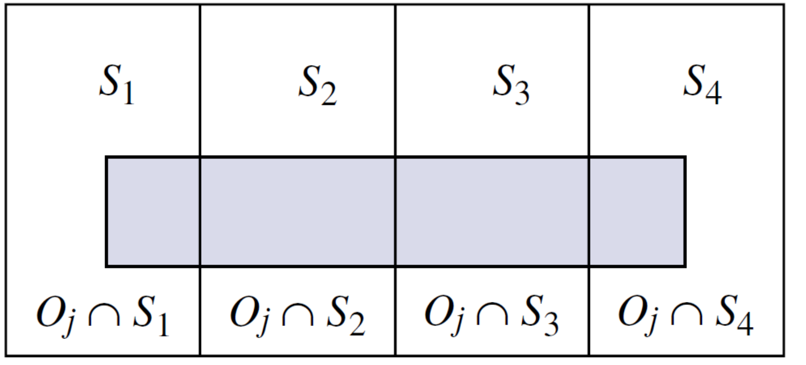
$$P\{O|S_1\}\ ,\ P\{O|S_2\},\ P\{O|S_3\},\ P\{O|S_4\}$$

$$P\{O\}\ ???\ \text{Is not know}$$

2. Review of Basic Probabilities

Baye's Rule:

P{O} for the population



$$P(O_j) = P(O_j \cap S_1) + P(O_j \cap S_2) + P(O_j \cap S_3) + P(O_j \cap S_4)$$

P{O∩S_i} is not given

We know that

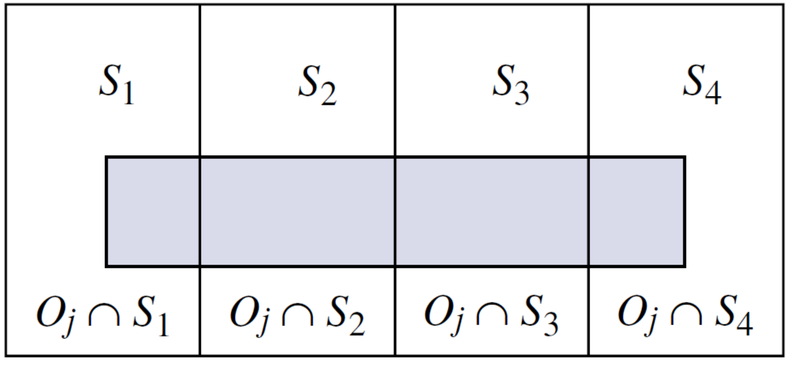
$$P\{O|S_i\} = \frac{P\{O \cap S_i\}}{P\{S_i\}}$$

Then $$P\{O \cap S_i\} = P\{O|S_i\}P\{S_i\}$$

2. Review of Basic Probabilities

Baye's Rule:

P{O} for the population



$$P(O_j) = P(O_j \cap S_1) + P(O_j \cap S_2) + P(O_j \cap S_3) + P(O_j \cap S_4)$$

Then, law of Total probability

$$P\{O\} = P\{O|S_1\}P\{S_1\} + P\{O|S_2\}P\{S_2\} + P\{O|S_3\}P\{S_3\} + P\{O|S_4\}P\{S_4\}$$

Then

$$P\{S_i|O\} = \frac{P\{O \cap S_i\}}{P\{O\}}$$

2. Review of Basic Probabilities

Baye's Rule:

Suppose that 1% of all children have tuberculosis (TB). When a child who has TB is given the Mantoux test, a positive test result occurs 95% of the time. When a child who does not have TB is given the Mantoux test, a positive test result occurs 1% of the time. Given that a child is tested and a positive test result occurs, what is the probability that the child has TB?

The states of the world are

$$S_1 = \text{child has TB}$$
$$S_2 = \text{child does not have TB}$$

The possible experimental outcomes are

$$O_1 = \text{positive test result}$$
$$O_2 = \text{nonpositive test result}$$

We are given the prior probabilities $P(S_1) = .01$ and $P(S_2) = .99$ and the likelihoods $P(O_1|S_1) = .95$, $P(O_1|S_2) = .01$, $P(O_2|S_1) = .05$, and $P(O_2|S_2) = .99$. We seek $P(S_1|O_1)$. From (7),

$$P(S_1|O_1) = \frac{P(O_1|S_1)P(S_1)}{P(O_1|S_1)P(S_1) + P(O_1|S_2)P(S_2)}$$

$$= \frac{.95(.01)}{.95(.01) + .01(.99)} = \frac{95}{194} = .49$$

2. Review of Basic Probabilities

Random Variables and Probability Distributions:

- **Definition:**

A ***random variable*** is a function that associates a number with each point in an experiment's sample space.

- **Definition:**

A random variable is discrete if it can take only discrete values x_1, x_2, ... with probability associated with each value of X written P{X= x_1}.

- **Definition:**

The cumulative distribution function F(x) for any random variable X is defined by $F(x)$= P(X≤ x). For a discrete random variable X,

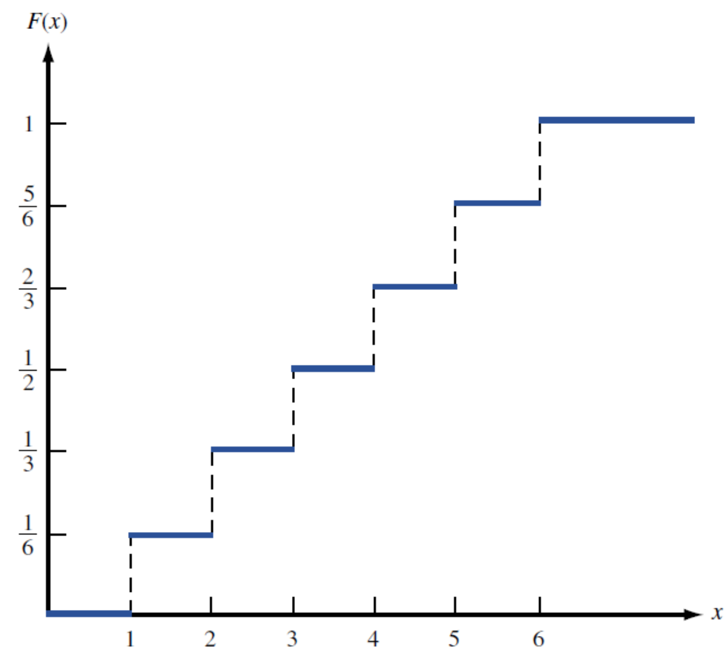
2. Review of Basic Probabilities

Random Variables and Probability Distributions:

For a **discrete** random variable X, P(X≤ x) = F(x)

$$F(x) = \sum_{\substack{\text{all } x \\ \text{having } x_k \le x}} P(\mathbf{X} = x_k)$$

Let X be the number of dots that show when a die is tossed.
Then for i = 1, 2, 3, 4, 5, 6,
P(X = i)=1/6.
The cumulative distribution function (cdf) for X is shown in the figure.



2. Review of Basic Probabilities

Random Variables and Probability Distributions:

For a **continuous** random variable X, the pdf of X is $f(x)$ such that

$$f(x) \geq 0 \quad and \quad \int_{-\infty}^{+\infty} f(x)dx = 1$$

Important Remark

For any continuous random variable, the probability at a point = 0

$$P\{X = a\} \neq f(x = a) \qquad always \qquad P\{X = a\} = 0$$

Important Remark

For any continuous random variable, the probability is always evaluated within an interval a < X < b at a point

$$P\{a < X < b\} = \int_{a}^{b} f(x)dx$$

2. Review of Basic Probabilities

Random Variables and Probability Distributions:

For a **continuous** random variable X, the CDF of X is $F(x)$ such that

$$F(x = a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx$$

Example

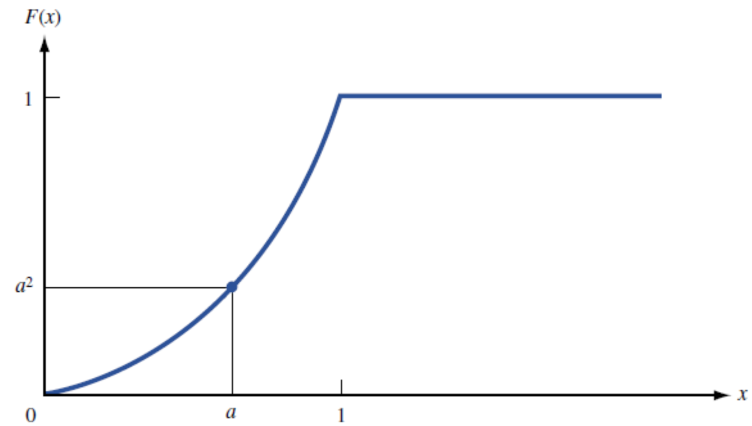
Consider a continuous random variable **X** having a density function $f(x)$ given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Random Variables and Probability Distributions:

Example

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Find the cdf for $\mathbf{X}$. Also find $P(\frac{1}{4} \le \mathbf{X} \le \frac{3}{4})$.

For $a \le 0$, $F(a) = 0$. For $0 \le a \le 1$,

$$F(a) = \int_0^a 2x \, dx = a^2$$

For $a \ge 1$, $F(a) = 1$. $F(a)$ is graphed in Figure 6.

$$P(\tfrac{1}{4} \le \mathbf{X} \le \tfrac{3}{4}) = \int_{1/4}^{3/4} 2x \, dx = [x^2]_{1/4}^{3/4} = (\tfrac{9}{16}) - (\tfrac{1}{16}) = \tfrac{1}{2}$$

2. Review of Basic Probabilities

Mean and Variance :

- The mean (or expected value) and variance used to summarize information of a random variable's probability distribution.

- The mean of a random variable X (written **E[X]**) is a measure of central location

Mean of a Discrete Random Variable

For a discrete random variable $\mathbf{X}$,

$$E(\mathbf{X}) = \sum_{\text{all } k} x_k P(\mathbf{X} = x_k)$$

Mean of a Continuous Random Variable

For a continuous random variable,

$$E(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\, dx$$

2. Review of Basic Probabilities

Mean and Variance :

- For a function $h(X)$ of a random variable X (such as X^2 and e^X), the expected value of the random variable with function h is E[h(X)]

If X is a discrete random variable

$$E[h(\mathbf{X})] = \sum_{\text{all } k} h(x_k)\, P(\mathbf{X} = x_k)$$

If **X** is a continuous random variable,

$$E[h(\mathbf{X})] = \int_{-\infty}^{\infty} h(x) f(x)\, dx$$

2. Review of Basic Probabilities

Mean and Variance :

- The variance of a random variable X (written as Var[X]) measures the dispersion or spread of X about the mean E(X).

- Var[X] is defined to be $E[X - E(X)]^2$.

Variance of a Discrete Random Variable

For a discrete random variable **X**, (8′) yields

$$\text{var}\,\mathbf{X} = \sum_{\text{all }k} [x_k - E(\mathbf{X})]^2\, P(\mathbf{X} = x_k)$$

Variance of a Continuous Random Variable

For a continuous random variable **X**, (9′) yields

$$\text{var}\,\mathbf{X} = \int_{-\infty}^{\infty} [x - E(\mathbf{X})]^2 f(x)\, dx$$

2. Review of Basic Probabilities

Mean and Variance :

Example

Consider the discrete random variable X having P(X= i)=1/6 for i = 1, 2, 3, 4, 5, 6. Find E(X) and var[X].

$$E(\mathbf{X}) = (\tfrac{1}{6})(1 + 2 + 3 + 4 + 5 + 6) = \tfrac{21}{6} = \tfrac{7}{2}$$

$$\begin{aligned}\text{var}\,\mathbf{X} = (\tfrac{1}{6})[&(1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 \\ &+ (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2] = \tfrac{35}{12}\end{aligned}$$

2. Review of Basic Probabilities

Mean and Variance :

Example

During the first week of each month, I (like many people) pay all my bills and answer a few letters. I usually buy 20 first-class mail stamps each month for this purpose. The number of stamps I will be using varies randomly between 10 and 24, with equal probabilities. What is the average number of stamps left?

The pdf of the number of stamps used is

$$p(x) = \frac{1}{15}, x = 10, 11, \ldots, 24.$$

The number of stamps left is given as

$$h(x) = \begin{cases} 20 - x, & x = 10, 11, \ldots, 19 \\ 0, & \text{otherwise} \end{cases}$$

2. Review of Basic Probabilities

Mean and Variance :

Example

$$E[h(x)] = \sum_{x=10}^{19} (20 - x) P(X = x)$$

$$E[h(x)] = E[20 - X] = E[20] - E[X] = 20 - E[X]$$

$$E\{h(x)\} = \tfrac{1}{15}[(20 - 10) + (20 - 11) + (20 - 12) + \cdots + (20 - 19)] + \tfrac{5}{15}(0)$$
$$= 3\tfrac{2}{3}$$

The product $\frac{5}{15}(0)$ is needed to complete the expected value of $h(x)$. Specifically, the probability of being left with *zero* extra stamps equals the probability of needing 20 stamps or more—that is,

$$P\{x \geq 20\} = p(20) + p(21) + p(22) + p(23) + p(24) = 5\left(\tfrac{1}{15}\right) = \tfrac{5}{15}$$

2. Review of Basic Probabilities

Mean and Variance :

<u>Example</u>

Find the mean and variance for the continuous random variable **X** having the following density function:

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Review of Basic Probabilities

Mean and Variance :

Example

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(\mathbf{X}) = \int_0^1 x(2x)\, dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\text{var}\ \mathbf{X} = \int_0^1 \left(x - \frac{2}{3}\right)^2 2x\, dx = \int_0^1 \left(x^2 - \frac{4x}{3} + \frac{4}{9}\right) 2x\, dx$$

$$= \left[\frac{2x^4}{4} - \frac{8x^3}{9} + \frac{8x^2}{18}\right]_0^1 = \frac{1}{18}$$

3. Common Probability Distributions

1. Binomial Distribution

Suppose that a manufacturer produces a certain product in lots of n items each. The fraction of defective items in each lot, *p*, is estimated from historical data. We are interested in determining the pdf of the number of defectives in a lot

the probability of *k* defectives in a lot of n items is

$$P\{x = k\} = C_k^n p^k (1 - p)^{n-k}, k = 0, 1, 2, \ldots, n$$

$$E\{x\} = np$$

$$\text{var}\{x\} = np(1 - p)$$

3. Common Probability Distributions

1. Binomial Distribution

Example:

John Doe's daily chores require making 10 round trips by car between two towns. Once through with all 10 trips, Mr. Doe can take the rest of the day off, a good enough motivation to drive above the speed limit. Experience shows that there is a 40% chance of getting a speeding ticket on any round trip.

a) What is the probability that the day will end without a speeding ticket?
b) If each speeding ticket costs $80, what is the average daily fine?

3. Common Probability Distributions

1. Binomial Distribution

Example:

The probability of getting a ticket on any one trip is $p = .4$. Thus, the probability of not getting a ticket in any one day is

$$P\{x = 0\} = C_0^{10}(.4)^0(.6)^{10} = .006$$

This means that there is less than 1% chance of finishing the day without a fine. In fact, the average fine per day can be computed as

$$\text{Average fine} = \$80E\{x\} = \$80(np) = 80 \times 10 \times .4 = \$320$$

3. Common Probability Distributions

2. Poisson Distribution

Customers arrive at a bank or a grocery store in a "totally random" fashion, meaning that we cannot predict when someone will arrive. The pdf describing the ***number*** of such arrivals during a specified period is the Poisson distribution.

$$P\{x = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots$$

The mean and variance of the Poisson are

$$E\{x\} = \lambda$$

$$\text{var}\{x\} = \lambda$$

3. Common Probability Distributions

2. Poisson Distribution

Example:

Repair jobs arrive at a small-engine repair shop in a totally random fashion at the rate of 10 per day.

a) What is the average number of jobs that are received daily at the shop?
b) What is the probability that no jobs will arrive during any 1 hour, assuming that the shop is open 8 hours a day?

2. Poisson Distribution

Example:

The average number of jobs received per day equals λ = 10 jobs per day.

To compute the probability of no arrivals per hour, we need to compute the arrival rate per hour-namely,

λ_{hr} = 10/8 = 1.25 jobs per hour. Thus

$$P\{\text{no arrivals per hour}\} = \frac{(\lambda_{\text{hour}})^0 e^{-\lambda_{\text{hour}}}}{0!} = \frac{1.25^0 e^{-1.25}}{0!} = .2865$$

3. Exponential Distribution

If the ***number*** of arrivals at a service facility during a specified period follows the Poisson distribution, then, automatically, the distribution of the time interval between successive arrivals must follow the ***exponential distribution***.

Let λ is the rate at which Poisson events occur, then the distribution of time between successive arrivals, *x*, is

$$f(x) = \lambda e^{-\lambda x} \; , \qquad x > 0$$

$$E[X] = \frac{1}{\lambda} \quad and \quad Var[X] = \frac{1}{\lambda}$$

3. Exponential Distribution

Example:

Cars arrive at a gas station randomly every 2 minutes, on the average. Determine the probability that the interarrival time of cars does not exceed 1 minute. The desired probability is of the form P{x ≤ A}, where A = 1 minute in the present example.

Let λ is the rate at which Poisson events occur, then the distribution of time between successive arrivals, *x*, is

$$F(A) = \int_0^A f(x)dx = \int_0^A \lambda e^{-\lambda x}dx = \left[-e^{-\lambda x}\right]_0^A = 1 - e^{-\lambda A}$$

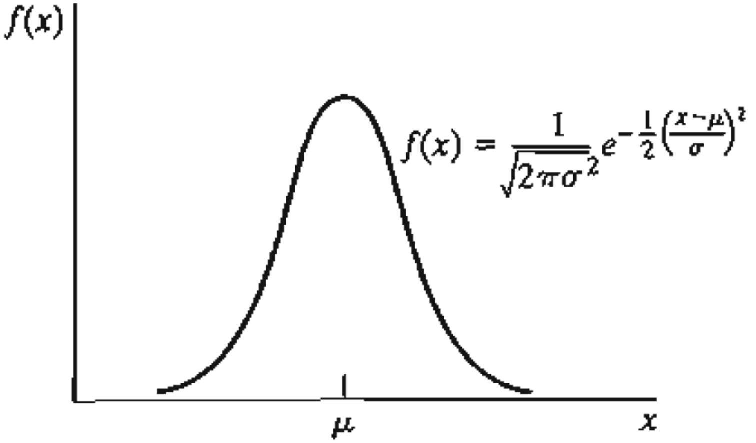
$$F(A=1) = 1 - e^{-2(1)} = 0.3934$$

3. Common Probability Distributions

4. Normal Distribution

The normal distribution describes many random phenomena that occur in everyday life, including test scores, weights, heights, and many others. The pdf of the normal distribution is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} \quad , \qquad -\infty < x < +\infty$$



Distribution Parameters

$$E[X] = \mu$$

$$Var[X] = \sigma^2$$

4. Normal Distribution

Central Limit Theorem

Let x_1, x_2, … and x_n be independent and identically distributed random variables, each with mean μ and standard deviation σ. The sample average

$$S_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

As n becomes large ($n \to \infty$), the distribution of S_n becomes asymptotically normal with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\sigma/n}$, regardless of the original distribution of x_1, x_2, … and x_n .

4. Normal Distribution

Standard Normal Distribution

- No closed form for the CDF of the normal random variable
- Normal tables have been prepared for CDF of standard normal with mean zero and standard deviation 1.
- Any normal random variable, X with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$, can be converted to a standard normal, Z, by using the transformation

$$Z = \frac{X - \mu}{\sigma}$$

- **6-sigma limits:** Over 99% of the area under any normal distribution is enclosed In the confidence interval

$$\boldsymbol{\mu - 3\sigma \le X \le \mu + 3\sigma}$$