Chapter : Dimension of vector spaces:

Definition: Dimension of vector space, $Dim(V) = |B|$ where B is a basis of V.

Remark: ① Let V be the zero vector space. Then $B_V = \phi$, and therefore $Dim(V) = |B_V| = 0$

② Let B_1 and B_2 be any two basis of V. Then $|B_1| = |B_2|$.

Some Rules and Properties

1. Let S be a set of vectors such that $0 \in S$. Then S is linear dependent $\Rightarrow$ S is not a basis.

2. Let S be a set of vectors such that there exists a vector in S can be written as the linear combination of the remain vectors in S. Then S is linear dependent $\Rightarrow$ S is not basis.

   Also, If S is linear dependent then there is a vector can be written as linear combination of the remain vectors in S.

   Example: $\{(0,1), (3,5)\}$ is linear indepent because $(0,1) \neq \lambda(3,5)$ and $(3,5) \neq a(0,1)$

3. The basis of a vector space is existed and is not necessary to be unique.

* standered (or natural) basis of some known vector spaces :

1) $B_{R^2} = \{(1,0), (0,1)\} \Rightarrow Dim(R^2) = 2$

2) $B_{R^3} = \{(1,0,0), (0,1,0), (0,0,1)\} \Rightarrow Dim(R^3) = 3$

3) $BP_1 = \{1, x\} \Rightarrow Dim(P_1) = 2$

4) $BP_2 = \{1, x, x^2\} \Rightarrow Dim(P_2) = 3$

5) $BM_2 = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\} \Rightarrow Dim(M_2) = 4$

(Ex) Let $M = \{(x,y,z) : 2x - y + z = 0\}$. It is easy to prove that M is vector subspace of R^3. Find $Dim(M)$?

Solution : Notice that $(a,b,c) \in M$ iff (a,b,c) is a solution of $2x - y + z = 0$.

Now,

$2x - y + z = 0$ is a (1) equation in (3) veriables $\Rightarrow$ there are ∞ many solutions $\Rightarrow$ the solution can be written by (2) parameters.

Let $x = s$ and $y = t \Rightarrow z = t - 2s$

$$So,\ M = \left\{\begin{bmatrix}s\\t\\t-2s\end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix}s\\0\\-2s\end{bmatrix} + \begin{bmatrix}0\\t\\t\end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix}1\\0\\-2\end{bmatrix}s + \begin{bmatrix}0\\1\\1\end{bmatrix}t\right\}$$

So, $\boxed{\{(1,0,-2), (0,1,1)\}}$ spans M and

Basis

$$\left.\begin{array}{l}(1,0,-2) \neq \lambda(0,1,1) \\ (0,1,1) \neq a(1,0,-2)\end{array}\right\} \Rightarrow \{(1,0,-2), (0,1,1)\} \text{ is Linear independent.}$$

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Example: Let $M = \{ a+bx+cx^2+dx^3 : a+b = c-2d = 0 \}$.
$M \trianglelefteq P_3(\mathbb{R})$. Find the basis of M?

Solution: Notice that

$$a+b=0 \Rightarrow a=-b$$
$$c-2d=0 \Rightarrow c=2d$$

So, any polynomial in M can be written as:

$$-b+bx+2dx^2+dx^3$$

Now

$$b(-1+x) + d(2x^2+x^3)$$

Let

$$V = \{ -1+x \,,\, 2x^2+x^3 \}.$$

Clear that ① V spans M
② V is Linear indep (why)

Hence, V is a basis of M. and

$$\text{Dim}(M) = 2 \quad \blacksquare$$

** Some Rules:

① Let $B = \{ v_1, \dots, v_n \}$ spans the vector space V.
If $S = \{ C_1, \dots, C_m \}$ is Linearly indep in V then

$$|S| \leq |B|$$

(ex) Let $\text{Dim}(V) = 3$ and S be set of vectors where $|S| = 4 \Longrightarrow S$ is Linear dependent (since $|S| > \text{Dim}(V)$).

② Let $B = \{v_1, —, v_n\}$ be a basis of V. If v is a vector in $V \Rightarrow v = a_1 v_1 + \dots + a_n v_n$.
The $(a_1, \dots, a_n)$ is called the <u>Coordinate</u> of v. which is unique.

Example: $B = \{(1,3), (0,1)\}$ is a basis of $\mathbb{R}^2$. Find the Coordinate of $(2,5)$?

<u>Solution</u>

Suppose that $(2,5) = a(1,3) + b(0,1)$

$$\Rightarrow \begin{cases} a = 2 \\ 3a + b = 5 \end{cases}$$

$$\Rightarrow \begin{array}{l} a = 2 \\ b = -1 \end{array} \Rightarrow (2,-1) \text{ is the coordinate of } (2,5) \text{ respect to the basis } B.$$

③ If $|S| = \text{Dim}(V)$ and S spans $V \Rightarrow S$ is a basis.

④ If $|S| = \text{Dim}(V)$ and S is Linear independent $\Rightarrow S$ is a basis.

(5) Let $S = \{v_1, \dots, v_n\}$ be Linear indep where $|S| < \text{Dim}(V) = m$. Then there exists $v_{n+1}, \dots, v_m$ s.t. $\{v_1, \dots, v_n, v_{n+1}, \dots, v_m\}$ is a basis of V.

Rule: If $M \trianglelefteq V$ then $Dim(M) \leq Dim(V)$.

Coordinates and change of basis:

Let $B = \{v_1, \ldots, v_n\}$ be a basis of V. $(a_1, \ldots, a_n)$ is called the coordinate of v if

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

we write:

$$[v]_B = (a_1, \ldots, a_n)$$

Example: $B = \{(1,2), (-1,4)\}$ is a basis of R^2. Find $[(5,6)]_B$?

Solution Let $(5,6) = a(1,2) + b(-1,4)$

$$\Rightarrow \begin{cases} a - b = 5 \\ 2a + 4b = 6 \end{cases}$$

$$\Rightarrow 6b = -4 \Rightarrow b = \frac{-2}{3}$$

$$a = \frac{13}{3}$$

$$\therefore [(5,6)]_B = \frac{13}{3}.$$

Example: $\{v_1 = 3,\ v_2 = -1 + x,\ v_3 = x^2\}$ is a basis of $P_2(R)$. Find $[1 - x^2]_B$?

solution: Suppose that

$$1 - x^2 = a v_1 + b v_2 + c v_3$$

$$= (a3) + (-b + bx) + cx^2$$

$$= (3a - b) + bx + cx^2$$

$$\Rightarrow \left.\begin{matrix} 3a - b = 1 \\ \boxed{b = 0} \\ \boxed{c = -1} \end{matrix}\right\} \Rightarrow \boxed{a = \frac{1}{3}}$$

Hence $[1 - x^2]_B = (\frac{1}{3}, 0, -1)$

** Transmission matrix :

Suppose B_1 and B_2 are two basis of V. In the next example will show how to find the transmission matrix from basis to a nother.

(Ex) Let $S = \{(2,1), (0,3)\}$ and $S^* = \{(-1,0), (3,3)\}$ be two basis of $\mathbb{R}^2$. write the transmission matrix from S into S^* ?

Solution

STEP 1 we will find $[(2,1)]_{S^*}$

suppose $(2,1) = a(-1,0) + b(3,3)$

$$\Rightarrow \begin{cases} -a + 3b = 2 \\ 3b = 1 \end{cases} \Rightarrow \begin{matrix} b = 1/3 \\ a = -1 \end{matrix}$$

$$[(2,1)]_{S^*} = (-1, 1/3)$$

STEP 2 we will find $[(0,3)]_{S^*}$

suppose $(0,3) = a(-1,0) + b(3,3)$

$$\Rightarrow \begin{cases} -a + 3b = 0 \\ 3b = 3 \end{cases} \Rightarrow \begin{matrix} b = 1 \\ a = 3 \end{matrix}$$

$$[(0,3)]_{S^*} = (3,1)$$

STEP 3

the transmission matrix from S into S^* $= P = \begin{bmatrix} -1 & 3 \\ 1/3 & 1 \end{bmatrix}$

** Rule

Let B_1 and B_2 be two basis of V. Then

$$[V]_{B_2} = P_{B_2 B_1} \cdot [V]_{B_1}$$

Example Let $X = (1,2,-1)$ the coordinate vector of v with respect to the basis $S = \{ \frac{1}{2}, -x, 2x^2 \}$. Find v.

If $\underset{s^*s}{P} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, find $[v]_{s^*}$?

Solution

$$v = 1(1/2) + 2(-x) + (-1)(2x^2)$$
$$= 1/2 - 2x - 2x^2.$$

Now

$$[v]_{s^*} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} [v]_S$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

Example : Let $S = \{(2,1), (0,3)\}$ be a bais of $\mathbb{R}^2$. If $S^* = \{A_1, A_2\}$ is a nother basis such that

$$\underset{s^*s}{P} = \begin{bmatrix} 1/\sqrt{5} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

find S^* ?

Solution : Let $A_1 = (a,b)$ $A_2 = (c,d)$. From the definition of $\underset{s^*s}{P}$, we get

$$\begin{cases} (2,1) = \frac{1}{\sqrt{5}}(a,b) - \frac{2}{\sqrt{5}}(c,d) \\ (0,3) = \frac{2}{\sqrt{5}}(a,b) + 1/\sqrt{5}(c,d) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{5}}a - \frac{2}{\sqrt{5}}c = 2 \quad \text{---} (1) \\ 1/\sqrt{5}\, b - \frac{2}{\sqrt{5}}d = 1 \quad \text{---} (2) \\ 2/\sqrt{5}\, a + 1/\sqrt{5}\, c = 0 \quad \text{---} (3) \\ 2/\sqrt{5}\, b + 1/\sqrt{5}\, d = 3 \quad \text{---} 4 \end{cases}$$

we can solve (1) and (3) togather $\Rightarrow$, and (2) and (4) togather

$\Rightarrow A_1 = \left(\frac{2}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$

$A_2 = \left(4/\sqrt{5}, \frac{1}{\sqrt{5}}\right)$

Rules

① Let $_{s^*}P_s$ and $_{s'}P_{s^*}$ then

$$\boxed{_{s'}P_s = {}_{s'}P_{s^*} \cdot {}_{s^*}P_s}$$

② If $_{s^*}P_s$ then $\boxed{\left({}_{s^*}P_s\right)^{-1} = {}_{s}P_{s^*}}$

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** Rank of matrix

Def : Rank of matrix, rank(A) = Dim of the subspace spanned by columns of A = the maximal number of linearly independent columns of A.

For example: $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$

rank(A) = 2 because $\{(1,-2,3), (0,-3,3)\}$ is linearly independent and $(1,1,0) = 1(1,-2,3) - 1(0,-3,3)$ (Linear combination).

Rule: $\text{rank}(A) = \text{rank}(A^t)$.

** Finding rank of matrix from Row-Echolon form:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow[-3R_1+R_3]{2R_1+R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-Echolon form

number of non zero rows = 2

$\Rightarrow$ rank(A) = 2

Rule

① Let size(A) = $m \times n$ where $m < n$, so rank(A) $\leq m$

② Let size(A) = $m \times n$ where $n < m$, so rank(A) $\leq n$

(EX) Find rank$\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 5 & 8 \end{bmatrix}\right)$

★ null space of a matrix A.

The null space of any matrix A is the set of all Vectors B s.t. $AB = 0$ (i.e. the set of all solutions of the system $AX = 0$).

★ nullity of a matrix A.

It is the number of vectors which are consider as a basis of the null space of a matrix A (i.e. $\text{nullity}(A) = \text{Dim}(V)$ where V is the null space of A)

★ Rank - Nullity Theorem:

$\text{Nullity}(A) + \text{rank}(A) =$ The Number of Columns.

★ Remark

$\text{Nullity}(A^t) = \text{Nullity}(A)$.

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