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ON SOME $(2, m, n)$-GROUPS

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Abstract

In an earlier work [3], we proved that the symmetric group S_n is a (2, 3, 16)-group, if $16 \leq n \leq 37$. In this paper, the result is found to be true beyond 37 and upto 45.

1. Introduction

Let $G = \langle x, y \rangle$ be a finite group generated by x and y and l, m, n be positive integers satisfying $l \leq m \leq n$, $x^l = y^m = (xy)^n = 1$. Then following the notations of Coxeter and Moser [5], we define an (l, m, n)-group by

$$(l, m, n) = \{G \mid G = \langle x, y \rangle : x^l = y^m = (xy)^n = 1\}.$$

Such groups have also been used in [6] and [9] for getting interesting

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relations between the order of the group G and the orders of the generators. Note that the study of symmetric group S_n is very important as every finite group admits an embedding in a symmetric group S_n for a suitable n (cf. Herstein [7]). Moreover, in the notations of Wielandt [11], in a symmetric group S_n choosing an element a of order 3 and two $x, y \in S_n$ of order 2 with x even and y odd such that $\langle a, x\rangle$ and $\langle a, y\rangle$ are primitive on the n symbols and both contain some cyclic permutation of prime order p with $p < n - 2.$ Then a well known theorem of Jordon implies that $\langle a, x\rangle = A_n$ the alternating group and $\langle a, y\rangle = S_n.$ Also, it is remarked in [6] that the alternating group A_n is a factor group of the (2, 3, 7)-group (see also [8]). An interesting relation between order of the group $G = \langle x, y\rangle$ and (l, m, n)-group has been obtained in [9], as the order μ of the group G can be expressed as $\mu = nt,$ it is shown that $n \leq lmt^l.$ It is known that if G is a primitive group of degree $n = p + k,$ where p is a prime and $k \geq 3,$ and has element of degree and order p, then G is either the symmetric group S_n or the alternating group A_n (cf. Wielandt [11]). Following the importance of the symmetric group $S_n,$ in this paper, we are interested in studying the structure of the symmetric group S_n under certain restrictions on n. In an earlier work [3], we proved that the symmetric group S_n is a (2, 3, 16)-group, if $16 \leq n \leq 37.$ One of the main results of this paper is that, we have proved for $38 \leq n \leq 45,$ the symmetric group $S_n \in (2, 3, 16),$ that is $S_n = \langle x, y\rangle,$ where x, y are of orders 2, 3 and the product xy has order 16, thus the result in [3] is true beyond 37 and upto 45. The smallest prime numbers 2 and 3 play significant role in this paper, as many important finite groups can be generated by two elements of orders 2 and 3. For instance, we have determined the structure of finite (2, 3, 6)-groups (cf. Al-Salman and Al-Thukair [1, 4]).

2. Main Results

We begin with fixing some notations:

D_n: Dihedral group of degree n, $G \cong \overline{G} : G$ is isomorphic to $\overline{G}$, V: Klein 4-group, S_n: Symmetric group on n objects, $\mathbb{Z}_n$: Cyclic group under addition modulo n.

As, in this paper, we are interested in (l, m, n)-groups, first we study the cases $l = 2$, $m = 2$ or 3, and $n = 2$ or $n \geq 3$.

Case 1. $l = m = n = 2$. In this case, we prove the following:

Proposition 1. *The Klein* 4-*group* $V \in (2, 2, 2)$.

Proof. Let $G = \langle x, y : x^2 = y^2 = (xy)^2 = 1 \rangle$, where $x = (0, 1)(2, 3)$, $y = (0, 2)(1, 3)$, $xy = (0, 3)(1, 2)$. It is clear that $G \cong V$, that is, $G \cong V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, it is an abelian group of order 4 and therefore a subgroup of S_4.

Case 2. $l = m = 2$ and $n \geq 3$. In this case, we prove the following:

Proposition 2. *The Dihedral group of degree n*, $D_n \in (2, 2, n)$.

Proof. Let $G = \langle x, y : x^2 = y^2 = (z = xy)^n = 1 \rangle$. Since

$$xzx^{-1} = x(xy)x^{-1} = x^2(yx^{-1}) = y^{-1}x^{-1} = (xy)^{-1} = z^{-1},$$

we get that $G \cong D_n$, where $n \geq 3$, and D_n is a non-abelian group.

Case 3. $l = 2$, $m = 3$ and $n = 16$. In this case, we prove the following proposition, which is the main result of this paper.

Proposition 3. *The symmetric group* $S_n \in (2, 3, 16)$, *for* $38 \leq n \leq 45$.

Proof. The proof is divided into several lemmas, and the proof for each lemma will depend on the following two steps:

Step 1. Finding two elements *x* and *y*, satisfying:

(a) Singerman's formula [2, 10].

(b) The relations: $x^2 = y^3 = z^{16} = 1$, where $z = xy$.

Step 2. To prove that $G = \langle x, y \rangle = S_n$, for $38 \leq n \leq 45$.

Lemma 1. $S_{38} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(4, 14)(5, 31)(6, 7)(8, 30)(9, 35)(11, 34)(12, 19)(13, 16)$$

$$(17, 18)(20, 33)(21, 28)(22, 24)(23, 36)(25, 27)(26, 37)$$

$$(29, 32)\underline{0}\,\underline{1}\,\underline{3}\,\underline{10} : 2^{17}.1^4,$$

$$y = (0, 1, 2)(3, 4, 15)(5, 16, 14)(6, 8, 31)(9, 32, 30)(10, 11, 35)$$

$$(12, 20, 34)(13, 17, 19)(21, 29, 33)(22, 25, 28)(23, 36, 24)$$

$$(26, 37, 27)\underline{7}\,\underline{18} : 3^{12}.1^2$$

and

$$z = xy = (0, 1, 2, 3, ..., 15)(16, 17, ..., 31)(32, 33, 34, 35)\underline{36}\,\underline{37} : 16^2.4.1^2.$$

Step 2.

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)$$

$$(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)$$

$$(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37} : 2^{16}.1^6.$$

Let $\sigma = yz^8$. Then we have

$$\sigma = (0, 9, 32, 22, 17, 27, 18, 26, 37, 19, 5, 24, 31, 14, 13, 25, 20, 34, 4, 7, 15, 11, 35, 2, 8, 23, 36, 16, 6)(1, 10, 3, 12, 28, 30)(29, 33)\underline{21} : 29.6.2.1.$$

It follows from z^8 and σ^2 that G is 2-transitive (fixing $\underline{33}$) and thus it is primitive. Since the cyclic type of σ is 29.6.2.1, it follows that σ^6 is an element of degree and order 29. This proves that $G = S_{38}$ (cf. Wielandt [11]).

Lemma 2. $S_{39} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (1, 2)(3, 15)(4, 5)(6, 14)(7, 31)(8, 9)(10, 30)(11, 35)(12, 20)(13, 16)(17, 19)(18, 36)(21, 34)(22, 24)(23, 37)(25, 33)(26, 28)(29, 32)(27, 38)\underline{0} : 2^{19}.1,$$

$$y = (0, 1, 3)(4, 6, 15)(7, 16, 14)(8, 10, 31)(11, 32, 30)(12, 21, 35)(13, 17, 20)(18, 36, 19)(22, 25, 34)(23, 37, 24)(26, 29, 33)(27, 38, 28)\underline{2}\,\underline{5}\,\underline{9} : 3^{12}.1^3$$

and for $z = xy$,

$$z = (0, 1, ..., 15)(16, ..., 31)(32, 33, 34, 35)\underline{36}\,\underline{37}\,\underline{38} : 16^2.4.1^3.$$

Step 2. We have

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37}\,\underline{38} : 2^{16}.1^7.$$

Let $\sigma = xz^8$. Then we have

$$\sigma = (0, 8, 1, 10, 22, 16, 5, 12, 28, 18, 36, 26, 20, 4, 13, 24, 30, 2, 9)$$
$$(3, 7, 23, 37, 31, 15, 11, 35)(17, 27, 38, 19, 25, 33)$$
$$(21, 34, 29, 32)\underline{6}\,\underline{14} : 19.8.6.4.1^2.$$

It follows that G is 2-transitive by z and σ^{19} (fixing $\underline{36}$). As the cycle type of σ is $19.8.6.4.1^2$, it follows that σ^{24} is an element of degree and order 19. Hence, $G = S_{39}$ (cf. Wielandt [11]).

Lemma 3. $S_{40} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(3, 4)(5, 14)(6, 31)(7, 9)(8, 36)(10, 30)(11, 35)(12, 20)$$
$$(13, 16)(17, 19)(18, 37)(21, 34)(22, 24)(23, 38)(25, 33)(26, 28)$$
$$(27, 39)(29, 32)\underline{0}\,\underline{1} : 2^{19}.1^2,$$

$$y = (0, 1, 2)(3, 5, 15)(6, 16, 14)(7, 10, 31)(8, 36, 9)(11, 32, 30)$$
$$(12, 21, 35)(13, 17, 20)(18, 37, 19)(22, 25, 34)(23, 38, 24)$$
$$(26, 29, 33)(27, 39, 28)\underline{4} : 3^{13}.1$$

and for $z = xy$,

$$z = (0, 1, ..., 15)(16, ..., 31)(32, 33, 34, 35)\underline{36}\,\underline{37}\,\underline{38}\,\underline{39} : 16^2.4.1^4.$$

Step 2. We have

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)$$
$$(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)$$
$$(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37}\,\underline{38}\,\underline{39} : 2^{16}.1^8.$$

Let $\sigma = z^8 y$. Then it follows that

$$\sigma = (0, 36, 9, 2, 31, 38, 24, 14, 16, 23, 7, 3, 32, 30, 25, 20, 27, 18, 29, 35, 12, 4, 21, 33, 26, 37, 19, 39, 28, 13, 15, 10)(1, 8)(5, 17, 34, 22, 11)\underline{6} : 32.5.2.1.$$

Thus, x and σ^2 show that G is 2-transitive (fixing $\underline{1}$) and therefore G is primitive. Since σ^{32} is an element of degree and order 5, it follows that $G = S_{40}$ (cf. Wielandt [11]).

Lemma 4. $S_{41} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(4, 14)(5, 31)(6, 7)(8, 30)(9, 10)(11, 29)(12, 39)(13, 16)(17, 38)(18, 19)(20, 37)(21, 22)(23, 36)(24, 26)(25, 40)(27, 35)(28, 32)(33, 34)\underline{0}\,\underline{1}\,\underline{3} : 2^{19}.1^3,$$

$$y = (0, 1, 2)(3, 4, 15)(5, 16, 14)(6, 8, 31)(9, 11, 30)(12, 32, 29)(13, 17, 39)(18, 20, 38)(21, 23, 37)(24, 27, 36)(25, 40, 26)(28, 33, 35)\underline{7}\,\underline{10}\,\underline{19}\,\underline{22}\,\underline{34} : 3^{12}.1^5$$

and

$$z = xy = (0, 1, ..., 15)(16, ..., 31)(32, ..., 39)\underline{40} : 16^2.8.1.$$

Step 2. It is clear that G is primitive group as 41 is a prime number. We have

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37}\,\underline{38}\,\underline{39}\,\underline{40} : 2^{16}.1^9.$$

Now let $\sigma = yz^8$. Then it follows that

$$\sigma = (0, 9, 3, 12, 32, 21, 31, 14, 13, 25, 40, 18, 28, 33, 35, 20, 38, 26, 17, 39, 5, 24, 19, 27, 36, 16, 6)(1, 10, 2, 8, 23, 37, 29, 4, 7, 15, 11, 22, 30)\underline{34} : 27.13.1.$$

As σ has a cycle type 27.13.1, it follows that σ^{27} is an element of degree and order 13. Hence, $G = S_{41}$ (cf. Wielandt [11]).

Lemma 5. $S_{42} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(4, 14)(6, 13)(7, 31)(8, 9)(10, 30)(11, 39)(12, 16)(17, 38)(18, 19)(20, 37)(21, 23)(22, 40)(24, 36)(25, 33)(26, 28)(29, 32)(34, 35)(27, 41)\underline{0}\,\underline{1}\,\underline{3}\,\underline{5} : 2^{19}.1^4,$$

$$y = (0, 1, 2)(3, 4, 15)(5, 6, 14)(7, 16, 13)(8, 10, 31)(11, 32, 30)(12, 17, 39)(18, 20, 38)(21, 24, 37)(22, 40, 23)(25, 34, 36)(26, 29, 33)(27, 41, 28)\underline{9}\,\underline{19}\,\underline{35} : 3^{13}.1^3$$

and for $z = xy$

$$z = (0, 1, ..., 15)(16, ..., 31)(32, ..., 39)\underline{40}\,\underline{41} : 16^2.8.1^2.$$

Step 2. We have

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37}\,\underline{38}\,\underline{39}\,\underline{40}\,\underline{41} : 2^{16}.1^{10}.$$

Let $\sigma_1 = yz^8$. Then we get

$$\sigma_1 = (0, 9, 1, 10, 23, 30, 3, 12, 25, 34, 36, 17, 39, 4, 7, 24, 37, 29, 33, 18,$$
$$28, 19, 27, 41, 20, 38, 26, 21, 16, 5, 14, 13, 15, 11, 32, 22, 40, 31)$$
$$(2, 8)\underline{6}\,\underline{35} : 38.2.1^2.$$

The elements z^8 and σ_1 show that G is 2-transitive (fixing $\underline{35}$) and hence G is primitive group. Now let $\sigma_2 = yz^3$. Then we have

$$\sigma_2 = (0, 4, 2, 3, 7, 19, 22, 40, 26, 16)(1, 5, 9, 12, 20, 33, 29, 36, 28,$$
$$30, 14, 8, 13, 10, 18, 23, 25, 37, 24, 32, 17, 34, 39, 15, 6)$$
$$(11, 35, 38, 21, 27, 41, 31) : 25.10.7$$

and the cycle type of σ_2 is 25.10.7, consequently, σ_2^{50} is an element of degree and order 7. Hence, $G = S_{42}$ (cf. Wielandt [11]).

Lemma 6. $S_{43} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(4, 14)(5, 31)(6, 39)(7, 36)(8, 10)(9, 40)(11, 35)(12, 20)$$
$$(13, 16)(17, 19)(18, 41)(21, 34)(22, 24)(23, 42)(25, 33)(26, 29)$$
$$(30, 32)(37, 38)\underline{0}\,\underline{1}\,\underline{3}\,\underline{27}\,\underline{28} : 2^{19}.1^5,$$

$$y = (0, 1, 2)(3, 4, 15)(5, 16, 14)(6, 32, 31)(7, 37, 39)(8, 11, 36)$$
$$(9, 40, 10)(12, 21, 35)(13, 17, 20)(18, 41, 19)(22, 25, 34)$$
$$(23, 42, 24)(26, 30, 33)(27, 28, 29)\underline{38} : 3^{14}.1$$

and for $z = xy$

$$z = (0, 1, ..., 15)(16, ..., 31)(32, ..., 39)\underline{40}\,\underline{41}\,\underline{42} : 16^2.8.1^3.$$

Step 2. We have

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37}\,\underline{38}\,\underline{39}\,\underline{40}\,\underline{41}\,\underline{42} : 2^{16}.1^{11}.$$

Let $\sigma = yz^8$. Then we have

$$\sigma = (0, 9, 40, 2, 8, 3, 12, 29, 19, 26, 22, 17, 28, 21, 35, 4, 7, 37, 39, 15, 11, 36)(1, 10)(5, 24, 31, 14, 13, 25, 34, 30, 33, 18, 41, 27, 20)(6, 32, 23, 42, 16)\underline{38} : 22.13.5.2.1.$$

Since 43 is a prime number, G is a primitive group. Moreover, as the cycle of σ is 22.13.5.2.1, σ^{110} is an element of degree and order 13. This proves that $G = S_{43}$ (cf. Wielandt [11]).

Lemma 7. $S_{44} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(3, 4)(5, 14)(6, 7)(8, 13)(9, 31)(10, 39)(11, 35)(12, 16)(17, 34)(18, 20)(19, 41)(21, 33)(22, 29)(23, 25)(24, 42)(26, 28)(27, 43)(30, 32)(36, 38)(37, 40)\underline{0}\,\underline{1} : 2^{21}.1^2,$$

$$y = (0, 1, 2)(3, 5, 15)(6, 8, 14)(9, 16, 13)(10, 32, 31)(11, 36, 39)(12, 17, 35)(18, 21, 34)(19, 41, 20)(22, 30, 33)(23, 26, 29)(24, 42, 25)(27, 43, 28)(37, 40, 38)\underline{4}\,\underline{7} : 3^{14}.1^2$$

and for $z = xy$,

$$z = (0, 1, ..., 15)(16, ..., 31)(32, ..., 39)\underline{40}\ \underline{41}\ \underline{42}\ \underline{43} : 16^2.8.1^4.$$

Step 2. We have

$$z^4 = (0, 4, 8, 12)(1, 5, 9, 13)(2, 6, 10, 14)(3, 7, 11, 15)(16, 20, 24, 28)$$
$$(17, 21, 25, 29)(18, 22, 26, 30)(19, 23, 27, 31)(32, 36)(33, 37)$$
$$(34, 38)(35, 39)\underline{40}\ \underline{41}\ \underline{42}\ \underline{43} : 4^8.2^4.1^4.$$

Let $\sigma = yz^4$. Then we have

$$\sigma = (0, 5, 3, 9, 20, 23, 30, 37, 40, 34, 22, 18, 25, 28, 31, 14, 10, 36, 35)$$
$$(1, 6, 12, 21, 38, 33, 26, 17, 39, 15, 7, 11, 32, 19, 41, 24, 42, 29,$$
$$27, 43, 16)(2, 4, 8)\underline{13} : 21.19.3.1.$$

The elements y and σ^3 show that G is 2-transitive (fixing $\underline{4}$) and therefore G is primitive. Since the cycle type of σ is 21.19.3.1, we have that σ^{21} is an element of degree and order 19. Hence, $G = S_{44}$ (cf. Wielandt [11]).

Lemma 8. $S_{45} \in (2, 3, 16)$.

Proof.

Step 1. Let $G = \langle x, y \rangle$, where

$$x = (2, 15)(4, 14)(5, 31)(6, 39)(7, 9)(8, 40)(10, 38)(11, 34)(12, 20)$$
$$(13, 16)(17, 19)(18, 42)(21, 33)(22, 29)(23, 25)(24, 43)(26, 28)$$
$$(27, 44)(30, 32)(35, 37)(36, 41)\underline{0}\ \underline{1}\ \underline{3} : 2^{21}.1^3,$$

$$y = (0, 1, 2)(3, 4, 15)(5, 16, 14)(6, 32, 31)(7, 10, 39)(8, 40, 9)$$
$$(11, 35, 38)(12, 21, 34)(13, 17, 20)(18, 42, 19)(22, 30, 33)$$
$$(23, 26, 29)(24, 43, 25)(27, 44, 28)(36, 41, 37) : 3^{15}$$

and

$$z = xy = (0, 1, ..., 15)(16, ..., 31)(32, ..., 39)\underline{40}\,\underline{41}\,\underline{42}\,\underline{43}\,\underline{44} : 16^2.8.1^5.$$

Step 2. We have

$$z^8 = (0, 8)(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(16, 24)$$

$$(17, 25)(18, 26)(19, 27)(20, 28)(21, 29)(22, 30)$$

$$(23, 31)\underline{32}\,\underline{33}\,\underline{34}\,\underline{35}\,\underline{36}\,\underline{37}\,\underline{38}\,\underline{39}\,\underline{40}\,\underline{41}\,\underline{42}\,\underline{43}\,\underline{44} : 2^{16}.1^{13}.$$

Now, letting $\sigma = yz^8$, we find

$$\sigma = (0, 9)(1, 10, 39, 15, 11, 35, 38, 3, 12, 29, 31, 14, 13, 25, 16, 6, 32$$

$$23, 18, 42, 27, 44, 20, 5, 24, 43, 17, 28, 19, 26, 21, 34, 4, 7, 2, 8, 40)$$

$$(30, 33)(36, 41, 37)\underline{22} : 37.3.2^2.1.$$

The elements z and σ^3 show that G is 2-transitive (fixing $\underline{41}$). Since the cycle type of σ is $37.3.2^2.1,$ it follows that σ^6 is an element of degree and order 37. Therefore, $G = S_{45}$ (cf. Wielandt [11]).

Remark. If $G = \langle x, y\rangle = S_n$ and $\alpha \in S_n,$ then $\langle \alpha^{-1}x\alpha, \alpha^{-1}y\alpha\rangle = S_n$ holds.

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References

[1] S. A. Al-Salman and F. A. Al-Thukair, On finite (2, 3, 6) groups, Math. Japon. 33(1) (1988), 1-15.

[2] S. A. Al-Salman, On some (2, 3, 16)-groups of degree n, $16 \leq n \leq 25$, Arab Gulf J. Sci. Res. 3(1) (1985), 249-258.

[3] S. A. Al-Salman, On some 2-generator groups of degree n, $16 \leq n \leq 37$, J. Coll. Sci., King Saud Univ. 18(1) (1987), 115-123.

[4] S. A. Al-Salman and F. A. Al-Thukair, On transitive (2, 3, 6)-groups of degree $n \leq 23$, Arab Gulf J. Sci. Res. 4(2) (1986), 573-585.

[5] H. S. M. Coxeter and W. O. J. Moser, Generators and Relations for Discrete Groups, 4th ed., Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[6] I. M. S. Dey and Weigold James, Generators of symmetric and alternating groups, J. Aust. Math. Soc. 12(1) (1971), 63-68.

[7] I. N. Herstein, Topics in Algebra, 2nd ed., John Wiley and Sons, New York, 1975, pp. 71-73.

[8] G. A. Miller, On the groups generated by two operators, Bull. Amer. Math. Soc. 7(10) (1901), 424-426.

[9] Morris Newman, 2-generator groups and parabolic class numbers, Proc. AMS 31(1) (1972), 51-53.

[10] D. Singerman, Subgroups of Fuchsian groups and finite permutation groups, Bull. London Math. Soc. 2 (1970), 319-323.

[11] H. Wielandt, Finite Permutation Groups, 2nd ed., Academic Press, Inc., 1968, p. 39.