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Mathematics of finance and valuation

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Undergraduate study in
Economics, Management,
Finance and the Social Sciences

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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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This is one of a series of subject guides published by the University. We regret that due to pressure of work the author is unable to enter into any correspondence relating to, or arising from, the guide. If you have any comments on this subject guide, favourable or unfavourable, please use the form at the back of this guide.

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Preface

This subject guide is not a course text. It sets out the logical sequence in which to study the topics in the course. Where coverage in the main texts is weak, it provides some additional background material. Further reading is essential

Notes

Chapter 1: Introduction

1.1 Relationship to previous mathematics courses

If taken as part of a BSc degree, **116 Abstract mathematics** must be passed before this course may be attempted.

You should therefore have knowledge and understanding of basic linear algebra and computational aspects of calculus with regard both to single variable integration and to the basic essentials of multiple-variable differential calculus with optimisation in mind (including the use of Lagrange Multipliers. At some critical points later in the course you will need to rely on a facility to perform calculations with integrals, so we highlight the level of difficulty in some learning activities in Chapter 2. Chapter 8 relies on knowing how to define the Riemann integral, a matter that is covered in either **116 Abstract mathematics** or **117 Advanced calculus** (although the latter is not a prerequisite for this course). Elementary notions of Probability or Statistics are also depended upon.

In most of this course we use the language of Linear Algebra. This does, of course, mean that we will also be guided by the usual geometric arguments that are associated with ideas from Linear Algebra. That is so even when the arguments are motivated by probability. Note that the approach taken in this course does not rely on any use of measure theory.

We will think of the component of a vector as describing various possible sums of money payable according to future prevailing circumstances. That is, if $x = (x_1, ..., x_n)$ is a vector in $\mathbb{R}^n$, we will want to think of that vector as referring to a model of future events arising as n different possible circumstances, labelled 1,2, ..., n and called 'the states of nature'. Then the component x_i is interpreted as a sum of money payable when the state of nature is i. Thus x models what is known in Probability Theory as a 'random variable' X with the states of nature described by the sample space $\Omega = \{1,2, ..., n\}$ and the realization of X when the state of nature is i is x_i.

In Chapter 2 we help you to line up your assumed knowledge of Linear Algebra with your assumed knowledge of Basic Probability and Statistics. This is an important step which enables us to use geometric intuition to solve problems originating in the valuation of financial contracts.

By financial contract we mean reasonably standard financial arrangements such as insurance policies, or a contract for the execution of building works which may involve foreseeable but currently unknown complications. In all these we ask how much to pay 'today' for a contractually specified payment (albeit uncertain sum of money) to be received at some specified future date such as the proverbial 'same time' next year.

The aim of this course is to identify the fundamental concepts and methods of Financial Mathematics. We will thus learn two approaches to representing the uncertain evolution of asset prices, first in discrete time and then in continuous time. We will formulate a basis for valuing well-defined future payments that depend on one of a number of specified circumstances occurring when it is not known in advance which of these circumstances will arise.

The mathematical arguments in discrete time will be conducted rigorously, meaning that terms will be precisely defined and results will be proved. Of course plenty of motivation and informal explanations will be given.

We will rely on the rigorous development in discrete time to support a much more informal approach to the continuous time approach. In the second part (the continuous time context) the emphasis will be on using calculus to obtain valuations of what are called European call and put options.

So, for this subject, you will not only have to solve problems: you will have to be able to reason abstractly, and be able to **prove** or **justify** things. In most of this subject, we need to work with **precise definitions**, and you will need to know these. Not only will you need to know these, but you will have to understand them, and be able (through the use of them) to demonstrate that you understand them. Simply learning the definitions without understanding what they mean is not going to be adequate. One hopes that these words of warning won't discourage you, but it's important to make it clear that this is a subject at a higher level than some of its prerequisites.

1.2 Aims of the course

This subject is designed to introduce the main mathematical ideas involved in the modelling of asset price evolution and the valuation of contingent claims (such as call and put options) in a discrete and continuos framework.

1.3 Learning outcomes

At the end of the course and having completed the essential reading and activities, you should have:

- knowledge, understanding and formulation of the principles of risk-neutral valuation including some versions of the No-Arbitrage Theorem
- knowledge of replication and pricing of contingent claims in certain simple models (discrete and continuous)
- knowledge of the derivation of the Black-Scholes equation, its solution in special cases, the Black-Scholes formula.

You should be able to:

- demonstrate knowledge of the subject matter, terminology, techniques and conventions covered in the subject
- demonstrate an understanding of the underlying principles of the subject
- demonstrate the ability to solve problems involving understanding of the concepts.

1.4 Syllabus

This is an introduction to an exciting and relatively new area of mathematical application. It is concerned with the valuation (i.e pricing) of 'financial derivatives'. These are contracts which are bought or sold in exchange for the promise of some kind of payment in the future, usually contingent upon a share-price then prevailing (of a specified share) or alternatively the level achieved by a share index, i.e by a certain weighted average of share prices. They are called 'derivatives' because they are **derived** from some **underlying** financial asset such as a share.

The course reviews the financial environment and some of the financial derivatives traded on the market. It then introduces the mathematical tools which enable the modelling of the fluctuations in share prices. Inevitably these are modelled by equations containing a random term. It is this term which introduces risk; it is shown how to counterbalance the risks by putting together portfolios of shares and derivatives so that risks temporarily cancel each other out and then this strategy is repeated over time. As this procedure resembles hedging a bet – i.e betting both ways – one talks of dynamic hedging. A very intuitive valuation argument (albeit now regarded as only of historical interest, see Section 9.7 for details) is based on 'hedging' and begins like this: the yield of a temporarily riskless portfolio must equal the rate of return offered by a safe deposit bank account (i.e. a riskless bank rate). The latter, of course, needs to be assumed to exist. This equation assumes that the market which values shares and derivatives actually is in equilibrium and hence eliminates the opportunities of 'arbitrage' (such as making a sure-thing profit from, say, buying cheap and selling dear).

The 'riskless hedge' argument, just mentioned, implies in the continuous-time model that the price of a derivative is the solution of a differential equation. One may either attempt to solve the differentiab equation by standard means such as numerical techniques or via Laplace transforms, though this is not always easy or feasible. However, there is an alternative route which may provide the answer: a calculation of the expected payment to be obtained from the contract by using what is known as the synthetic probability (or the risk-neutral probability). One proves that, regardless of what an investor believes the expected growth rate in the share price will be, the dynamic hedging has the effect that he might as well believe the growth rate te be the riskless growth rate. Though this may seem obvious is retrospect it does require some careful reasoning to justify.

The course considers two approaches to risk-neutral calculation, using discrete time and using continuous time. Continuous time requires the establishment of a second-order volatility correction term when using standard first-order approximation from calculus. This leads to what is known as the Itô Rule/Formula. Finite time arguments need some apparatus from Linear Algebra such as the Separating Hyperplane Theorem. We enter the subject from the discrete-time model for an easier discussion of the main issues.

1.5 Organisation of the subject guide

After this introductory chapter, in which we also discuss the reading list, the next chapter recalls some basic background in mathematics. Chapter 3 introduces you to some financial ideas with which the later mathematics will be concerned.

The course proper begins, in effect, with Chapter 4, which is dedicated to a study of a very simple framework. There is just one risky asset and money ('cash' to be precise), just one future date, and only two possible states of nature.that might occur. One state is such that an economic agent may regard it as 'favourable' for himself and another that he regards as 'unfavourable'. Of course another economic agent may view these two states the other way about. What matters is that there are only two states. The aim is to explain two central ideas. One is 'replication' of a claim by means of a portfolio, which enables the claim to be valued by the cost of purchasing the portfolio. This means that once this cost is incurred the holder of the portfolio is not exposed to any risks associated with settling his future liabilities. He is made neutral to risk. The other idea is valuation using expected values with reference to a 'purpose-made' probability measure called a 'risk-neutral' measure which is sometimes referred to as the 'synthetic probability' (for an emphasis of its specific purpose). Thus the expected value of the claim under this measure is to agree with the purchase cost of the replicating portfolio.

Once understood in the simplest concept, we re-establish in Chapter 5 the two central ideas in a one-period model with several assets and several states. We prove the Fundamental Theorem of Asset Pricing, namely that a risk-neutral measure exists if and only if there are no arbitrage opportunities.

In Chapter 6 the two ideas are extended to cover multiple periods aided by the idea of a 'self-financing' trading strategy. This is intellectually the hardest part of the course as it uses ideas from several quarters. The main tool is conditional expectation and this requires quite a lot of technical apparatus to define rigorously what information is known at certain points in time.

Whereas Chapter 6 assumes a quite general framework for the evolution of asset price, we discuss in Chapter 7 a specific approach to modelling asset price, the Binomial model. In this model, price changes are generated at each period in a uniform way by two constant factors, one factor inducing a possible price move up, the other an alternative price move down. This uniform behaviour permits easy formulas for valuation of European calls and puts. Furthermore it offers easy computation procedures for evaluating American puts. The significance of the model comes from limit considerations. In an appropriately constructed limit (as the number of periods tends to infinity and the time between periods tends to zero) the models European call and put valuations tend to the corresponding valuations in the Black-Scholes continuous-time model.

The theme of Chapter 8 is modelling price uncertainty in continuous time. As its base the model assumes an anticipated (constant) rate of growth μ for the price of an asset and a noise of constant 'amplitude' σ. The growth rate is modelled along the lines of the deterministic interest rate reviewed in Chapter 3. Thus the anticipated percentage growth in price over a time span of Δt is $\mu\,\Delta t$. The further ingredient in the model is 'standard' noise as generated by a 'stochastic process', that is by a family of random variables indexed by time, denoted here z_t. The standard amount of noise over the time span Δt starting at time t is Δz_t which is defined to be $z_{t+\Delta t} - z_t$. It is assumed that Δz_t is normally distributed with variance Δt. The noise added to the anticipated growth is then $\sigma\Delta z_t$ so that the 'standard noise' is amplified or attenuated by a factor σ. Thus the noise element has variance $\sigma^2\Delta t$ which is thus proportional to the time span Δt. The central tool developed in this Chapter is Itô's Formula. This is concerned with functions of price and allows the computation of the increment in a differentiable function when the stochastic price changes over a time interval. The Formula gives a stochastic form of Taylor's Theorem.

The techniques of Chapter 8 are put to use in the Chapter 9. Time is discretized so that the time span between periods is fixed but arbitrarily small, along lines familiar in courses **116 Abstract mathematics** or **117 Advanced calculus**. The Itô Formula is used to approximate increments in value. Valuation proceeds by the familiar route of replication over the individual periods between the discrete time points. The emphasis is on deriving in the limit (as the time spans tend to zero) a valuation of a European option contract written on some asset. This will be a deterministic function which identi.es the option value in terms of two free variables: the current asset price, and the time left to the expiry date of the contract. The value function may thus be interpreted as the conditional expected value of the option pay-off under a risk-neutral probability with conditioning on the current asset price and the time to expiry. We are able to identify what this risk-neutral measure is once we derive the celebrated Black-Scholes equation. This is the partial differential equation satisfied by the value function, and we obtain it from a consideration of increments in value. So we are finally able to identify the risk-neutral measure. Armed with the risk-neutral measure we can value a European call option. The final chapter is concerned with American long-dated options, that is, ones which have a relatively long life left (their expiry date is 'far' into the future). For the purposes of a tractable analysis of such options we place their expiration date at infinity. Options which expire at infinity are called perpetual. They satisfy an ordinary differential equation version of the Black-Scholes equation and the equation may be solved explicitly. Thus we are able to trace the graphs of perpetual options and have an indication of the value of a long-dated American option.

Finally, an epilogue sums up, in slightly more technical jargon, what we hope you will have achieved by the end of this course.

Not all chapters of the guide are the same length. It should not be thought that you should spend the same amount of time on each chapter. I will not try to specify how much relative time should be spent on each: that will vary from person to person, and we do not want to be prescriptive. I can however indicate roughly what proportion of time might be spent by a lecturer teaching the material in this guide. A lecturer would judiciously pick which examples to include in a lecture, and which to leave to the students to study on their own. Though the mathematics of subsequent chapters is not any harder in this course than in any other, the table below articulates the significant need to spend time in Chapter 3 on building up an understanding of the financial environment.

Chapter 2	Vector space approach	5%
Chapter 3	Financial environment	20%
Chapter 4	One risky asset and two states	10%
Chapter 5	One period many assets	10%
Chapter 6	Multi-period models	20%
Chapter 7	The binomial model	15%
Chapter 8	Continuous-time modelling	5%
Chapter 9	Black-Scholes model	10%
Chapter 10	Perpetual options	5%

1.6 Reading advice

Notes on the reading lists

Most topics in this course are covered in great detail by a large number of books. For that reason, we have resisted the temptation to specify **essential** reading in each chapter of the guide. What is true, however, is that **textbook reading is essential**. Textbooks will provide more in-depth explanations than you will find in this guide (which is explicitly not meant to be a textbook), and they will also provide many more examples to study, and many more exercises to work through.

The following books are the ones I have referred to in this guide, listed roughly in order of usefulness and grouped according to their level. In the first group, the first text, by Pliska, is one of the earliest textbooks on the 'undergraduate market'. Our treatment in Chapters 5 and 6 is inspired in part by Pliska. The first Shreve text is a very accessible alternative for the core material of Chapters 4–6. We follow the Notation variant to Pliska's notation established in Pliska with two exceptions. Firstly, when considering a trading strategy or portfolio H we write $Vt(H)$ for its value t rather than Vt. Secondly, we refer to Pliska's 'dominant strategies' as 'sure-thing arbitrage strategies'. We strongly recommend that the student reads through the book by Hull to understand the institutional arrangements governing stock-exchanges and the contracts that they issue. A very good second text for this material is Cvitanić and Zapatero. It is hard to be quite as prescriptive in relation to the material of Chapters 7–10 because our approach is informal – thus we describe, rather than define, stochastic integrals as limiting sums avoiding the requisite formalities. Hull contains an account of the informal theory. Luenberger is exquisite in clarity. Baxter and Rennie is admirable on this front, though written for the 'practitioners'.in the finance houses. That has been replaced by its 'classroom' version, a more technically demanding text, by Alison Etheridge aimed at university students with measure theory as a prerequisite. The second Shreve text is accessible but more formal. Finally, for a good background for linear algebra and advanced calculus see the books by Ostaszewski, or Binmore and Davies.

The second group of books on the reading list is meant for a very special reader: students should know by now that any course such as this is an introduction to the collective knowledge amassed by an army of mathematicians, so these books are a pointer for ambitious students looking for further developments. The books in the third group are popularisations and provide either anecdotal background or a historical perspective.[1]

[1] *Earlier editions than those listed here are equally useful.*

Detailed reading references in this subject guide refer to the editions of the set textbooks listed above. New editions of one or more of these textbooks may have been published by the time you study this course. You can use a more recent edition of any of the books; use the detailed chapter and section headings and the index to identify relevant readings. Also check the virtual learning environment (VLE) regularly for updated guidance on readings.

Recommended reading

Shreve, S. Stochastic *Calculus for Finance I, The Binomial asset pricing model.* (Springer, 2004) [ISBN 9780387249681].

Shreve, S. *Stochastic Calculus for Finance II, Continuous-time models.* (Springer, 2004) [ISBN 9780387401010].

Pliska, S.R. *Introduction to Mathematical Finance - Discrete Time Models.* (Blackwell, 1998) [ISBN 9781557869456].

Hull, J.C. *Options, Futures and other Derivative Securities.* (Prentice Hall, 2005) sixth edition [ISBN 9780131499089]. See also the URL: www.rotman.utoronto.ca/~hull/

Cvitanić, J. and F. Zapatero *Introduction to the Economics and Mathematics of Financial Markets.* (MIT, 2004) [ISBN 9780262033206, 9780262532594 (solutions manual)].

Luenberger, D. *Investment Science.* (Oxford University Press, 1997) [ISBN 9780195108095].

Roman, S. *Introduction to Mathematics of Finance.* Undergraduate Texts in Mathematics. (Springer, 2004) [ISBN 9780387213644].

Baxter, M. and A. Rennie *Financial calculus.* (Cambridge University Press: Cambridge, 1996) [ISBN 9780521552899].

Etheridge, A. *A Course in Financial Calculus.* (Cambridge University Press: Cambridge, 2002) [ISBN 9780521890779].

Ostaszewski, A. *Advanced Mathematical Methods.* (Cambridge University Press: Cambridge, UK, 1991) [ISBN 9780521289641].

Binmore, K. and J. Davies *Calculus:Concepts and Methods.* (Cambridge University Press: Cambridge, UK, 2001) [ISBN 9780521775410].

Intermediate reading

Bingham, N.H. and R. Kiesel *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives.* (Springer, 1998) first edition [ISBN 1852330015] second edition, 2004 [ISBN 9781852334581].

Campbell, J.Y., A.W. Lo and A.C. MacKinlay *The econometrics of financial markets.* (Princeton University Press) [ISBN 9780691043012].

Janson, S. *Gaussian Hilbert Spaces.* Cambridge Tracts in Mathematics[2] 129 (Cambridge University Press, 1997) [ISBN 9780521561280 (hbk)] in connection with the geometric view of probability sketched in Chapter 2.

Williams, D. *Probability with martingales.* (Cambridge, 1991) [ISBN 9780521406055].

Øksendal, B. *Stochastic differential equations.* (Springer, 1998) [ISBN 978 3540047582].

Wilmott, P., S. Howison and J. Dewynne *Mathematics of Financial Derivatives.* (Cambridge University Press: Cambridge, 1995) [ISBN 9780521497893].

Whittaker, E.T. and G.N. Watson *A course of modern analysis.* (Cambridge University Press, 1984) [ISBN 9780521588072].

Durrett, R. *Stochastic Calculus – A Practical Introduction.* (CRC Press, 1996) [ISBN 9780849380716]. An excellent though moderately difficult account of Itô integration, properties of Brownian motion, solution to stochastic differential equations.

Evans, L.C. *Partial Differential Equations.* (American Mathematical Society, Providence, 1998) [ISBN 9780821807729].

Schuss, Z. *Theory and Applications of Stochastic Differential Equations.* (J. Wiley, 1980) [ISBN 9780471043942]. A very down-to-earth text but directed at applications in physics. Discusses Stratonovich integration as well.

Merton, R.C. *Continuous-time Finance.* (Blackwell, 1996) [ISBN 9780631185086].

Samuelson, P. 'Proof that properly anticipated prices fluctuate randomly', *Industrial Management Review*, 6(2) 1965, pp.41–49.

[2] *Earlier editions than those listed here are equally useful.*

Further reading (historical, anecdotal,or popular texts)

Please note that as long as you read the Essential reading you are then free to read around the subject area in any text, paper or online resource. You will need to support your learning by reading as widely as possible and by thinking about how these principles apply in the real world. To help you read extensively, you have free access to the VLE and University of London Online Library (see below).

Other useful texts for this course include:

Bernstein, P. *Against the Odds.* (J. Wiley, 1998) [ISBN 0471295639].

Bernstein, P. *Capital Ideas – The Improbable origins of modern Wall Street.* (J. Wiley, 1998) [ISBN 9780029030127]. A wonderful history of the growth of ideas in this area – a very good read.

Davis, M. and A. Etheridge Louis Bachelier. *Theory of Speculation: the origins of modern finance.* Translated and with commentary (Princeton, 2006) [ISBN 9780691117522].

Dunbar, N. *Inventing Money: The Story of Long-Term Capital Management and the Legends Behind It.* (J. Wiley, 2001) [ISBN 9780471498117].

Kay, J. *Foundations of Corporate Success.* (Oxford University Press, 1995) [ISBN 9780198289883].

Kay, J. *The truth about markets.* (Allen Lane, Penguin Press 2003) [ISBN 9780140296723]. ('Everything you wanted to know about economics, but were afraid to ask'. Mervyn King, Governor, Bank of England).

Lewis, M. *Liar's Poker.* (Coronet Books, 1989) [ISBN 9780340767009].

Lowenstein, R. *When Genius Failed: The Rise and Fall of Long-Term Capital Management.* (Random House, 2000) [ISBN 9780375758256].

Smithers, A. *Valuing Wall Street: Protecting Wealth in Turbulent Times.* (McGraw-Hill, 2002) [ISBN 9780071354615 (hbk) 9780071387835 (pbk)].

Bachelier, L. *Louis Bachelier's Theory of Speculation: the origin of modern finance.* Translated and with commentary by Davis, M and A. Etheridge (Princeton: Princeton University Press, 2006) [ISBN 9780691117522].

1.7 Online study resources

In addition to the subject guide and the reading, it is crucial that you take advantage of the study resources that are available online for this course, including the VLE and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at: http://my.londoninternational.ac.uk

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged in to the Student Portal in order to register! As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you forget your login details at any point, please email uolia.support@ london.ac.uk quoting your student number.

The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of subject material.
- Electronic study materials: The printed materials that you receive from the University of London are available to download, including updated reading lists and references.
- Past examination papers and *Examiners' commentaries*: These provide advice on how each examination question might best be answered.
- A student discussion forum: This is an open space for you to discuss interests and experiences, seek support from your peers, work collaboratively to solve problems and discuss subject material.
- Videos: There are recorded academic introductions to the subject, interviews and debates and, for some courses, audio-visual tutorials and conclusions.
- Recorded lectures: For some courses, where appropriate, the sessions from previous years' Study Weekends have been recorded and made available.
- Study skills: Expert advice on preparing for examinations and developing your digital literacy skills.
- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login: http://tinyurl.com/ollathens

The easiest way to locate relevant content and journal articles in the Online Library is to use the **Summon** search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please see the online help pages: www.external.shl.lon.ac.uk/summon/about.php

1.8 Using the subject guide

I have already mentioned that this guide is not a textbook. It is important that you read textbooks in conjunction with the guide and that you try problems from the textbooks. The learning activities throughout the guide, and the sample questions at the end of the chapters, are a very useful resource. You should try them once you think you have mastered a particular chapter. Do really try them: don't just simply read the solutions where provided. Make a serious attempt before consulting the solutions. Note that the solutions are often just sketch solutions, to indicate to you how to answer the questions, but in the examination, you must show all your calculations. It is vital that you develop and enhance your problem-solving skills and the only way to do this is to try lots of examples.

1.9 Examination advice

Important: the information and advice given here are based on the examination structure used at the time this guide was written. Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current *Regulations* for relevant information about the examination, and the VLE where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

A Sample examination paper is given as an appendix to this guide. There are no optional topics in this syllabus: you should do them all.

Please be aware that a few sections in this guide have been starred (*) to indicate that the material of the section is not examinable. The material has been included in the guide either for its interest value or in order to help you to see the connection with a wider view of the subject.

Please do not assume that the questions in a real examination will necessarily be very similar to these sample questions. An examination is designed (by definition) to test you. You **will** get examination questions that are unlike any questions in this guide. The whole point of examining is to see whether you can apply knowledge in both familiar **and** unfamiliar settings. The Examiners have an obligation to surprise you. For this reason, it is important that you try as many examples as possible: from the guide and from the textbooks. This is not so that you can cover any possible type of question the examiners can think of! It's so that you get used to confronting unfamiliar questions, grappling with them, and finally coming up with the solution.

Do not panic if you cannot completely solve an examination question. There are many marks to be awarded for using the correct approach or method.

A final point about the examination for this course is to reassure you that examination questions will not assume familiarity with the contents of any of the starred sections in this subject guide. The purpose of such examples is to motivate the mathematics and to persuade you that the mathematics you are learning has important uses. Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and *Examiners' commentaries* for the course which give advice on how each question might best be answered.

Acknowledgement

This guide is based on an established lecture course at LSE of the same name. I am also very grateful to my colleagues Nick Bingham and Michalis Zervos for a careful reading of the manuscript and numerous suggestions.

Adam Ostaszewski

Chapter 2

A vector space approach to uncertainty

2.1 Introduction and aims of the chapter

This chapter begins with some activities helping to remind you of some basic concepts from Probability such as mean, variance, moment generating function and the standard normal distribution. In particular you are guided to calculate a few important integrals here, ahead of their natural appearance towards the end of the course. This should help you to check on the kind of calculus that will be required. The main matter, however, is concerned with how to employ the language of vector spaces to talk about random variables; we will use them to describe possible future prices of 'assets' and actual 'portfolios', that is collections of assets held by an economic 'agent'.

2.2 Learning outcomes

At the end of this chapter and the relevant readings, you should be able to:

- compute expectations (integrals) using 'log-normal densities'
- be prepared to use vectors to study random variables, asset prices and portfolios of assets.

2.3 Essential reading

There is no **specific** essential reading for this chapter. It is essential that you do some reading, but the topics discussed in this chapter are adequately covered in so many texts on the 'applications of calculus' that it would be artificial and unnecessarily limiting to specify precise passages from precise texts. The list below gives examples of relevant reading. (For full publication details, see Chapter 1.)

Ostaszewski, A. *Advanced Mathematical Methods.* Chapter 21.

2.4 Concepts from probability and statistics

The activities in the first subsection are meant to make you aware of the practical difficulties in applying some of the ideas in the Black-Scholes model of Chapter 9.

2.4.1 Activities referring to basic statistics

Learning activity 2.1

Let $X_1, ..., X_n$ be independent random variables. For each i assume that $E[X_i] = \mu$ and $var[X_i] = \sigma^2$. Define

$$\begin{aligned} \bar{X} &= \frac{1}{n}(X_1 + ... + X_n), \\ V &= \frac{1}{n-1}[(X_1 - \bar{X})^2 + ... + (X_n - \bar{X})^2]. \end{aligned}$$

Check that
(i) $E[\bar{X}] = \mu$ and so $E[(X_1 - \bar{X})^2] = var(X_1 - \bar{X})$,
(ii) $cov(X_1, \bar{X}) = \frac{1}{n} var(X_1)$,
(iii) $var[\bar{X}] = \frac{1}{n}\sigma^2$.
Show that $var(X_1 - \bar{X}) = \frac{(n-1)\sigma^2}{n}$ and so deduce that

$$E[V] = \sigma^2.$$

[Hint: $var(X_1 - \bar{X}) = var(X_1) + var(\bar{X}) - 2cov(X_1, \bar{X})$.]

Learning activity 2.2

Show that when the variables of the last question X_i are normally distributed

$$var(V) = \frac{2\sigma^4}{n-1}.$$

Thus the standard deviation is small when the sample is large.

Comments on the activity

Suppose X_i are observed rates of return on investment per period of a single year, there being a largish number n of periods (perhaps months, or days). Then $S = nA$ will be the annual rate (at least to a good first approximation) and $E[S] = n\mu$. The corresponding variance is $var[S] = n\sigma^2$ and so the standard deviation for the year is $\sigma_S = \sigma\sqrt{n}$. Hence

Mean blur.

$$\sigma/\mu = (\sigma_S/\sqrt{n})/(E[S]/n) = \sqrt{n} \cdot (\sigma_S/E[S]).$$

Suppose the annual rate is reasonably accurate, as measured by the ratio of its standard deviation to the expected value, for instance $\sigma_S/E[S]$ might be of order 1 or 2. However, this ratio scales upwards hugely (by a factor of $\sqrt{n}$) when the annual rates are scaled down to per-period rates. See Luenberger pages 214–216.

2.4.2 Concepts of probability requiring calculus

The activities in this subsection revise the kind of use of integral calculus that is required in this unit. Reference is made to the probability density function of the standard normal variable denoted

$$\varphi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

and the corresponding cumulative distribution function

$$\Phi(t) = \int_{-\infty}^{t} \varphi(x)dx = \int_{-\infty}^{t} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}.$$

Learning activity 2.3

The **moment generating function** is defined as $M(t) = E[e^{tX}]$. When X is a standard normal variable show that $M(t) = e^{\frac{1}{2}t^2}$. [Hint:

$$E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx}\varphi(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2+tx}dx$$

and complete the square.]

Learning activity 2.4

Using the identity $E[e^{tX}] = e^{\frac{1}{2}t^2}$, show that for a standard normal random variable

$$E[X^4] = 3.$$

[Hint: Expand both sides of the identity as series, thus:

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\{1+tx+\frac{t^2x^2}{2}+\frac{t^3x^3}{6}+\frac{t^4x^4}{24}+...\}dx = 1+\frac{1}{2}t^2+\frac{1}{8}t^4+...,$$

and compare both sides.]

Learning activity 2.5

Moment generating function

If Z is a log-normally distributed random variable with mean m and variance s show that $E[Z] = e^m e^{\frac{1}{2}s^2}$. [Hint: The random variable $X = (\ln Z - m)/s$ is normally distributed with zero mean and unit variance (i.e. is a standard normal random variable). Use $E[Z] = E[e^{sX+m}]$ and the previous activity.]

Learning activity 2.6

Show that if Z is a random variable such that

$$\ln Z = m + aU + bV$$

where U and V are independent random variables with standard normal distributions then

$$E[Z] = e^{m+\frac{1}{2}(a^2+b^2)}.$$

Hint: $E[Z] = E[e^{m+aU+bV}] = e^m E[e^{aU}]E[e^{bV}]$. Why?

Comments on the activity

The next activity shows how to deal with $\ln Z$ *when the two normal random variables are correlated. The trick is to regard the random variables as two vectors and to replace them with two orthogonal ones with the same linear span (recall the Gram-Schmidt process in Linear Algebra). In this Gaussian variables context orthogonality has the same meaning as zero correlation.*

Learning activity 2.7

If Z is a random variable such that

$$\ln Z = m + X + Y,$$

where X and Y are correlated random variables with $Cov(X, Y) = \rho\sigma_X\sigma_Y$ normal distributions both with mean zero, then the trick is to replace X by rescaling it to a new random variable U with variance unity and to rewrite Y as a sum of U and another standard normal variable V. Writing

$$Y = \alpha U + \beta V,$$

where $X = \gamma U$, show that

$$\gamma = \sigma_X, \alpha = \rho\sigma_Y, \beta = \sqrt{1-\rho^2}\sigma_Y.$$

Deduce that

$$E[Z] = e^{m+\frac{1}{2}(\sigma_X^2+\sigma_Y^2+2\rho\sigma_Y\sigma_Y)}.$$

Learning activity 2.8

Find the expected value of the function of a random variable X given by $g(X) = \max\{X, 0\}$, when the probability density function of X, here denoted $f(x)$, is log-normal. That is, evaluate

$$E[g(X)] = \int_k^\infty (x-k)f(x)dx = \int_k^\infty xf(x)dx - k\int_k^\infty f(x)dx.$$

This calculation is the basis of the Black-Scholes formula for valuing a 'call-option'.

A Black-Scholes calculation

[Hint. Make a change of variable by putting

$$w = w(x) = \frac{\ln x - m}{\sigma},$$

and write the two integrals in terms of the cumulative normal distribution function Φ. It may be helpful to refer to '*sectional expectation*', $E[g(X); X > k]$, as defined below, and how that changes under a change of variable:

$$\begin{aligned} E[g(X); X \;\; > \;\; k] &=_{def} \int_k^\infty g(x)f(x)dx = E[g(e^{\sigma W+m}); W > w(k)] \\ &= \int_{w(k)}^\infty g(e^{\sigma w+m})\varphi(w)dw. \end{aligned}$$

Learning activity 2.9

The function Erf is defined in Mathematica as

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Check that

$$\Phi(x) = (1 + Erf(x/\sqrt{2}))/2.$$

2.5 Linear algebra: the language of vectors

In investment analysis one studies the possible prices an asset might have at some future point in time. The term 'asset' is wide ranging and might be a share in a company, but it can also be the pay-out offered by some form of insurance contract. Since in both cases the future value is affected by 'circumstances' we can draw attention to such dependence, if the need arises, by using the more general term 'contingent claim'. This term is used interchangeably with 'asset'.

Primary and 'derivative' assets

Contingent claims most often considered in this course take the form of a payment in an amount which depends on the value of some specified asset where the evolution in time of the value of the specified asset is given. Since the value of the specified asset is a 'given', the asset is said to be a 'primary asset'. The contingent claim is then said to be a derivative asset, or just a 'derivative'.

It may be necessary to contemplate various plans of action now, or in the future, and the action could well depend on the realized future price. In order to be specific about various future scenarios it becomes necessary to list what possible prices should be considered in this kind of forward planning. Say the prices might be $X_1, X_2, ..., X_n$. We will initially think of these numbers as being positive (or, sometimes, just non-negative). It is therefore possible to think about the information currently available concerning the future of an asset, or the information currently assumed about the future, as being described by the 'vector of prices' $(X_1, X_2, ..., X_n)^t$.

2.5.1 States of nature

In order to discuss more complicated scenarios, say involving several assets at once, it is more helpful as a starting point to describe the future as just a set Ω listing all 'possible future **atomic** events of interest'. Any one such event is typically denoted ω and no specification at this stage is given as to what is to be an 'atomic event'. These events, in whatever way selected, will become 'atomic' by sheer dint of definition only. Of course in any particular discussion of future events we would need to be specific about the list Ω. For instance, if there are only two assets of interest and only one future date, an atomic event ω would be defined by: 'the first asset will take the price

X_i and the other asset the price Y_j'. Here, as above, $X_1, X_2, ..., X_n$ lists all possible prices of the first asset and $Y_1, Y_2, ..., Y_m$ those of the second. In fact we could identify ω with the pair of values (X_i, Y_j). This example explains why the set Ω is called the **sample space**: it is the set of all possible data samples that may occur. It is also referred to as a 'population' from which a sample of data may be drawn.

It is customary to refer to the atomic events also as '**states of nature**'. The original asset under discussion is now thought of as having price $X(\omega)$ when the future is given by the event ω. The asset is thus best described by correspondence to the states of nature, that is by the vector $(X(\omega_1), ..., X(\omega_m))^t$ if $\Omega = \{\omega_1, ..., \omega_m\}$. When the context implies the use of Ω we refer also to this vector as the 'vector of possible prices' and denote it by X. A word of warning here: it is sometimes convenient, but really very careless (clearly not a habit to be encouraged), to speak of 'asset X'; but it is very important to distinguish conceptually between the name of an asset and the vector of its prices at some given point in time. Assets, like any other products in the market, though different in name could in principle have equal vectors of possible prices; when there is danger of a mix-up we will be more careful with the notation.

Example 1 (Potential gains and losses as vectors). An investor with M dollars of cash in his account is considering whether to buy an asset now when its unit price is 1 dollar or to wait to the end of the day when he thinks the price is likely to go up or down by at most one cent with equal probability. He can think of this as a coin-tossing game with a fair coin. The result of the toss is either $\omega_1 = Heads$ or $\omega_2 = Tails$ with $Heads$ indicating a rise. Thus $\Omega = \{\omega_1, \omega_2\}$. He models the unit dollar-price of the asset with the vector X where $X(\omega_1) = 1.01$ and $X(\omega_2) = 0.99$. If the investor decides to wait to the close of business, the unit-price in dollars of the asset in his account – namely his cash – is modelled by B, where $B(\omega_1) = 1$ and $B(\omega_2) = 1$, i.e. $B = (1, 1)$ since irrespective of the toss the denomination of the currency is unchanged. If he bought the asset his gain G from waiting is thus either $G(\omega_1) = +0.01M$ or $G(\omega_2) = -0.01M$. In **vector** form the gain G satisfies $G = M \cdot (X - B)$. The investor reckons he has a 50% chance of winning by waiting (but also a 50% chance of losing).

Example 2 (Effects of diversification). Our investor has in mind to split his cash evenly between two assets, X, Y both currently selling at 1 dollar per unit. He thinks that either of the two assets will independently rise or fall by one cent by the end of the day. This time it is a two-coin tossing game and so the outcomes will be
$\omega_1 = (Heads, Heads)$, $\omega_2 = (Heads, Tails)$,
$\omega_3 = (Tails, Heads)$, $\omega_4 = (Tails, Tails)$ with *Heads* indicating a rise. As before, we have a vector of prices for the first asset, listed according to state, given by
$X = (X_1, X_2, X_3, X_4) = (1.01, 1.01, 0.99, 0.99)$, and a vector for the second asset given by $Y = (Y_1, Y_2, Y_3, Y_4) = (1.01, 0.099, 1.01, 0.99)$. This time the investor's gain will be
$G = \frac{M}{2}(X + Y) - M \cdot B = M(0.01, 0, 0, -0.01)$. By diversifying his acquisitions he will lose only in one state of nature; the chance of this he assesses as 25%. But then he also narrows his chances of a win.

We refrain from going through any more complicated examples of Ω at this stage. Instead we merely let Ω name the members of the set of possible simple events; we suppose that there are at most m possible simple events and we label them $\omega_1, ..., \omega_m$. Thus $\Omega = \{\omega_1, ..., \omega_m\}$.

Although we are interested here in asset prices, the more general situation is that of modelling some numerical property, call it X, which is associated with every member of a '**population**' Ω. Thus a member ω of Ω drawn from the population displays a value $X(\omega)$. The vector of potential values obtained by drawing an individual member from the population is thus given by $(X(\omega_1), ...X(\omega_m))^t$. In this context such a vector is called a '**random variable**'. The term owes its name to two facts: that a numerical property may in general be variable over the population; and, that, when the population is large, rather than conduct a census in order to learn its values, it may be more economical to observe the values on a representative sample of individuals drawn at 'random'. A simple example is that of 'height' of people in a population. We will not discuss appropriate ways to select a sample of individuals at random, which is a matter for statistics textbooks. But we will show how mathematical ideas may be applied profitably to random variables. Most often we will be concerned with the random variable model of asset prices.

2.5.2 Compound events

The notation adopted in this section does imply that the components of X need not be distinct (as in Example 2). So a future event such as 'the price of the asset will be x' need no longer be a simple event. This is because one and the same price x may occur in several states of nature. The event that the price is x is a 'compound event'; we denote that event by $[X = x]$. The definition is

$$[X = x] := \{\omega \in \Omega : X(\omega) = x\}.$$

Evidently for distinct x the compound events $[X = x]$ are disjoint and, as x varies through all the possible prices, these events together exhaust all the states of nature. We therefore say that the asset values **partition** Ω. In Example 2 of the last subsection we have

$$[X = 1.01] = \{\omega_1, \omega_2\} \text{ and } [X = 0.99] = \{\omega_3, \omega_4\}.$$

On the other hand notice that by referring to Ω the language we have adopted allows us to describe **any** other asset Y by the vector of its prices $(Y(\omega_1), ..., Y(\omega_m))$. This is a vector with the **same** number of components as X.

2.5.3 Portfolios

If someone holds u units of asset X the value of the quantity u of the asset in state ω_i will be

$$uX(\omega_i).$$

Long and short positions

If u is negative, this number will be negative and may be interpreted as the value of a debt resulting from having initially borrowed $|u|$ units of the asset. When u is negative the holder is said to be in a position which is **short** of $|u|$ units of the asset X, or more conveniently 'short $|u|$ units X'. If u is positive the holder is correspondingly said to be **long** u units in X. Whilst you may regard as natural the use of negative quantities in this context, you should be aware that this use of mathematics coincides with the system of 'double-entry' book-keeping promoted during the Italian Renaissance. Its first exposition, due to the mathematician Pacioli in 1494, was in a **printed** book, at a time when printing was expensive (a mere quarter of a century after Guttenberg's invention of metal type). It is one of the foundation stones of capitalism.

Double entry book-keeping

Consider now holding u units of asset X and v units of asset Y; this is referred to in financial terms as a **portfolio** of u units of X and v units of Y. The portfolio may be neatly represented by the vector $(u, v)^t$, it being understood that a fixed order of listing the assets is being referred to. With a two-asset list X, Y the portfolio $(u, v)^t$ lies in $\mathbb{R}^2$ though the assets themselves lie in $\mathbb{R}^m$. The value at the future date of this portfolio in state ω_i will be

$$uX(\omega_i) + vY(\omega_i)$$

and this is simply the i-th co-ordinate of the vector $uX + vY$ in $\mathbb{R}^m$.

Example 3. A person who borrows, without interest, the M dollars from our investor of Example 1 and exchanges them for asset X when the unit price is 1 dollar holds the portfolio which is M units long in asset X (so $u = M$) and short M units (i.e. $v = -M$ units) in asset B (the cash which he exchanged for the asset). Thus $uX + vB$ represents his gain that day at close of business. This equals the vector G considered earlier.

2.5.4 The portfolio space: replication and completeness

If we allow both long and short positions in the assets X and Y, the set of pay-offs $uX + vY$ of all portfolios $(u, v)^t$ is simply the subspace in $\mathbb{R}^m$ spanned by the two vectors X, Y. This is called the **portfolio space** of the assets X and Y. We will be concerned in future examples with several assets $S_1, ..., S_N$ and with the assessment of the current value of **portfolios** $(u_1, ..., u_N)^t$. The portfolios are vectors in $\mathbb{R}^N$ and represent u_1 units of asset S_1, u_2 units of asset $S_2, ..., u_N$ units of asset S_N. The portfolio has payoff vector

$$u_1S_1 + ... + u_NS_N.$$

When $\Omega = \{\omega_1, ..., \omega_m\}$ this vector is in $\mathbb{R}^m$. Clearly the vector lies in $Lin\{S_1, ..., S_N\}$, a subspace of $\mathbb{R}^m$, to be called the **portfolio space** of $S_1, ..., S_N$. The assets $S_1, ..., S_N$ are not necessarily assumed to be linearly independent as vectors in $\mathbb{R}^m$.

It is easy to confuse the distinction between the portfolio $(u_1, ..., u_N)^t$ which is in $\mathbb{R}^N$ and its pay-off in $\mathbb{R}^m$, so the reader should beware.

We will refer to $S_1, ..., S_N$ as the spanning assets, or even basis assets (if the vectors happen to be linearly independent), and to a general member of the portfolio space $Lin\{S_1, ..., S_N\}$ as an asset X when

$$X = u_1S_1 + ... + u_NS_N.$$

We can write this last line as

$$X = Su.$$

Here $u = (u_1, ..., u_N)^t$ and S is the $m \times N$ matrix

$$\begin{bmatrix} S_1(\omega_1) & S_2(\omega_1) & ... & S_N(\omega_1) \\ S_1(\omega_2) & S_2(\omega_2) & ... & S_N(\omega_2) \\ ... & ... & ... & ... \\ S_1(\omega_m) & S_2(\omega_m) & ... & S_N(\omega_m) \end{bmatrix}.$$

When X in $\mathbb{R}^m$ is in the portfolio space, and $X = Su$, for some portfolio u, we say that u *replicates* the claim. If *each* vector X in $\mathbb{R}^m$ may be replicated by some portfolio we say that the portfolio space is **complete**.

We will return to this point in Section 5.

2.5.5 The riskless asset

We let $\mathbf{1}$ denote the vector $(1, 1, ..., 1)^t$ in $\mathbb{R}^m$ having a one in each co-ordinate. Thus $u\mathbf{1}$ represents an asset which has price u in each state of nature. Since there is no risk regarding the value later in time, an asset represented by $u\mathbf{1}$ is termed a **riskless asset**. By contrast any vector not of this form is said to be **risky**. A dollar deposited in a bank when the interest rate is fixed in advance provides an example of a riskless asset, since the dollar together with interest accrual is paid back by the bank no matter what the state of nature. (This is true provided we believe the bank is safe from failure, which we do throughout this unit.)

Of course the portfolio space spanned by the set of assets $\{S_1, ..., S_N\}$ might contain $\mathbf{1}$ but need not Figure 2.1. The riskless asset in a two-state model.

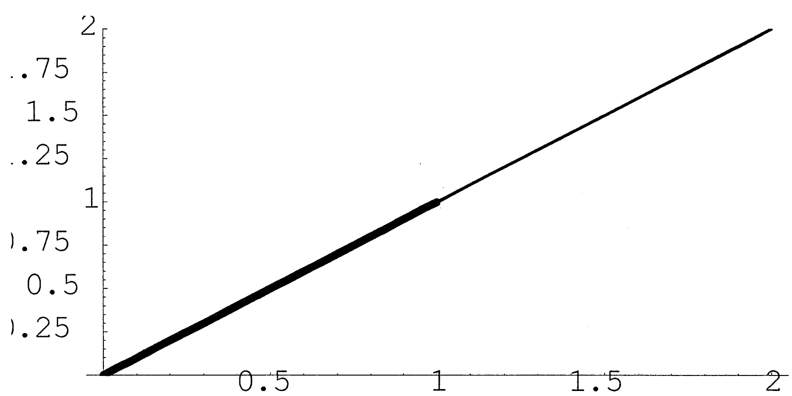

Figure 2.1: The riskless asset in a two- state model.

2.5.6 The call option: replication and linear span

Strike/exercise price

A 'call option' is a financial contract issued with reference to a specified asset, called the 'underlying' asset, and a price known as the 'exercise price' or 'strike price', which we denote by K; it is issued (often by an exchange) at some initial date, here taken to be $t = 0$, and it promises at a later date, here taken to be $t = 1$, a payment which when the underlying asset has price $S(1)$ is equal to $S(1) - K$, **provided** this is positive, but is zero otherwise. Thus the payment has value $C = \max\{S(1) - K, 0\}$, the larger of $S(1) - K$ and zero.

A related option is the 'put option'. This again is issued (often by an exchange) at some initial date, which we take to be $t = 0$, and it promises at a later date, which we take as $t = 1$, a payment equal to $K - S(1)$ provided this is positive (as before the underlying asset has price $S(1)$) and zero otherwise. Thus the payment has value $P = \max\{K - S(1), 0\}$, the larger of $K - S(1)$ and zero.

We will use the notation

$$(x)^+ = \max\{x, 0\}.$$

Thus the call option pay-off is $(S(1) - K)^+$. The corresponding put option pay-off is $(K - S(1))^+$.

Learning activity 2.10

Plot the following pay-off functions in Mathematica obtained by considering combinations of puts and calls:

$$\begin{array}{ccc} (x-1)^+, (1-x)^+, & (x-1)^+ - (1-x)^+, & (x-1)^+ - (x-2)^+, \\ (1-x)^+ - (2-x)^+, & (x-1)^+ - 2*(x-1.5)^+ \quad +(x-2)^+, & (x-1)^+ + (1-x)^+, \\ (x-1)^+ + (2-x)^+, & (x-1)^+ + 2(1-x)^+, & 2(x-1)^+ + (x-1)^+ \end{array}$$

Say in your own in words what are the circumstances under which profits arise to the holder of these contracts.

Comments on the activity

Reading from left to right, these represent pay-offs to contracts described by the following names:
on the first line: a long call, a long put, forward contract, bull spread,
on the middle line: bear spread, butterfly spread, bottom straddle,
on the bottom line: strangle, strip, strap.
Notes:

1. The function $(x)^+ = \max\{x, 0\}$ may be defined in Mathematica by:
plus[x_] := If[x<=0,0,x].

2. The pay-off for a long call option with exercise price k is given by $(x - k)^+$, since this pays $x - k$ when $x > k$ and zero otherwise.

What is the use of the call option? Whenever the asset has a terminal price $S(1)$ above K, the call, by way of a cash compensation (paid out whenever the contract is exercised), effectively gives the holder the right to buy a unit of the asset from the exchange at the lower price K. (The holder 'calls' for a sale at price K.) If the asset price is below K the holder need not exercise the contract. It is intuitively clear that the option is a valuable form of insurance. But what should its fair price be?

We pause to consider how to illustrate the contract, as this will help us value it.

There are three natural approaches to illustrating the payment scheme.

The first approach is to plot payment against **all** values that we could assign to $S(1)$, as in the first figure below. This picture helps us to see how the payment formula works.

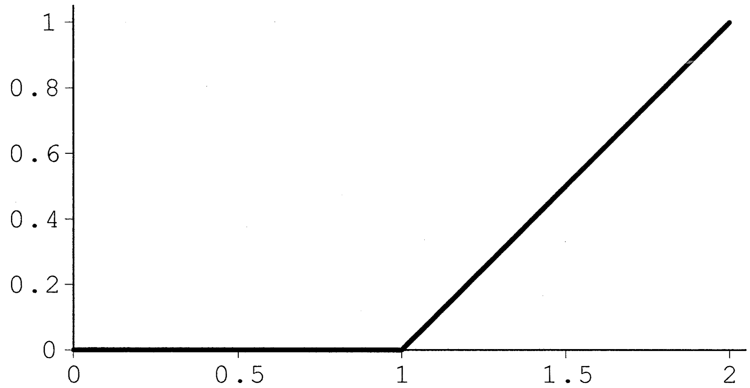

Fig. 2.2: Payment from a call option with strike price $K = 1$ against all values of $S(1)$.

The second approach recognises that, at least for the time being, we want to have only a finite number of states of nature to deal with. So one way to illustrate this finiteness is to indicate on the price axis, say with dots as on Figure 2.3, just those future prices which may be of interest to us in some forward planning exercise and then to graph only the corresponding option values. For instance, we might conceive of two economic scenarios, a good one in which the underlying asset has a high price S_H, and a poor one in which the asset has a low price S_L,with $S_L < S_H$. What are the good and poor economic scenarios defined by? Just as you please; for instance according to whether the chosen asset has a high or low price (for example, the asset might be a portfolio of shares such as makes up a market indicator, like a FTSE-100 index).

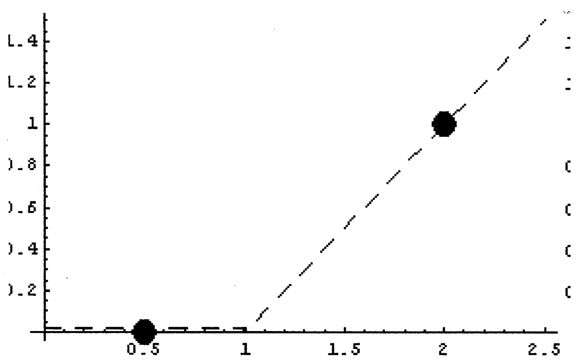

Fig. 2.3: Payment from a call option with strike price K against two 'reference values' for $S(1)$.

The third approach stresses the fact that we want $S(1)$ to be represented as a vector, with its components corresponding to states. So in the third figure, Figure 2.4, we suppose that there are only two states of nature ω_1 and ω_2 (corresponding to the poor and good economic scenarios referred to above), and we put up two axes one for each of these two states. The idea is that the two values in the two states arising from any payment scheme X can be shown as vector (x_1, x_2), with $x_1 = X(\omega_1)$, and $x_2 = X(\omega_2)$. This makes the x_1 axis correspond to state ω_1 and x_2 axis correspond to state ω_2. Now the payment from a sale of one unit of the underlying asset is represented by a vector with two components (S_L, S_H). The prices paid in the two states can then be read off according to state from one or other axis. In Figure 2.4 this is illustrated with the vector $S = (S_L, S_H) = (\frac{1}{2}, 2)$.The riskless asset is also shown in the diagram as $\mathbf{1} = (1, 1)$.

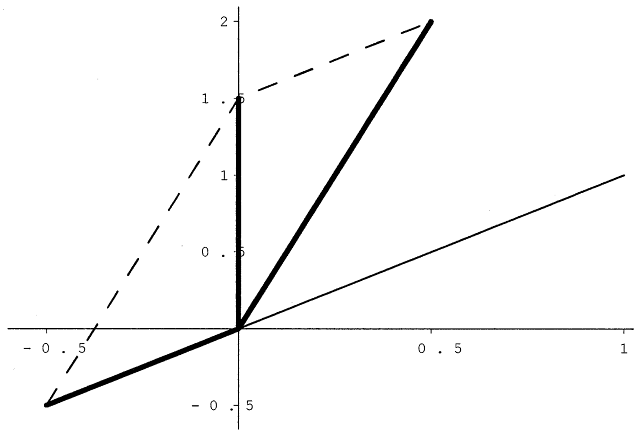

Fig. 2.4: Replicating the call $C = (0, 1.5)$ using the riskless vector $\mathbf{1} = (1, 1)$ and the risky vector $S(1) = (.5, 2)$ as spanning vectors.

On the same diagram we show a call option. It pays $C(\omega)$ in state ω where $C(\omega) = \max\{S(1, \omega) - K, 0\}$. For our illustration we took $K = \frac{1}{2}$. Thus K here satisfies $S_L \le K < S_H$. In the poor economic state $C(\omega_1) = 0$, and in the good economic state $C(\omega_2) = S_H - K = \frac{1}{2}$; thus C lies along the ω_2-axis.

This third diagram represents how we intend to model contingent claims at their simplest. We take Ω to consist of two states. We take for our portfolio space the span of two assets: the riskless bank deposit $\mathbf{1}$ and a 'risky' underlying asset with price vector S, i.e. one with prices $S_L < S_H$. Thus $S \notin Lin\{\mathbf{1}\}$.

What is special about this model? First of all, we choose these two assets, because we know their value (price) at time $t = 0$ on the market. The market trades asset S for $S(0)$ and the bank offers the riskless asset $\mathbf{1}$ for $1/(1 + r)$. Secondly $Lin\{S, \mathbf{1}\} = \mathbb{R}^2$, so the model is complete; this means that any claim can be expressed as a sum of the two spanning assets in appropriate amounts. We hope to be able to price the claim correspondingly, as the cost the two summands. In other words, we use completeness to price claims.

Our first step is to see about spanning the call.

In this context it is natural to relabel the two states so that $\omega_H = High$ and $\omega_L = Low$ corresponding to the risky asset being valued high or low. So the asset price is $S_H = S(\omega_H)$ or $S_L = S(\omega_L)$ with $S_L < S_H$. As above, assume $S_L \le K < S_H$; so then

$$C(\omega_H) = S_H - K, \quad C(\omega_L) = 0.$$

We find a portfolio (u, v) so that the call is a combination of $\mathbf{1}$ and S:

$$C = u\mathbf{1} + vS.$$

We say the portfolio '**replicates**' the contract. Comparing both sides of this equation, we see that for replication to hold we need

$$\begin{aligned} u + vS_H &= S_H - K, \\ u + vS_L &= 0. \end{aligned}$$

So

$$v(S_H - S_L) = S_H - K,$$

hence

$$u = -S_L \cdot \frac{S_H - K}{S_H - S_L}, \quad v = \frac{S_H - K}{S_H - S_L}.$$

2.5.7 Arbitrage and linearity

Suppose the assets in their totality are given a definite numbering $1, 2, ..., k$ (so that their future prices of assets $X_1, ..., X_k$ are represented by the vectors $X_1, ..., X_k$). We can denote a portfolio of these assets by a vector whose i-th component gives the quantity of 'asset' X_i held by an investor. We will denote this vector by $H = (h_1, h_2, ..., h_k)$. The notation appears to imply that we are treating $X_1, ..., X_k$ as though they formed a basis and the question arises whether we are entitled to do so. This in turn depends on what assumptions we make about the initial prices of the assets. Denote the initial prices respectively by $p(X_i)$. We now show that when the asset market is in *equilibrium* these initial prices are connected to the future prices.

Suppose for the sake of argument that

$$X_1 = X_2 + X_3,$$

i.e. in all states of nature the price of asset X_1 is the sum of the prices of the other two assets. This says that the portfolio $H_0 = (1, 0, 0, \ldots)$ and $H_0' = (0, 1, 1, 0, ...0)$ are of equal value in one period's time. One might expect that in this case the current market value of both portfolios is also the same, i.e.

$$p(X_1) = p(X_2) + p(X_3).$$

Suppose, however, that in fact, for some reason $p(X_1) > p(X_2) + p(X_3)$, i.e. $D = p(X_1) - p(X_2) - p(X_3) > 0$. This inequality suggest a method for making money, called **arbitrage**, based on the idea of buying cheap and selling dear and pocketing the difference D. This idea is very simple to comprehend if done 'instantaneously'. However, here we must wait for a period of time to elapse and so must worry about interest, which is either paid on borrowed money (and tied-up in stock bought), or earned on a deposit.

To explain this point consider again the riskless asset $\mathbf{1}$. If the interest rate over the period is r, then in exchange for 1 dollar deposited initially, an amount $1 + r$ will be received one period later; equivalently, an amount $1/(1 + r)$ needs to be deposited initially to receive one period later 1 dollar in all states of nature. Thus the price to be paid for receiving the asset $\mathbf{1}$, denoted $p(\mathbf{1})$, satisfies

$$p(\mathbf{1}) = \frac{1}{1 + r}.$$

Let X_{k+1} denote the riskless asset $\mathbf{1}$. Thus $H = (h_1, ..., h_{k+1}) \in \mathbb{R}^{k+1}$.

Armed with this notation, here is the idea of arbitrage across time.

First find someone who owns a unit of X_1 and has no intention of selling it during the following period. Borrow this unit of X_1 and sell it for $p(X_1)$. This amounts to borrowing the portfolio H_0, defined above, in order to sell it. Next buy (for less) the portfolio H'_0 comprising one unit each of assets X_2 and X_3 for the total cost $p(X_2) + p(X_3)$ and deposit the difference D in a bank. At the end of the period, when the state of nature is ω, sell H'_0, which gives an amount of money $X_2(\omega) + X_3(\omega)$; this equals $X_1(\omega)$ and so enables a unit of the asset X_1 to be bought back, and given back to the lender. The overall effect is that this behaviour leaves the enterprising borrower with $D(1 + r)$ in his bank account at the end of the period.

We draw attention to the fact that the model of market securities in which this argument works must include either a riskless security, or a portfolio of securities with pay-off equal to $\mathbf{1}$.

A similar strategy exists if $p(X_1) < p(X_2) + p(X_3)$, i.e. $D = p(X_2) + p(X_3) - p(X_1) > 0$, but based this time on borrowing H'_0, selling it, acquiring H_0, depositing the difference D in the bank and waiting the one period. (Figure this out for yourself.)

We conclude that if our market example is to have no arbitrage, we must have

$$p(X_1) = p(X_2) + p(X_3).$$

A second important conclusion is that, in the absence of an arbitrage as just discussed, a contingent claim which pays nothing in some states and a positive amount in at least one state is prevented from having an initial price of zero. (Otherwise you could buy for nothing a chance of positive profit with no risk.)

This argument clearly generalizes and we may conclude as follows.

Conclusion. *If the assets $X_1, ..., X_k$ do not give rise to an arbitrage and an asset X is linearly dependent on the given assets with*

$$X = \sum_{i=1}^{k} u_i X_i$$

then its price satisfies

$$p(X) = \sum_{i=1}^{k} u_i p(X_i).$$

Thus in particular

$$p(\sum h_i X_i) = \sum h_i p(X_i).$$

Moreover, for any portfolio H, we have

$$V_0(H) = \sum h_i p(X_i).$$

If a model is free of arbitrage the price functional p is thus a **linear** transformation.

2.5.8 Valuation if a linear transformation is an expectation

We return to the point of view that portfolios form a vector space (compare with Section 2.3.8).

If we think of the states of nature ω_i occurring with probability p_i then the expected value of asset X is

$$E_p[X] = \sum_i p_i X(\omega_i).$$

This is also referred to as the mean value of X, denoted m_X.

Evidently the portfolio holding u units of X and v units of Y defines an asset $Z = uX + vY$, the expected value of which is

$$\begin{aligned} E[Z] &= \sum_i p_i\{uX(\omega_i) + vY(\omega_i)\} = uE[X] + vE[Y\}, \\ m_Z &= u \cdot m_X + v \cdot m_Y. \end{aligned}$$

Thus, unsurprisingly, expectation is a linear transformation from the portfolio space to the reals $\mathbb{R}$.

Now we set about establishing a converse: that valuation is an appropriate expectation.

Suppose that the portfolio space $Lin\{S_1, ..., S_N\}$ is complete, i.e. coincides with $\mathbb{R}^m$ (where m is the number of states of nature). Then each of the natural base vectors $e_1 = (1, 0, 0, ...)^t$, $e_2 = (1, 0, 0, ...)^t, ...$ represents a claim. For example, e_1 pays one dollar if and only if state one occurs, but is otherwise worthless. Note that in the absence of arbitrage opportunities the price p_1 of the claim e_1 is positive and similarly p_2, $p_3, ...$ are all positive. These claims are known as **Arrow-Debreu securities**.

Relative to the basis of Arrow-Debreu securities, an arbitrary claim $X = (x_1, ..., x_m)^t$ may be represented in the form

$$X = x_1 e_1 + x_2 e_2 + ... + x_m e_m.$$

In the absence of arbitrage, the pricing function p is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}$, and so we may write

$$p(X) = x_1 p(e_1) + x_2 p(e_2) + ... + x_m p(e_m),$$

i.e.

$$p(X) = (p_1, p_2, ..., p_m) \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_m \end{bmatrix},$$

and the matrix representing matrix is just the row vector $(p_1, ..., p_m)$.

Notice that if the interest rate is zero then $p(\mathbf{1}) = 1$ and, since

$$\mathbf{1} = e_1 + e_2 + ... + e_m,$$

we have

$$\begin{aligned} 1 &= p(\mathbf{1}) \\ &= p(e_1) + p(e_2) + ... + p(e_m) \\ &= p_1 + p_2 + ... + p_m. \end{aligned}$$

So $\mathbf{p} = (p_1, ..., p_m)^t$ is a probability vector and

$$\begin{aligned} E_{\mathbf{p}}[X] &= \sum_i p_i X(\omega_i) \\ &= x_1 p(e_1) + x_2 p(e_2) + ... + x_m p(e_m) \\ &= p(X). \end{aligned}$$

Thus p is a 'probability measure' on the states under which the market price of a claim agrees with its expected value.

2.6 A geometric view of probability

In linear algebra courses one is taught to think geometrically about vector spaces. We have seen above that vector space language is a natural language for describing uncertainty when working with a finite number of states of nature. We will be reaping the benefits of thinking about probability geometrically in a considerable part of the course – especially so in Chapter 6. We see in this section, as a more modest application, why *correlation* is to be viewed through the spectacle of orthogonality and how hyperplanes correspond to a fixed expected monetary outcome. Later, in Chapter 6 in a converse kind of way hyperplanes take a central role in constructing probabilities.

Let $\mathbf{p} = (p_1, ..., p_n)^t$ be a probability vector such that $p_i > 0$ for each i. Thus $p_1 + ... + p_n = 1$.

Example 1. The formula

$$\langle X, Y \rangle_{\mathbf{p}} = E_{\mathbf{p}}[XY] = \sum_{i=1}^{n} p_i X_i Y_i,$$

where X, Y are in $\mathbb{R}^n$, defines an inner-product. Here we think of $X = (X_1, ..., X_n)^t$ as describing the possible pay-offs $X_i = X(\omega_i)$, according to the possible outcomes $\omega_1, ..., \omega_n$, with associated probabilities of the outcomes given by $(p_1, ..., p_n)^t$.

Since all the probabilities p_i are positive, we have

$$\langle X, X \rangle = 0 \text{ if and only if } X = 0,$$

so that $||X||_{\mathbf{p}} = \sqrt{\langle X, X \rangle}$ defines a norm.

Example 2. Find $||\mathbf{1}||_{\mathbf{p}}$ where $\mathbf{1} = (1, ..., 1)^t$ is the riskless pay-off of one dollar in all states of nature.

Solution. We have

$$||\mathbf{1}||_{\mathbf{p}}^2 = \sum p_i \cdot 1^2 = 1.$$

Example 3. Relate the inner-product $\langle X, Y \rangle_{\mathbf{p}}$ when $p_1 = p_2 = ... = p_n = 1/n$ to the usual Pythagorean inner-product $\langle X, Y \rangle_2$.

Solution. We have

$$\begin{aligned} E_{\mathbf{p}}[XY] &= \sum_i \frac{1}{n} X_i Y_i = \frac{1}{n} \langle X, Y \rangle_2, \\ ||X||_{\mathbf{p}} &= \frac{1}{n} ||X||_2. \end{aligned}$$

Example 4. Relate the norm $||X||_{\mathbf{p}}$ and inner product $\langle X, Y \rangle_{\mathbf{p}}$ to the Pythagorean inner product $\langle X, Y \rangle_2$ and norm $||X||_2$ by considering the transform $Z = Z(X)$ of the random variable X given by

$$Z = S^{1/2} X,$$

where $S^{1/2}$ denotes the diagonal matrix $diag\{\sqrt{p_1}, \sqrt{p_2}, ...\}$, i.e. a matrix with non-zero entries only along its diagonal. Thus

$$Z_i = \sqrt{p_i} \cdot X_i.$$

Solution. This transformation amounts to a scale-change in the independent directions. We first note that $S = S^{1/2}S^{1/2} = diag\{p_1, p_2, ...\}$ and that $S^{1/2}$ is symmetric, i.e. $(S^{1/2})^t = S^{1/2}$. Now we see that

$$\langle X, Y \rangle_{\mathbf{p}} = \sum_{i=1}^{n} p_i X_i Y_i = X^t S Y,$$

so $X^t Y = \langle X, Y \rangle_2 = \langle X, S^{-1}Y \rangle_{\mathbf{p}}$. Thus a hyperplane $\{X : X^t Y = 1\}$ with normal Y translates to the hyperplane $\{X : \langle X, S^{-1}Y \rangle_{\mathbf{p}} = 1\}$ with normal $S^{-1}Y$. Of course $S^{-1} = diag\{1/p_1, 1/p_2, ...\}$. Thus the hyperplane of portfolios X of initial value one, $X^t\mathbf{1} = 1$, translates to the hyperplane with normal $S^{-1}\mathbf{1} = (1/p_1, 1/p_2, ...)$.

Lastly, we compute the Pythagorean norm of $||Z||_2$ to be

$$||Z||_2 = X^t S^{1/2} S^{1/2} X = X^t S X = ||X||_{\mathbf{p}}.$$

Of course this just says

$$||Z||_2 = \sum p_i X_i^2 = ||X||_{\mathbf{p}}.$$

Obviously the transform of the riskless vector is

$$S^{1/2}\mathbf{1} = (\sqrt{p_1}, ..., \sqrt{p_1})^t.$$

From this last example it should come as no surprise that purely geometric facts concerned with norms and inner-products translate into probabilistic statements and, conversely, that probabilistic statements about covariances and variances yield under specialization standard geometric facts.

Application of Example 1. This inner product is used to measure the co-dependence of two vectors relative to the possible outcomes ω in Ω. Indeed if the distinct values taken by X are x_i for $i = 1, ..., k$ and the distinct values taken by Y are y_j for $j = 1, .., k'$ then

$$E_{\mathbf{p}}[XY] = \sum_{i,j} x_i y_j \cdot P[X = x_i \& Y = y_j],$$

Here $[X = x_i \ \& \ Y = y_j]$ denotes the compound event

$$E_{ij} = \{\omega : X(\omega) = x_i \ \& \ Y(\omega) = y_j\},$$

which is equal to

$$\{\omega : X(\omega) = x_i\} \cap \{\omega : Y(\omega) = y_j\}.$$

Its probability is just

$$\sum_{\omega \in E_{ij}} p(\omega).$$

Stretching conventions to allow the summation operation $\sum$ to be applied to a set of real numbers one might have written this more elegantly as

$$\sum \{p(\omega) : X(\omega) = x_i \ \& \ Y(\omega) = y_j\}.$$

We define X and Y to be **independent** if for each i and j

$$P[X = x_i \,\&\, Y = y_j] = P[X = x_i] \cdot P[Y = y_j].$$

This corresponds to the intuitive notion that when dealing with two events involving independent properties one multiplies their probabilities.

Comment. A more careful interpretation is this. Suppose we partition Ω up according to what value X takes. Then for any value x the 'contour' of Ω defined by the function X, namely

$$[X = x] = \{\omega : X(\omega) = x\},$$

may be regarded as a population in its own right, denoted Ω_x. Then on Ω_x we may define the probability

$$p_x(\omega) = \frac{p(\omega)}{P[X = x]}.$$

We may extend the definition so that $p_x(\omega) = 0$ for ω not in Ω_x. This is known as the '**conditional probability**' (conditional on the premise $X = x$). Recall that X takes the distinct values $x_1, ...x_k$. If $x = x_i$ then under p_x the random variable Y takes the value y_j with a probability equal to

$$p_x[Y = y_j] = \sum_{\omega \in E_{ij}} p_x(\omega)$$

since

$$\{\omega : \omega \in \Omega_x \,\&\, Y(\omega) = y_j\} = \{\omega : X(\omega) = x_i \,\&\, Y(\omega) = y_j\}.$$

But

$$\begin{aligned} \sum_{\omega \in E_{ij}} p_x(\omega) &= \sum_{\omega \in E_{ij}} \frac{p(\omega)}{P[X = x]} = \frac{1}{P[X = x]} \sum_{\omega \in E_{ij}} p(\omega). \\ &= \frac{P[X = x \,\&\, Y = y_j]}{P[X = x]}. \end{aligned}$$

So

$$p_x[Y = y_j] = \frac{P[X = x \,\&\, Y = y_j]}{P[X = x]}.$$

If the number $p_x[Y = y_j]$ does not depend on the choice of contour $[X = x]$ and agrees with the value $P[Y = y_j]$ (i.e. the probability that $Y = y_j$ on the whole population), we have every reason to say that X and Y vary independently of each other over Ω.

Example 5. Multiplication Theorem. Show that if X and Y are independent then

$$E_{\mathbf{p}}[XY] = E_{\mathbf{p}}[X]E_{\mathbf{p}}[Y].$$

Solution. We have

$$\begin{aligned} E_{\mathbf{p}}[XY] &= \sum_{i,j} x_i y_j \cdot P[X = x_i \& Y = y_j], \\ &= \sum_{i,j} x_i y_j \cdot P[X = x_i] \cdot P[Y = y_j], \\ &= \sum_{i} x_i \cdot P[X = x_i] \cdot \sum_{j} y_j \cdot P[Y = y_j] \\ &= E_{\mathbf{p}}[X]E_{\mathbf{p}}[Y]. \end{aligned}$$

Any equation immediately offers a way for measuring the validity of the assumptions leading to it by a comparison of its two sides. So here, when X, Y are arbitrary, the discrepancy $E_{\mathbf{p}}[XY] - E_{\mathbf{p}}[X]E_{\mathbf{p}}[Y]$ is used to measure the amount of dependence between X and Y. In fact, as we shall see in a moment, this discrepancy can be rewritten as an inner-product as follows:

$$E_{\mathbf{p}}[(X - E_{\mathbf{p}}[X])(Y - E_{\mathbf{p}}[Y])],$$

i.e. using $\langle , \rangle_{\mathbf{p}}$. This last expression is all the more rewarding by reason of its geometric interpretation as measuring the cosine of the angle between the vectors $X' = X - E_{\mathbf{p}}[X]$ and $Y' = Y - E_{\mathbf{p}}[Y]$. When the random variables X, Y are independent, the cosine of the angle between X' and Y' is zero, and so the latter two vectors are at right angles. By contrast, when the angle between them is zero, the vectors are collinear, and so for some scalar λ

$$X - E_{\mathbf{p}}[X] = \lambda(Y - E_{\mathbf{p}}[Y]).$$

We reconsider this matter at the end of the current Section.

Definition. The formula

$$cov(X, Y) = E_{\mathbf{p}}[(X - E_{\mathbf{p}}[X])(Y - E_{\mathbf{p}}[Y])],$$

where X, Y are in $\mathbb{R}^n$ defines the **covariance** of X and Y. Thus if X, Y are independent then $cov(X, Y) = 0$. Note that $var(X) = cov(X, X)$ is the **variance** of X, namely $E_{\mathbf{p}}[(X - E_{\mathbf{p}}[X])^2]$.

Example 6. Verify that

$$E[XY] = E[X]E[Y] + cov(X, Y).$$

Solution. Clearly we have

$$\begin{aligned} E[(X - E[X])(Y - E[Y])] &= E[XY] - E[XE[Y]] - E[YE[X]] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]. \end{aligned}$$

We note that by expanding $cov(X + Y, X + Y)$ we may obtain the identity

$$var(X + Y) = var(X) + var(Y) + 2cov(X, Y).$$

(See the Learning activities.) Thus if X and Y are independent, as defined above, $cov(X, Y) = 0$ and we have

$$var(X + Y) = var(X) + var(Y).$$

We will need this result in Chapter 8.

When $p_i > 0$ for all states i, it follows from the definition that

$$cov(X, X) = 0 \text{ if and only if } X = E[X] \cdot \mathbf{1},$$

i.e. the variance of X is zero if and only if $X_i = E[X]$ for all i making X a constant.

We define the **standard deviation** of X to be

$$\sigma_X = \sqrt{cov(X, X)}.$$

Unfortunately the formula

$$||X|| = \sigma_X = \sqrt{cov(X,X)}$$

does **not** in general define a norm. But, as the remaining norm properties listed in Section 2.4 of Ostaszewski do hold, it is said to be a **pseudo-norm**.

All is not lost, however: the covariance formula does define a norm when it is applied to vectors X lying in any subspace of $\mathbb{R}^n$ that does not include $Lin\{\mathbf{1}\}$. We note that the largest dimensional such subspace has dimension $n-1$. The most obvious candidate is the set of random variables

$$\{X : E_{\mathbf{p}}[X] = 0\}.$$

This is indeed a vector subspace, indeed it is the null space (or kernel) of a linear transformation: expectation is after all a linear transformation.

Comment. There is an alternative approach to covariance which allows us to regard σ_X as a norm. We simply have to be prepared to identify (i.e. regard as equal) two vectors when their difference lies in $Lin\{\mathbf{1}\}$.

Such an identification involves a subtlety. Properly speaking, in the circumstances envisaged, if $X - Y \in Lin\{\mathbf{1}\}$, one ought to say that X and Y are not so much 'equal' as 'equivalent', written in symbols $X \equiv Y$. Reason: under this definition if $X \equiv Y$ then Y is in the affine set $X + Lin\{\mathbf{1}\}$ consisting of all vectors of the form $X + Z$ where $Z \in Lin\{\mathbf{1}\}$, i.e. a line through X parallel to the subspace/line $Lin\{\mathbf{1}\}$ (for parallel affine subspaces refer to Section 2.1).

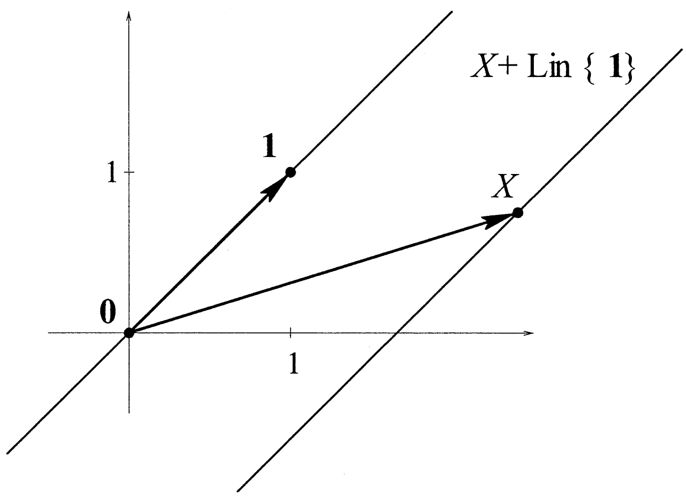

Fig. 2.5: The line through X parallel to Lin$\{\mathbf{1}\}$.

As noted above, instead of working in $\mathbb{R}^n$ we can drop down by one dimension and consider any $n-1$ dimensional subspace $\mathcal{W}$ not passing through $\mathbf{1}$. Then σ_W is a norm for $\mathcal{W}$. Evidently such a subspace $\mathcal{W}$ is a hyperplane (see end of Section 2.6). One can select for any vector X a unique vector $W(X)$ in $\mathcal{W}$ equivalent to X namely the unique intersection point of the line $X + Lin\{\mathbf{1}\}$ and $\mathcal{W}$. (See the Learning activities.) In the obvious sense $W(X)$ is the 'projection' of X onto $\mathcal{W}$ parallel to $Lin\{\mathbf{1}\}$. We devote the whole of a later chapter to this important idea.

It is helpful to consider the case $\mathcal{W} = \{X : E_{\mathbf{p}}[X] = 0\}$. Being the kernel of a linear mapping with rank one it is a subspace of dimension $n-1$. As $E_{\mathbf{p}}[\mathbf{1}] = 1$ the subspace $\mathcal{W}$ does not pass through $\mathbf{1}$. We find a formula for $W(X)$. Since $W(X)$ is in $X + Lin\{\mathbf{1}\}$, put $W(X) = X + \alpha\mathbf{1}$ with α a scalar. Now $E_{\mathbf{p}}[W(X)] = 0$, so $E_{\mathbf{p}}[X] + \alpha = 0$ i.e. $\alpha = -E_{\mathbf{p}}[X]$, and hence

$$W(X) = X - E_{\mathbf{p}}[X]\mathbf{1}.$$

In this notation $cov(X,Y) = E_{\mathbf{p}}[W(X)W(Y)]$, and so covariance measures the angle between the representatives of X and Y. When X and Y are independent their representatives are at right angles.

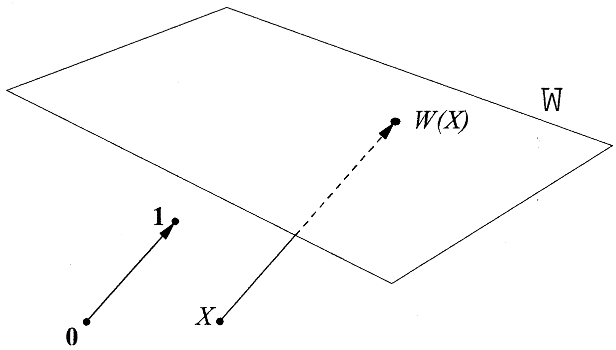

Fig. 2.6: Selecting $W(X)$ in $\mathcal{W}$ equivalent to X.

2.7 Solutions to selected activities

Solution to learning activity 2.1

$$var[\bar{X}] = \frac{1}{n^2}\sum var[X_i] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

By independence,

$$cov(X_1, \bar{X}) = \frac{1}{n}cov(X_1, X_1 + ... + X_n) = \frac{1}{n}var(X_1) = \frac{\sigma^2}{n}.$$

Since $E[X_1 - \bar{X}] = 0$ we have, using the hint,

$$\begin{aligned} E[(X_1 - \bar{X})^2] &= E[X_1 - \bar{X}]E[X_1 - \bar{X}] + var(X_1 - \bar{X}) \\ &= 0 + \sigma^2 + \frac{\sigma^2}{n} - 2\frac{\sigma^2}{n} = \frac{(n-1)\sigma^2}{n}. \end{aligned}$$

Solution to learning activity 2.7

Thus V is a combination of X and Y independent of U. We have

$$\sigma_X^2 = var(X) = \gamma^2 var(U) = \gamma^2,$$

i.e. $\gamma = \sigma_X$. Moreover,

$$\begin{aligned} \rho\sigma_X\sigma_Y &= cov(X,Y) \\ &= < \gamma U, \alpha U + \beta V > \\ &= \alpha\gamma < U, U > + \beta\gamma < U, V > \\ &= \alpha\gamma, \end{aligned}$$

i.e. $\alpha = \rho\sigma_Y$. Hence

$$\beta = \sqrt{1-\rho^2}\sigma_Y,$$

since $\sigma_Y^2 =< \alpha U + \beta V, \alpha U + \beta V >= \alpha^2 + \beta^2$.

The idea of rescaling X and replacing Y is taken straight from the Gram-Schmidt orthogonalization procedure. Indeed it is enough to view $||X|| = var(X)$ and $< X, Y >= cov(X, Y)$ as the norm and inner-product on the space of zero-mean random variables. Here we apply the fact that the span of X, Y is the same as the span of U, V.

Finally, since $X + Y = (\alpha + \gamma)U + \beta V$ we have

$$E[Z] = e^{m+\frac{1}{2}(\gamma^2+2\alpha\gamma+\alpha^2+\beta^2)} = e^{m+\frac{1}{2}(\sigma_X^2+\sigma_Y^2+2\rho\sigma_Y\sigma_Y)}.$$

A reminder of your learning outcomes

At the end of this chapter and the relevant readings, you should be able to:

- compute expectations (integrals) using 'log-normal densities'
- be prepared to use vectors to study random variables, asset prices and portfolios of assets.

Chapter 3

The financial environment: a preliminary discussion

3.1 Aims of the chapter

The purpose of this chapter is twofold. Firstly, we revise the basic ideas of interest rates and discounting within a deterministic framework. You should be aware of these ideas from earlier courses, so this is a refresher section. Next we describe informally the financial transactions for which we will later create formal mathematical models. By 'modelling' is meant something quite different from calculating or drawing mathematical inferences. The action of describing some aspects of reality in mathematical terms is in itself a separate creative act. To be able to move backwards and forwards between some mathematical calculations (resembling quite often either some linear algebra, or some calculus) and a description of financial markets is an acquired skill. We thus urge the reader to take time to understand the financial environment which is modelled by the mathematics in this course. For completeness, we give a rapid account here of the features that will be studied in the course. But the student is advised to read more plentifully. At the end of this chapter you will be able to answer a question such as this:

A leitmotif for this chapter

'A zero-coupon treasury-bond with face value £100 is a document which may be exchanged for £100 at the Bank of England after a given period of time , at which time it is said to mature. If a trader is willing to sell a bond maturing in a quarter of a year currently for £97.5, what does this imply about the trader's belief about the bank interest rate applicable for the current quarter? The trader offers two similar government bonds each with face value £100 one with a maturity of one year the other with a maturity of two years. He sells the first at a price as though he believes the interest rate applicable for the next year will be 10% per annum. What is the price? He sells the second bond as though the interest were constant at 10.5% per annum. What does he therefore believe the bank interest rate will be for the second year?'

3.2 Learning outcomes

At the end of this chapter and the relevant readings, you should be able to:

- understand what is meant by continuosly compounded interest
- compute the terminal value in a deposit account of an initial deposit and of a deposit stream
- compute the present value of a single future payment and of an income stream

- compute the expected value of a defaulting income stream.

3.3 Suitable reading for this chapter

Cvitanić, J. and F. Zapatero *Introduction to the Economics and Mathematics of Financial Markets.* Chapter 1.

Hull, J.C. *Options, Futures and other Derivatives.* Chapters 1–4.

Luenberger, D. *Investment Science.* Chapters 1, 2, 3 and 7.

3.4 Basic deterministic models

3.4.1 Savings and loans: compounding

One of the services offered by a bank is the use of a deposit account. This offers individuals the opportunity to lend money to the bank. Implicitly this is a contract, in which the bank announces that it will periodically 'add money to the account', an activity known as interest accrual. The interest period length is of course announced as part of the contract, and a simple proportionality-rule for interest accrual is applied on the assumption that no withdrawals occur in the period. To be precise: the bank specifies (i) what the interest period length is (or, equivalently, the **compounding frequency** with which in each year it adds interest to the deposit account) and (ii) a rate of interest accrual, denoted by r, measured per annum (i.e. the unit of time is a year). Thus when the interest is added once a year and the interest rate is r_{year} a sum D deposited into the account for one period grows to $D(1 + r_{\text{year}})$. It is assumed that the depositor cannot withdraw the deposit during the year.

Banks do, of course, offer various kinds of deposit contract, in the form of differently named deposit accounts. These account will thus vary both in relation to the frequency of interest payments and interest rates applied to the account. For example, the bank may offer on one account an interest rate r_{year} per annum with one interest payment per annum, and on another account an interest rate r_{semi} per annum with interest paid semi-annually. In the second case an amount D deposited for half a year grows to $D\left(1 + r_{\text{semi}}\frac{1}{2}\right)$ since the annual rate is r_{semi} and the amount of time (measured in years) is $\frac{1}{2}$. After one year the deposit therefore grows to $D\left(1 + \frac{r_{\text{semi}}}{2}\right)^2$ .Thus a depositor willing to deposit D dollars for the duration of one year will compare how he is rewarded by the bank with the two potential balances

Annual and semi-annual accruals: r_{year} and r_{semi}.

$$D\left(1 + r_{\text{year}}\right) \text{ and } D\left(1 + \frac{r_{\text{semi}}}{2}\right)^2.$$

The two rates will offer an equivalent reward if and only if

$$1 + r_{\text{year}} = 1 + r_{\text{semi}} + \frac{r_{\text{semi}}^2}{4}.$$

In general one expects the semi-annually paying account to give a lower reward, because the bank permits the depositor to withdraw from the contract after the first six months. Put another way, the bank offers a higher reward because the deposit may not be withdrawn earlier.

As a part of the contract the bank may announce for how long the rate is to be held fixed into the future, or it will identify at what points in time it will vary the interest rate.

In this subsection we will examine the case of a fixed annually compounded rate which we denote by $r = r_{\text{year}}$. This means that we are considering deposit-making activity over a time horizon in which the rate stays at r and compounding occurs just once a year. See subsection 3.4.4 of this section to see how to adapt the arguments when rates are allowed to change.

One further comment is in order. The calculations that follow assume that the bank does not fail to honour its contract. In reality some banks do fail. The discussion here has to be interpreted as referring to a safe bank, or at worst a time interval over which the bank remains safe!

Terminal deposit account value

At the end of N years the account holds

$$D(1+r)^N,$$

where $r = r_{\text{year}}$.

Let's measure time so that beginning of the first period is denoted by $t = 0$ and the end of the N-th year is denoted $t = N$. Suppose the amount D is deposited at some time $t = k$ (with $0 \le k \le N$). Then the terminal value (i.e. at time $t = N$) of the amount in the deposit account is instead

$$D(1+r)^{N-k}.$$

More generally, suppose that at the start of each of the consecutive periods numbered $t = 0, 1, 2, 3, \ldots$ an amount D_t is deposited. The last of these deposits occurs at time $t = N$, and so is made at the end of the N-th period. In this case the terminal amount in the account, i.e. at the end of N periods, is

$$D_0(1+r)^N + D_1(1+r)^{N-1} + \ldots + D_{N-1}(1+r) + D_N.$$

The t-th term for $t = 0, 1..., N$ is

$$D_t(1+r)^{N-t}.$$

Higher frequency interest payments

The period length may vary. Recall that it is normal to measure the period in units of time with one year as the unit. Similarly, to enable comparison it is normal to quote the rate not per period but per unit of time. Thus if a rate of r_{quart} per annum is quoted and the period is a quarter of the year, then the interest paid per dollar per period is $r_{\text{quart}}/4$.

Quarterly and monthly accruals: r_{quart} and r_{month}.

Thus with quarterly payments a sum D deposited into the account for t years grows to

$$D\left(1 + \frac{r_{\text{quart}}}{4}\right)^{4t},$$

since the number of periods, N, is $4t$ as there are $4t$ quarters in t years.

It is important to understand what happens to interest payments as the period is shortened, i.e. as the frequency of payments in one year is increased. If the period is a month and the rate offered is r_{month} the formula changes to

$$D\left(1+\frac{r_{\text{month}}}{12}\right)^{12t},$$

as there are 12 months in a year. Similarly if the period is a day and the rate offered is r_{day} the formula alters to

Daily accruals: r_{daily}.

$$D\left(1+\frac{r_{\text{day}}}{365}\right)^{365t}.$$

In general if there are n periods in a year let us denote the rate offered by the bank by r_n. Thus r_1 denotes r_{year} and r_2 denotes r_{semi}. The formula in the general case becomes

$$D\left(1+\frac{r_n}{n}\right)^{nt}.$$

Special cases are always easier to understand. Let us consider the case $r_n = 1$. We ask what happens to the expression

$$\left(1+\frac{1}{n}\right)^{n}, \tag{3.1}$$

as n increases without bound. This is apparently an academic question, but it has an interesting answer. We look at this case despite the fact that one's first reaction is that you would not expect a bank to give one and the same rate for a range of contracts with increasing compounding frequency. Unsurprisingly, it turns out that the expression (3.1) is increasing in n (more interest is being paid!), however, the expression remains bounded and approaches the number $e = 2.7\ldots$ So, after all, if the bank 'generously' did not alter the contractual rate, it would still not be giving much away.

Why the number e

To see how the number e arises, consider the expansion

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n} &= 1+n\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^{2}+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^{3}+\ldots\\ &= 1+1+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\end{aligned}$$

The $(k+1)$ term for $k = 1, 2...$ is

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right).$$

As n increases unboundedly this term tends to the value

$$\frac{1}{k!}.$$

It is plausible now that

$$\left(1+\frac{1}{n}\right)^{n} \to 1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots = e.$$

Now suppose that a bank considers using a fixed deposit rate ρ and contemplates different compounding frequencies n which it intends to be 'large'. That is, in the notation of the last subsection, suppose that $r_n = \rho$ without fixing n. Using the substitution $m = n/\rho$, we have

$$\begin{aligned}\left(1+\frac{\rho}{n}\right)^{nt} &= \left(1+\frac{1}{m}\right)^{m\rho t} \\ &= \left(\left(1+\frac{1}{m}\right)^{m}\right)^{\rho t} \\ &\to e^{\rho t},\end{aligned}$$

as n, (or, equivalently, as m) increases without bound.

In particular, a dollar deposited for a year would grow to nearly e^{ρ} when n is large enough. Thus the bank can select ρ to control just how much it pays its depositors in this high-frequency compounding account.

We can revisit the problem a bank faces when it considers a range of contracts with compounding frequency n and corresponding rates r_n that in fact vary with n. Consider the more realistic situation of r_n varying with n in such a way that $r_n \to r_{\text{inst}}$. Thus r_{inst} takes its place as a limiting 'instantaneous' rate at the end of our hierarchy of, yearly, monthly, and daily rates. Now it is true, with r denoting r_{inst}, that

Instantaneous accruals: r_{inst}.

$$\left(1+\frac{r_n}{n}\right)^{n} \to e^{r} \text{ if } r_n \to r. \tag{3.2}$$

Not wishing to labour the point, let's note that a proper proof relies on the observation that, for any ε and all large enough n we shall have $r - \varepsilon \leq r_n \leq r + \varepsilon$. In consequence

$$\left(1+\frac{r-\varepsilon}{n}\right)^{n} \leq \left(1+\frac{r_n}{n}\right)^{n} \leq \lim_n \left(1+\frac{r+\varepsilon}{n}\right)^{n}.$$

But, we now know that

$$e^{r+\varepsilon} = \lim_n \left(1+\frac{r+\varepsilon}{n}\right)^{n} \text{ and } e^{r-\varepsilon} = \lim_n \left(1+\frac{r-\varepsilon}{n}\right)^{n}.$$

From here it is possible to deduce the result (3.2).

3.4.2 Continuous compounding

By the last section of work, since a good approximation to a very frequent compounding of interest account employing a fixed annual rate ρ is given by the formula

$$De^{\rho t},$$

we introduce a new mathematical model for high-frequency interest compounding.

We will say that a bank account pays interest with a constant **continuous compounding** rate ρ per annum, if after t units of time (measured in years) an initial deposit of D dollars causes the balance on the account to grow to

Continuous compounding rate: ρ.

$$De^{\rho t}.$$

Note that if the deposited amount D is made at a time t with $0 < t < T$, rather than at time $t = 0$, the deposit account's terminal value will instead be

$$De^{\rho(T-t)}.$$

Comment. Even if a bank does not quote a continuous compounding rate ρ, one can always compute an equivalent continuous rate. Indeed, if the bank quotes a rate r_{year} per annum and adds interest once a year we can compute an equivalent rate ρ such that

$$D(1 + r_{\text{year}}) = De^{\rho}.$$

Solving for ρ, we get

$$\rho = \log(1 + r_{\text{year}}).$$

In conclusion, there are a number of possible interpretations of the phrase 'a rate of interest r'. Interest may be added at discrete moments so that r refers to the rate denoted by r_n in the earlier work for some n. Alternatively, the rate may be applied according to the continuous compounding model, and then r stands for the rate which we have just denoted by ρ. The former interpretation will be applied in the discrete models of Chapters 3–7. The latter will be applied in the continuous-time models of Chapters 8–10.

3.4.3 Deposit rates

The idea behind the definition of the last subsection was to model what happens in the limit as the interest-period becomes increasingly smaller, that is, as the number of periods, n, in a year increases indefinitely.

We will want to study deposit making over time rather than a single solitary deposit payment.

In a context where the number of periods in T years, namely nT, becomes huge it makes more sense to describe deposit activity as a rate of payment, just like interest payment activity. So, instead of quoting the amount deposited in dollars, we will describe the deposit as occurring at a rate of so many dollars per annum.

We assume that interest is compounded at a frequency of n and that a rate r is applied to the account (i.e in the earlier notation the rate is r_n per annum and here $r = r_n$).

Thus when the deposit rate is quoted as q dollars per annum, the amount actually deposited at the beginning of a period is in fact

$$D = q \cdot \frac{1}{n},$$

since there are n interest periods in a year.

If the deposit rate is constant throughout the time of deposit making, the amount deposited at time t will always be

$$D_t = \frac{q}{n}.$$

Although the deposited amount is increasingly small as n increases, nevertheless the total amount deposited by the end of T years is the finite amount qT.

In the presence of interest accruing in intervals of length $1/n$, the terminal value of the account is the sum

$$\sum_{j=0}^{nT} \frac{q}{n} \left(1 + \frac{r}{n}\right)^{nT-j},$$

where j moves in integer steps. Alternatively, put $j/n = t$ and with t moving in steps of size $1/n$ the sum may be re-expressed as

$$\sum_{t=0}^{T} \frac{q}{n}\left(1+\frac{r}{n}\right)^{n(T-t)}.$$

Note that with the substitution $m = n/r$ we have

$$\begin{aligned}\left(1+\frac{r}{n}\right)^{n(T-t)} &= \left(1+\frac{1}{m}\right)^{mr(T-t)} \\ &= \left(\left(1+\frac{1}{m}\right)^{m}\right)^{r(T-t)} \\ &\approx e^{r(T-t)}.\end{aligned}$$

An explanation is given in Chapter 6 why the terminal value in the deposit account tends to the limit

$$\int_0^T qe^{r(T-t)}dt$$

and this of course is very simply computed to be

$$\left[q\frac{e^{r(T-t)}}{-r}\right]_0^T = \frac{q}{r}(e^{rT}-1).$$

Whilst very plausible, it is less obvious why if the deposit rate were to depend on time, and so take the form $q(t)$, the corresponding answer for the terminal value would be

$$\int_0^T q(t)e^{r(T-t)}dt.$$

To see the difficulty, note that when $q(t)$ is not constant it is no longer obvious what sense to make of the amount deposited at the the time t at the beginning of a time interval of length $1/n$.

If $q(t)$ is a continuous function, the amount may in fact be interpreted as any number $q(s)/n$ where $t \le s \le t + \frac{1}{n}$, without affecting the limiting value of the sums. Not an obvious fact: see Chapter 6 for motivation.

One is thus at liberty to opt for $q(t)/n$ as the interpretation.

3.4.4 Variable rates

Suppose D is deposited at time $t = 0$ and that a continuously compounded rate is held fixed at r until time $t = T_1$, whereupon the rate changes to s. If no withdrawals are made, what is the amount in the deposit account at time $t = T > T_1$? The answer is computed in two steps. The first step is to note that by time $t = T_1$ the amount in the deposit account is

$$De^{rT_1}.$$

This is now regarded as being deposited for the remainder of the time, namely for a length of time $T - T_1$, during which the rate s operates. The terminal value of the deposit is thus

$$\left(De^{rT_1}\right)e^{s(T-T_1)}.$$

What is the effective average rate if $T = 2T_1$? The amount in the deposit account is now

$$De^{rT_1}e^{sT_1} = De^{(r+s)T_1} = De^{\frac{1}{2}(r+s)T}.$$

Thus, the effective rate is the average of the two rates, i.e.

$$\frac{1}{2}(r+s).$$

This is a particularly appealing feature of continuous compounding.

3.4.5 Loan accounts

We can adapt the results of the preceding subsections by reversing the role of the deposit-maker and the deposit-taker. (Recall that until now the deposit-taker was the bank, but that now changes.) The reversal of roles changes the direction of the flow of money and now it is the new deposit-taker, officially the borrower, who pays interest to the bank. The interest rate is likely to be different, ultimately reflecting the higher risk due to the fact that an individual may fail to honour his contractual commitments. (Compare this with the opening comment on safe banks.)

We may nevertheless still denote the rate here by r. In this case all of the formulas developed in earlier sections may be interpreted as movements of money in the reverse direction, on the assumption that the borrower does not fail to honour his contractual obligations.

We note that differential rates applied according to the assessed risk class of customers is but one way in which banks may make money for themselves. Inevitably, they also make money on the difference between the borrowing and the lending rates.

3.5 Present value of future income: discounting

The fundamental problem in mathematical finance is to attach a meaningful value to a contract specifying a future payment of money. The first basic message is that value changes over time and so one looks to mathematics for tools with which to describe such changes. The second message is that at any time there are various notions of value: there may be valuations that are made individually (perhaps using private information, or reflecting the individual's taste for risk) and there are valuations that are public, being the recorded price achieved in a market transaction. In this section we concentrate on the latter.

The valuation problem is complex and intellectually challenging. It is beset with a variety of difficulties all of which mathematics seeks to model. For instance, details of the payment can vary: the future bank interest rates may be unknown (affecting the value of a deposit account), the date of payment may be certain or uncertain (e.g. an insurance policy claim for payment), likewise the amount for payment, especially if the amount is stated not as a number of currency units, but in terms of a quantity of assets whose price remains uncertain. In all cases the party supplying the asset or the party required to make a payment may default on its contractual obligations.

The public valuations are arrived at on markets where these contracts are themselves the subject of trade. In principle, therefore, in any given situation one needs 'only' to look up the reported value of the nearest equivalent contract in the market. This often provides a useful basis upon which to value other contracts.

But the reader needs to be warned that this approach avoids a basic question: how do the markets themselves (i.e. the market's participants) arrive at these valuations?

To answer this question, the first point to make is that the price at which such contracts trade is a result of the way in which the market mechanism responds to the supply and demand for these contracts by the market's participants. The participants' actions, in turn, depend on what may loosely be described as the 'information' they hold. That information includes knowledge and beliefs about the factors that may influence future trades in assets, including the valuation procedures followed by other participants.

Single future payment

In this section we consider the simplest situation: the amount is specified as a known number of currency units that is certain to be supplied at a known future date. Let the amount be F and the date be T years ahead.

The valuation problem can be turned into a contract. The contract we have in mind is to offer the future payment of F to the bank in exchange for a lump sum of money L at the present moment.

The bank views the contract as a standard loan. It offers a loan L and requires it to be paid back together with interest. We model interest as being compounded continuously. It thus requires the loan-taker to pay Le^{rT} and so if this is to be paid off by the sum to be received, namely F, then

$$F = Le^{rT}.$$

We conclude that the present value of receiving F in T years' time is given by

$$L = Fe^{-rT}.$$

It should be remembered that such a statement is just a limiting form of the situation in which the bank requires interest payments at the end of each period of length $1/n$ (measured in years), so that the present value of F received at time t must equate the repayment of loan with interest equal to

$$L\left(1+\frac{r}{n}\right)^{nt} = L\left(\left(1+\frac{1}{m}\right)^{m}\right)^{rt},$$

so that

$$L = F\left(\left(1+\frac{1}{m}\right)^{m}\right)^{-rt}.$$

Comment. This reference to a bank loan is really a beginner's textbook approach towards valuation. In the real world companies and governments issue 'bonds', i.e. they write contracts promising a future sum F and sell these contracts on the bond market. If the only contractual payment offered in the bond is the payment of F at the time T, and the market participants pay a price P for the bond, one may compute a notional interest rate at which a loan-based valuation would yield the price P. This is called the yield rate, y, and we have

$$P = Fe^{-yT}.$$

Future income stream

When we discuss loans in the continuous-time framework of Chapter 8 we find that we need to use **rates** of interest payment to describe what happens in the limit as the compounding of interest occurs with progressively higher frequency. In much the same way, if we want to describe income payments in continuous time we need to refer to payment **rates**.

Thus if a payment stream is asserted to be at a constant rate p, the interpretation is that in an interval of length Δt the actual payment is $p\Delta t$. Any such payment has present value

$$p\Delta t e^{-rt},$$

and so the present value of the entire payment stream is the limiting sum of the present values of all of these contributions, namely

$$\int_0^T pe^{-rt}dt = \frac{p}{r}(1 - e^{-rT}).$$

Of course this is an idealized way of thinking about what would happen in reality. That is, this integral is the limiting outcome from calculating the more complicated-looking sum:

$$\sum_{t=0}^{T} \frac{p}{n}\left(\left(1+\frac{r}{n}\right)^n\right)^{-t} = \sum_{t=0}^{T} \frac{p}{n}\left(\left(1+\frac{1}{m}\right)^m\right)^{-rt},$$

where t moves in steps of size $1/n$ in both summations.

Learning activity 3.1

An investor owns an asset whose cash value, $V(t)$, at time t he believes will be

$$V(t) = 1 + t.$$

He also believes that for the next 20 years, the continuously-compounded interest rate will on average be $r_{\text{initial}} = 5\%$, but will thereafter average at $r_{\text{final}} = 4\%$. Explain why the present value of a sale of the asset at time t is

$$P(t) = \begin{cases} V(t)e^{-t/20}, & \text{if } t < 20, \\ V(t)e^{-1}e^{-(t-20)/25} & \text{if } t \geq 20. \end{cases}$$

When should he sell the asset? [Note that the instantaneous interest rate is either r_{initial} or r_{final}, unless $t = 20$.]
If the investor changes his mind and now believes that the higher interest rate regime will in fact last 22 years, will his plans alter? You should justify your answer.

3.6 Payments under uncertainty: the paradigm

Under uncertainty, one way to value a claim is to regard it as one of a large number of independently arising claims that all have the same probability of arising. Then it is possible to invoke the law of averages (more accurately described as the law of large numbers) which suggests that, in practice, the average payout will be close to the 'expected pay-out' of a typical claim. This is all very well if the valuer is an insurer who insures a very large number of similar claims. Indeed, even if the insurer does not have a large customer base, he may seek so-called re-insurance which is obtained by creating a large enough pool from the total market for these claims.

Note that this approach to valuation is meaningless if there is just one claim. In such a case the law of large numbers says nothing in support of the proposed valuation. Such a valuation is a gamble and the gambler can lose out.

Here is an argument about a debt repayment default which is based on the law of averages approach. Here again we will model interest rates as continuously compounded at a rate r.

Suppose a borrower of a loan, L, is required to repay the loan by repaying at a constant rate q from time $t = 0$ to perpetuity.

Thus, in the absence of default, we should have

$$L = \int_0^\infty q e^{-rt} dt = \frac{q}{r},$$

so that

$$q = Lr.$$

Assuming the borrower pays only until time $t = T$ the repayment is worth

$$\int_0^T q e^{-rt} = \frac{q}{r}(1 - e^{-rT}).$$

Now suppose that the borrower may go broke at some time T and fails to pay anything after time T.

Suppose he is more likely to go broke earlier than later, and we assume that the probability density of the failure occurring at time τ is $p(\tau) = \alpha e^{-\alpha\tau}$, where α is some constant. This is supposed to mean that failure occurs in the interval $[\tau, \tau + \Delta\tau]$ with probability $p(\tau)\Delta\tau$. Notice that

A model of default.

$$\int_0^\infty p(\tau) d\tau = \int_0^\infty \alpha e^{-\alpha\tau} d\tau = 1.$$

The bank may compute an expected repayment as follows. With probability $p(\tau)$ the bank receives payment up to time τ only, and so receives a value

$$Q(\tau) = \int_0^\tau q e^{-rt} dt = \frac{q}{r}(1 - e^{-r\tau}).$$

The repayment it is to receive is thus valued in expectation as

$$\begin{aligned}\int_0^T Q(\tau)p(\tau)d\tau &= \int_0^\infty \frac{q}{r}(1-e^{-r\tau})\alpha e^{-\alpha\tau}d\tau \\ &= \frac{q\alpha}{r}\int_0^\infty (e^{-\alpha\tau}-e^{-(r+\alpha)\tau})d\tau \\ &= \frac{q\alpha}{r}\left(\frac{1}{\alpha}-\frac{1}{r+\alpha}\right) \\ &= \frac{q}{(r+\alpha)}.\end{aligned}$$

If the expected repayment is to equal L, then the bank requires that

$$q = L(r+\alpha),$$

i.e. as though the interest rate were $r+\alpha$.

Clearly this is a higher repayment rate than the safe rate r; if the bank chooses to apply a larger value of the parameter α (known as a loading factor) when agreeing the loan, then this choice reflects the bank's belief of a greater likelihood of an early default arising on the part of the borrower. Such a choice penalizes a particular class of borrowers more heavily, so that the bank may recoup the entire repayment of all loans at an average rate r.

Discriminatory lending rates.

3.7 Contracts with the Stock Exchange

We shall be concerned with the valuation of some basic financial contracts (also referred to as financial instruments) which can be entered into at stock exchanges or by finance houses. (However, we do not get to consider such financial instruments as for instance mortgages.) All of these are contracts for the supply by one party to another of some kind of asset, that is something capable of having 'value', say in money terms (see below). So long as the terms of such a contract are honoured (i.e. the counterparty does not default), possession of such a contract may therefore itself be regarded as having value, or being equivalent to possessing a sum of money. The asset in the contract could thus be not only a commodity, or some form of money (possibly foreign), but also another contract for an agreed payment at an agreed date (see later). A quotable literary example is in Flaubert's *Madame Bovary*, where the heroine gets into debt, and her creditor sells the debt. These contracts may and do change hands, so that though the liability for the supply rests with the original party the recipient may change. Of course the party liable for the supply can balance its liability with an opposite contract (agreed perhaps even at the same time, or at a later date which is prior to that in the first contract) for delivery with another party (or even the same party) usually on different terms. This balancing procedure is known as **closing out**. Thus payments for entry into various contracts may be made but no supply need ever occur.

Entering and closing out contracts.

Why all this continual play between swings and roundabouts? In essence these contracts and counter-contracts are made as a means of 'managing risk' – forms of insurance against loss of, say, savings or investment capital, or of income. For instance, a construction firm may quote terms for a contract to build an air liquefaction plant. The contract may be agreed during the period of the quote's validity, yet the cost of parts supplied by a foreign sub-contractor may change as a

A method for risk management.

result of adverse movements in exchange rates – a factor which is outside the direct control of the main contractor. At different points in time the risks may be viewed differently and call for protective action. It is true that in the end this is a musical chairs kind of game and someone is eventually caught out and left to pay for the loss if any does indeed occur. But so long as the payments levied each time a contract transfers the risk to some particular party are enough to cover the balance between gains and losses of the insuring party, this is a business proposition. Despite obvious differences there is a family resemblance to insurance against fire: for the right premium and a large enough number of policies the insurer does make money – and if by a rare accident he does not, then he recovers by raising the contract premiums later.

Standardized contracts.

The contracts at stock exchanges are drawn up to standards decided by the stock exchange, for example, specifying the unit size of a single order (e.g. one contract for corn will be for 5000 bushels), the date of delivery and in the case of commodities the quality and location of delivery. Those issued by finance houses need not follow the same standards. Contracts entered into at exchanges contain requirements as to methods of payments which are designed to protect both parties against default. For contracts that require payments to be made at future dates rather than solely at the time of issue, the contracting parties must maintain so-called 'margin accounts' into which payments must be made at regular times up to the expiry of the contract to reflect changes in the current value of the contract. This is both to ensure that the level of possible default loss is maintained nearly constant and low, and also so that if a default occurs to one party before the termination date of the contract the damage may be limited by closing out. We do not discuss the details (which vary between instruments), but strongly advise you to read about this in e.g. Hull (Section 2.2 pp. 23–25 and Section 8.8 pp. 194–195). It is also a good idea to read up how trades through a stock exchange are organized (instructions from your broker relayed to his representative at the exchange, from there transmitted to his trader on the floor of the exchange, who will buy/sell a contract at prevailing prices, or wait for the right price). See Hull Section 2.1 pp. 18–19 for an example.

In case you do not know anything about financial instruments let us review some basic facts about them.

3.8 Some financial instruments

First of all there are money loans or deposits which earn interest either at fixed-rates or at floating rates. Fixed rates are quoted for various time intervals: short term, long terms. See Hull Section 4.1 page 76 on the London Interbank Offer Rate (LIBOR). In regard to many central 'state banks' the deposit rate for a specific time interval is considered as risk-free. The idea that there is a **risk-free rate** is key to the mathematical assessment of risk – whenever we aim to protect a risk by holding a collection (or 'portfolio' to give the official term) of contracts and the portfolio is seen to be risk-free – albeit for only a short period – its growth in value over the period will be evaluated as though its initial value was 'money' deposited at the short-term rate then in force. This begs the question: which of the various competing interest rates on the market is regarded as **the** risk-free rate of our theory? One answer is the rate which banks use between themselves:

LIBOR. Since we may be calculating the effects of contracting activities to be undertaken in the future and we do not know the future short-term interest rate, one way out is to use as a substitute the currently available long-term rate for that time interval. Alternative procedures call for a mathematical model of the future interest rates. Here conceptually one needs to refer to the market for 'loans', that is the 'bond' market. A bond sells a legally-binding promise to pay a stated amount of money at a specified future date in exchange for an amount of money to be offered immediately by the agreed buyer of the bond. This is how one may obtain a loan for the specified 'term of years', and can in principle be done competitively by reference to candidate buyers, for instance by auction. The ratio of the amount to be received later to the necessarily smaller amount being offered immediately by the buyer of the bond can be interpreted (in various ways) as arising from an interest rate for the term of the loan. Thus one talks about the **term structure** of the interest rates (i.e. how the interest rate depends on the term of the bond).

Of course these interest rates are tied to the currency of the central national bank. The central bank sets its own currency's short-term interest rate guided by the performance of the economy as a whole and its relation to other economies (crudely speaking!). Evidently the rates being offered on the bond market are one of the indications of an economy's performance; indeed the government itself issues and sells bonds. The central bank rate is reflected in the rates quoted by the principal banks of the country. In turn the central bank's rate influences the amount bond buyers are willing to pay for bonds.

An exception to this story is the **Eurodollar**, which name is applied to the stock of US dollars held outside the US (see Hull, Section 6.4 pp. 137–142).

The exchange rates between different currencies are set by the market for supply and demand of currency between countries (some of it arising from a consumption demand and some speculatively).

Deposit accounts pay specified sums at specified dates to the account holder. In this respect the bond is superficially a similar instrument. The bond originates predominantly either from governments or from a large company (corporation) and is issued to raise funds for the originator's economic activity. The bond is characterized by its **face value**, namely the stated amount in the originator's currency which will be paid at term (at the maturity date bond) when it is said to be redeemed, and **coupons** stating amounts of money to be paid to the holder of the bond at specified dates (the coupon is specified as an annual amount but is paid in two installments semi-annually). These may be zero amounts in which case one speaks of a zero-coupon bond, or discount bond. Treasury bills and Treasury notes are the names given to government bonds of one's country. (We note that T-bills have a zero coupon and maturity up to two years, T-notes have a coupon and maturity of between 2 and 10 years, T-bonds have a maturity in excess of 10 years – all this at time of issue.) The buyer of the bond may choose to value the bond by reference to its present value using appropriate bank interest rates if it is his belief that these will be the interest rates in force in the future when determining whether or not to buy a bond at the offered price. Bonds once issued and sold may be sold on by their initial buyers. We may thus observe the prices at which bonds of various maturities are being traded in the market and ask what interest rates would give a bond its current price.

Bonds.

Bond yields as implied interest rates.

Such **implied interest rates** are called **yields**. A plot of the yield against the maturity date gives rise to a yield curve – see Hull, Section 4.5 pp. 82–84.

Note that we use only continuous compounding rates. Suppose a zero-coupon bond costs p when the principle sum/face value is f. Then the gain is $g = f - p$ and the yield is y over the remaining time to maturity t is given by

$$pe^{rt} = f.$$

Hence

$$r = \frac{1}{t}\ln(1 + \frac{g}{p}).$$

For instance a zero coupon bond with face value \$100 which is currently priced at \$97.5 and which matures in 3 months (=1/4 year) has a yield of $4\ln(1 + 2.5/97.5) = 0.1012 = 10.12\%$.

The following table identifies the observed yields on bonds of maturities of 1, 2, 3, 4, 5 years. The bonds all have a face value of 100 dollars and zero coupons.

Maturity (years)	Yield to Maturity	Implied rate in each year
1	10%	10%
2	10.5%	11%
3	10.8%	11.4%
4	11%	11.6%
5	11.1%	11.5%

Thus the price of the first bond in the table (expiring in one year and with yield 10%) is

$$100e^{0.1},$$

while the second bond (with two years' maturity and yield of 10.5% over the two years) is

$$100e^{2\cdot 0.105}.$$

In the absence of any further information we conclude that if the constant interest rate of 10% is applied over the first year and a constant interest rate of r is applied for the second year, then

$$100e^{2\cdot 0.105} = 100e^{0.1}100e^{r},$$

or

$$2 \cdot 0.105 = .01 + r.$$

One says that the implied forward rate for the second year is $r = 11\%$. The yield 10.5% is thus the average of the two interest rates of 10% and 11%. This simple averaging that we have just observed is the result of employing 'continuous compounding'.

We can plot the yield rates against maturity to obtain the following plot.

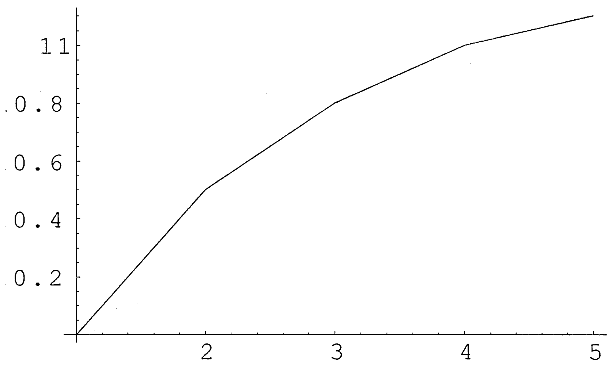

Fig 3.1: A plot of yield rates against maturity in years.

Thus the 'yield curve' obtained here is a rising one. The plot of the implied forward rates, however, is not rising.

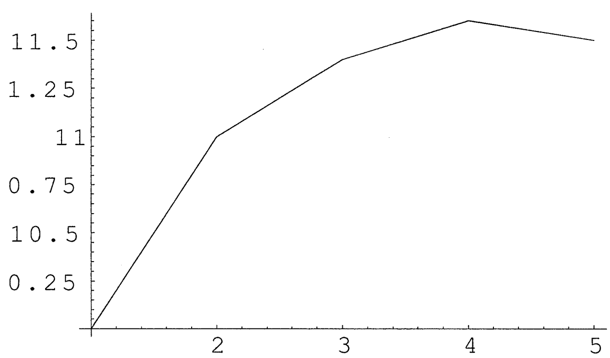

Fig 3.2: A plot of forward rates.

Learning activity 3.2

Here is a table similar to Hull, Section 4.5 page 42:

No.	Face value	Time to expiry	Annual coupon	Price of bond
1	100	1/4	0	97.5
2	100	1/2	0	94.9
3	100	1	0	90.0
4	100	1 1/2	8	96.0
5	100	2	12	101.6
6	100	2 3/4	10	99.8

Verify that the implied 3-month rate is 10.12% (p.a.),the 6-month rate is 10.47%, the 12-month rate is 10.54%. Use these rates to deduce the 18-month rate from No. 4 bond. Go on to find the 2-year rate using all the previous rates. For the final bond – recall that payments are made after 3, 9,... months up to maturity. Use a linear interpolation between $r =$ the 2 3/4 rate and the 2-year rate already established and derive a formula for the present value of the cash flow. Set this equal to the price and solve for r using Mathematica. Hint: The command

$$\text{FindRoot}[\phi[x], \{x, 0.1\}]$$

instructs Mathematica to solve $\phi(x) = 0$ for x starting with a guessed solution of $x = 10\%$. Note that throughout the question half the annual coupon is paid out semi-annual installments.

Stocks and shares.

We now consider the central notion of a 'share' and try to explain a variety of words used almost interchangeably. A **share,** or more properly, a '**share of the stock**' (i.e. share of the 'equity' capital, as defined below) of a limited company might be described jokingly as a perpetual bond with unspecified redemption value, unspecified coupon values, together with the right to participate proportionately to one's stake in the excess value of the company's assets should the company be wound up. This residual claim (i.e. to the excess of assets over liabilities, if there be any) comes with a legally-enshrined limitation as to the liabilities of the shareholders (for which see below). Seriously speaking and in a nutshell: shares give partial ownership of the company's assets and future income in return for money. Indeed when investors give money to a company they are described as having an 'equitable' interest in the company. Evidently they demand some form of 'guarantee' ('surety', or 'security') of having something for their money. The underlying notion of equitability or 'fairness' ('something in return for your money') gives rise to a number of different words which describe the paper certificates behind the contractual arrangements that exist between the investor and the firm. The words used include: share, stock, equity and security.

Asset is a very broad word indeed for all the means of value creation. You should consider this point in detail as the intricacy of the detail hides the extent to which the value of assets is subject to uncertainties. It includes not only the more obvious 'physical' assets which generate income, such as the means of production, both 'physical capital' (plant, inputs and outputs) and 'human capital', but also the associated 'architecture' of resources such as legal powers or rights (for example: patents, monopolies, ownership of 'brand names', contracts for supply). There are other so-called 'intangible' assets such as 'reputation' and 'goodwill' (a 'loyalty base' on the customer, or demand side, and on the supply side). Their creation and maintenance has very tangible costs, and these too play their part in income generation. See for example John Kay's inspirational book, *Foundations of Corporate Success.*

A share entitles the shareholder to vote at shareholders' meetings and so gives further rights, albeit in limited forms, over the choice of company strategy (for instance agreement to a merger) which include, in the extreme, the right to sack the directors. The assets owe their existence to the investment of cash in the company. Whilst this cash may be in exchange for shares, some of it can have its origin through either loans or through bond issuance. These actions create liabilities. However, a shareholder's liability is limited only to his 'stake' – that is he may lose only his investment and no more. This is the essence of the 'limited liability company', a foundation stone of 'capitalism'. Thus in the event of the company's liquidation its assets are used first to settle debts (such as bank loans) and the obligations of its bonds (prioritised by covenants or contracts) and then the excess value of assets, if there be any at all, is divided among the shareholders. (There may be prioritisation between different share issues.) In the worst case scenario the shareholders receive zero and need not concern themselves with outstanding debts. The liquidation pay-off to the shareholders graphed against asset value, as shown in Figure 3.3, thus exhibits a 'hockey-stick' shape, with the kink representing the debt level. This is the same as the graph for the pay-off from a call-options. This is the key feature behind any limitation on liabilities.

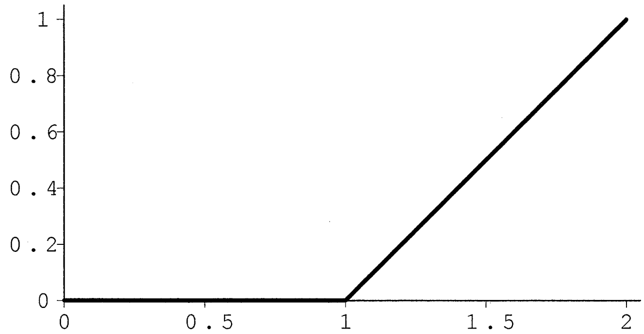

Fig. 3.3: Pay-off to shareholders against value of the firm when the debt is 1.

Debt and equity.

The liquidation procedure for dividing assets thus distinguishes between two kinds of liability: the one to the creditors (loan givers) and bond-holders which is called debt, the other to the shareholders which is called equity. The total of shares multiplied by the market price of shares is known as the market capitalization of the shares and so assesses the value of the equity. The ratio between the two forms of assets, known as leverage, may be assessed at any moment by reference to the current price on the market of the shares and bonds.

Dividends.

Turning to the brighter side of the matter: the shareholder receives payments called **dividends** which are made at specified (quarterly) dates, the amounts being determined on each occasion by the board of directors. The value of the share as assessed by its market price involves in part an estimate of the future dividend stream and also a view on the company's performance. It is therefore not surprising that the share price drops immediately after the share has 'gone ex-dividend' (the dividend has become payable). As to the view on the company's performance, this is based on various kinds of information such as market demand for its product, market supply for its inputs (including labour), as well as its management strengths. The share price is therefore influenced by the arrival of new information including government statistics and the fortunes of competitors including trading statements. Some of this information has a known arrival date and the official statistics may often be guessed from independent sources. Often information impacts on more than one company. Stock exchanges therefore compute weighted averages of the prices across various groups of companies which therefore act as indicators of the response of the market, or of relevant sectors of the market to new information. These are known as indices. See Hull, Section 3.5 page 60 for a discussion both of the weights and the composition of the best known indices.

3.9 A first look at futures and forwards

The older method of insurance against exposure to risk is to enter into a binding agreement for delivery or receipt of assets at a fixed price at a fixed future date. This takes out any uncertainty as to price. It is not necessarily an optimal procedure for risk management since on the delivery date the then current price for the asset may be either greater or less. One might say that 50% of the time one is worse off. Of course 50% of the time one is better off. Either way one has off-loaded the risk of being worse off onto another party whose job it is to manage risk. This is done at the cost of foregoing the 50% chance of being better off. There are in essence two versions of a contract for delivery in the future: a forward contract and a futures contract. Easier to understand is the forward contract since the date of delivery is specified exactly. The futures contract has a delivery date specified less narrowly – say during the course of a specified month: see Hull, Section 1.2 p.4 and Section 2.2, p.21 for details. The difference between the two contracts may appear small, and under certain assumptions their value/price is identical (Hull, Appendix 3A p.78), but one must compare them with care. It is of course the futures prices that are quoted on the exchange and in the newspapers. See Section 2.4 p.31 for an account of the published information which includes opening prices (just after the opening bell) and average closing prices – just before the closing bell (known as settlement prices – since margin accounts are settled at these prices) as well as daily highs and lows and lifetime highs and lows (prices traded so far since the initial appearance of the contract), together with the number of outstanding contracts (open interest) and estimated volume of trading.

The futures contract is said to be written on an **underlying asset** and is itself a **derivative security** of the underlying asset.

But could the risk be better managed? It will be helpful to introduce some technical terms. One talks about **taking a position** in an asset which is **long** – when contracting to buy – or **short** – when contracting to sell.(As a mnemonic note that the party in the short position in a contract will become short of whatever asset it sells off.) The pay-off to each of the two positions against the prevailing current price (**spot price**) is shown in the diagram given a delivery price K.

A long forward hedge.

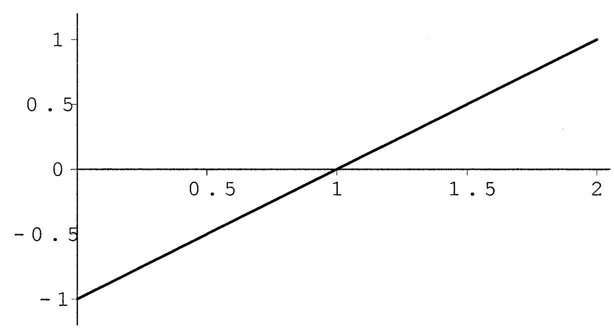

Fig. 3.4: Pay-off from a long position in a forward contract with delivery price of $K = 1$ dollar. Profitable when spot prices exceed 1 dollar.

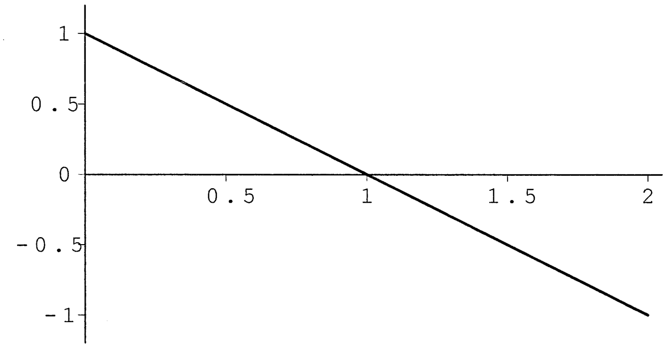

Fig. 3.5: Pay-off from a short position in a forward contract with delivery price of $K = 1$ dollar. Profitable when spot prices are below 1 dollar.

A short forward hedge.

Thus the futures contract for an asset exposed to uncertainty will evidently be in taken in the same position to the contemplated position of the company in the future.

For example, if the company is to sell an asset it will be in the short position (e.g. it may wish to sell assets which will come into its possession, or may need to sell off holdings to acquire funds). It therefore takes a short futures position (selling position) to safeguard funds to be received using a delivery price K. If the price at the delivery date is S and $S < K$ then a gain is made. See Hull, Example 3.1, Section 3.3, p.55.

Note in this example it is necessary to close out the short futures position somewhat before being in the anticipated short position, by picking a prior maturity date as close as possible to the original anticipated selling date. The acquisition of the short futures contract is called a **short hedge**.

Use of the word hedge indicates in the first instance an intention to act so as to protect against loss of value. The 'hedges' of ordinary day life refer to bushes; but these were planted around the perimeters (or edge) of fields to protect the crops in the field from wind. We will often find that holders of a portfolio of assets will want to supplement their portfolio with additional assets; often these additional assets are 'derivatives' – that is, contracts with pay-offs that are derived from the value of some simple asset such as a stock or bond.

3.10 Optimal hedges

The futures hedges considered above are only 'perfect hedges' if the future course of history turns out just as predicted. But, one can do better than hedge with a futures contract equal in size and position to the hedger's own liability. In the first place we can look for 'optimal hedges': measure the level of risk by variance and then minimize.

Suppose we know, or can estimate, the correlation coefficient between the change in the spot price of the asset and the change in the futures price. If we hold a portfolio Π comprising a quantity h of short futures contracts each quoting a price F and an asset of value S, then change in value of the portfolio is

$$\Delta\Pi = \Delta S - h\Delta F,$$

where for instance $\Delta S = S_2 - S_1$ is the gain in value of the asset between time 1 (now) and time 2 (the future). Note the negative sign against the futures contract signifying the short position. The underlying asset is thought of as currently in the possession of the hedger (as though bought) and therefore in a long position. Of course if the positions on the asset and the futures contract are reversed, the signs are also reversed on both quantities. In either case, however, the variance is

$$\begin{aligned} var(\Delta\Pi) &= cov(\Delta\Pi, \Delta\Pi) = \langle \Delta\Pi, \Delta\Pi \rangle \\ &= var(\Delta S) + h^2 var(\Delta F) - 2h cov(\Delta S, \Delta F) \end{aligned}$$

and minimizing this U-shaped quadratic polynomial in h over h yields (from the first-order condition of calculus)

Optimal hedge ratio.

$$h = \frac{cov(\Delta S, \Delta F)}{var(\Delta F)} = \frac{\rho\sigma_S\sigma_F}{\sigma_F^2} = \frac{\rho\sigma_S}{\sigma_F}.$$

Indeed, by definition

$$var(\Delta S) = \sigma_S^2, \qquad var(\Delta F) = \sigma_F^2, \qquad cov(\Delta S, \Delta F) = \rho\sigma_S\sigma_F.$$

The formula gives an optimal hedge ratio.

Learning activity 3.3

A company intends to hedge a forthcoming acquisition of 1 million gallons of jet fuel in 3 months' time. Over this period it believes that $\sigma_{jet} = 0.032$. The company considers using a futures contract on heating oil, for which in the same period it believes that $\sigma_{oil} = 0.04$. If $\rho_{jet,oil} = 0.8$, how many futures contracts should be bought if the stock-exchange unit of futures contract is 42,000 gallons? What sort (short/long) hedge position is required? [Compare with Hull, Example 3.3, Section 3.4, page 59].

3.11 The 'duration'-based hedge: a first-order hedge

Let us see another hedge calculation (Hull, Section 4.8 p.89) in regard to bonds. If the yield – calculated on a constant continuous compounding rate basis - is y, then by definition the bond price is related to the coupon payments and the face value by the formula

$$B = \sum_1^n c_i e^{-yt_i}. \qquad (3.3)$$

Here the coupon payments are c_i and are made at the dates t_i (and this includes as c_n the face vale of the bond). Let us calculate the amount by which the bond price will change if the yield were to change by a small amount Δy. We begin by computing that

$$\begin{aligned} \frac{dB}{dy} &= -\sum_1^n c_i t_i e^{-yt_i} \\ &= -B\sum_1^n t_i \frac{c_i e^{-yt_i}}{B}. \end{aligned}$$

Since the coefficients of the terms t_i in the second line are positive and by (3.3) sum to unity, they may be interpreted as weights applied to the times of coupon payments, thereby giving rise to an average time of payment defined by

$$D_B = \sum_1^n t_i \frac{c_i e^{-yt_i}}{B}.$$

This quantity is known as the **duration** of the bond. Thus we have approximately:

$$\Delta B = \frac{dB}{dy}\Delta y = -BD_B\Delta y.$$

See Hull, Section 4.6 p.101 for a calculation of duration.

Suppose a portfolio Π consists of bond currently priced at S and a quantity h units of a futures contract for delivery of bonds (other similar interest rate dependent asset) with a current ('spot') delivery price F of the bond. Then the change in the portfolio value consequent on the yield y changing by ΔY may be computed approximately by reference to duration to be

$$\begin{aligned} \Delta\Pi &= \Delta S - h\Delta F \\ &= SD_S\Delta y - hFD_F\Delta y \\ &= (SD_S - hFD_F)\Delta y. \end{aligned}$$

So taking

Duration-based hedge ratio.

$$h = \frac{SD_S}{FD_F},$$

where this ratio is known as the 'price sensitivity ratio' or 'duration-based ratio', we obtain a hedge which has, to first-order approximation, a zero change in value. One says that the hedge has a 'delta of zero' or is 'delta neutral'. Evidently the validity of this approach is dependent on the behaviour of $\Delta B/B$. The error in the duration approximation, given by Taylor's Theorem is governed by

Second order hedging.

$$\frac{d^2B}{d^2y},$$

so a better hedge may be obtained by constructing a wider portfolio which will enable off-setting control over this further quantity. Portfolio managers talk about the convexity measure of the portfolio; see e.g. Hull, Section 4.9 p.92 (a similar second-order idea will be discussed later under the name **gamma,** Hull, Section 15.6 p.355). The remark to be made here is that the delta neutrality has been obtained by balancing the effects of the same unknown change Δy through its effects on two distinct though related quantities.

For example see Hull, Section 4.8 p.89.

Before we give our next hedging example (the beta hedge) we need to study the forward contract.

3.12 Arbitrage arguments

We can identify the forward price in a forward contract by reference to what is known as an arbitrage argument. Originally the term arbitrage was used to describe the activity of buying a commodity at one price in one market and selling at a higher price at another market.

Setting the forward price by arbitrage argument

Our argument will be based on the current/spot price of the asset assuming a constant risk-free rate in the period between current time t and the delivery date T. We claim the forward price is

$$F = Se^{r(T-t)},$$

where S is the spot price. Thus

$$S = Fe^{-r(T-t)}.$$

In other words the 'present-value' of F agrees with the spot price (i.e. the price as at present). The forward price is the current price compounded up to what is called (by analogy with present value) its 'future value' at maturity.

Indeed, assume a price is offered of $F > Se^{r(T-t)}$. A risk-free opportunity to make money – known as an arbitrage opportunity – occurs as follows. Borrow S dollars, buy the asset for S dollars with the intention of selling the asset at the delivery date T for F dollars (this is called taking a short future position); the loan can be repaid from the proceeds of the sale since the debt amounts to only $Se^{r(T-t)}$. The positive excess $F - Se^{r(T-t)}$ is a sure profit.

If the inequality is reversed (i.e. $F < Se^{r(T-t)}$), we try to make a profit equal to the formula $Se^{r(T-t)} - F$. We can do this, by borrowing a unit of the asset at time t from someone who does not need it, and sell this asset for S. (This action is called shorting the asset, since we sell something we were short of!) Depositing S dollars until date T then yields a cash balance of $Se^{r(T-t)}$ in the deposit account, and this equals the first term in our hoped-for profit formula. We can also take a long position in a forward contract (i.e. offer to buy the asset) enabling us to buy back the asset for F dollars at time T. At maturity accept delivery of the asset paying F out of the invested monies. So paying F dollars out of the $Se^{r(T-t)}$ allows us to obtain a unit of the asset in order to settle the asset loan. (This action is called 'closing out the short position' – by returning the assets.) We are left with a positive excess profit of $Se^{r(T-t)} - F$.

For examples see Hull, Section 3.1 p.47.

Learning activity 3.4

Suppose that the underlying asset in the forward contract pays a known dividend of D dollars at time $t = 1$. Suppose that D is expressed relative to the initial time $t = 0$ asset price as $D = qS$, then q is called the 'dividend rate'. Verify by arbitrage argument that the forward price F agreed at time $t = 0$ is $S(1 + r - q)$.

Learning activity 3.5

Suppose that the underlying asset gives its holder a dividend paid in the asset (e.g. more shares if the asset is a share, or more foreign currency, if the asset represents a foreign currency deposit) and that at each moment of time t the dividend rate is a (known) constant q. This is taken to mean that over the time interval $[t, t+\Delta t]$ the dividend expressed relative to the current (spot) asset price S_t of the asset is worth $S_t e^{q\Delta t}$. Verify by arbitrage argument that the forward price F is $Se^{(r-q)T}$. How does this generalize the result of the preceding activity?

3.13 Forward contracts written on an index

Consider a futures contract on an index I which reflects the market – thought of as a grand portfolio. Since the index is a hypothetical portfolio, let us write $I = \sum_i w_i S_i$, where w_i are weights and S_i are the asset prices for the assets comprising the index. Thus the forward price F_I for delivery of the hypothetical portfolio at time T is

$$F_I = \sum_i w_i S_i e^{r(T-t)} = Ie^{r(T-t)}.$$

(This formula assumes that none of the assets yields a dividend in the interim.) We need to clarify here that the index itself is a valuation of a hypothetical portfolio of assets by reference to their spot prices. Consequently, the future delivery envisaged in the contract ought in theory to be that of the very same hypothetical portfolio of assets that defines the index; however, the practice is that the future is settled day by day in the money equivalent of the value of the hypothetical portfolio.

Now observe that to first order

$$\begin{aligned}\Delta F &= F(t+\Delta t) - F(t)\\ &= \frac{\partial F}{\partial I}\Delta I + \frac{\partial F}{\partial t}\Delta t\\ &= e^{r(T-t)}\Delta I + Ie^{r(T-t)}(-r)\Delta t,\end{aligned}$$

and so

$$\frac{\Delta F}{F} = \frac{\Delta I}{I} - r\Delta t.$$

Thus for small time changes it is approximately true that that the percentage growth in forward price is equal to the percentage growth in the market:

$$\frac{\Delta F}{F} = \frac{\Delta I}{I}.$$

But this is slightly misleading, in that the instantaneous indicators are connected as follows:

$$\frac{1}{F}\cdot\frac{dF}{dt} = \frac{1}{I}\cdot\frac{dI}{dt} - r.$$

Note that $\Delta I = \sum w_i \Delta S_i$ only if we assumes no changes have been made in the relative weights in the composition of the index. These weights are of course selected so as to reflect the total market capitalization of the share – along the lines of

$$w_i = \frac{N_i S_i}{\sum_j N_i S_j}.$$

Thus the weights may be subject to alteration in situations when an asset has a dramatic change in value. See Hull, Section 3.5, p.60.

3.14 Beta hedging: a perfect hedge

The next example of hedging is motivated by a theoretical result. The theory seeks to explain what is called the 'excess return' of an asset, or more generally a portfolio of assets, that is, the difference between the return of the asset and the riskless rate. The theory relates this excess to the 'market portfolio', that is, the quantities of all assets held by the totality of market agents. In an appropriately defined one-period model, known as the Capital Asset Pricing Model, or just 'CAPM' (pronounced 'cap-em'), it may be shown that if the market is in equilibrium, then the excess-return of any asset (or portfolio) is proportional to the excess return of the 'market portfolio'. The relevant coefficient of proportionality is called the β of the asset. This coefficient may be interpreted as measuring risks associated with the asset.

Thus, the beta β of a portfolio measures the ratio of the excess return on the portfolio over the risk-free rate on the one hand, to the excess return of the market as a whole over the risk-free rate on the other. It may be assessed using regression techniques from statistics.

Now suppose we want to protect a portfolio currently worth Π against excessive movements in stock prices over some not unreasonably long period of time. Let Δt denote the length of the time period. We regard the behaviour of the markets over this one period as being modelled by the CAPM.

Thus we have for any portfolio that

$$\text{return} - r\ \Delta t = \frac{\Delta\Pi}{\Pi} - r\Delta t = \beta(\frac{\Delta M}{M} - r\Delta t). \tag{3.4}$$

Here $\Delta\Pi$ is the change in the value of the portfolio at the end of the period and M represents the market regarded as a grand portfolio. The risk-free rate per unit time is r, so the return on risklessly-deposited dollars over the period is $r\Delta t$. Evidently $\frac{\Delta M}{M}$ is the change in value of 1 dollar invested in the market for the period of time Δt (since M dollars changes in value by ΔM). In practice we replace M in this relation by a suitable index, so we come to regard M as though it were the hypothetical portfolio defining the index. Now we may rewrite the equation (3.4) defining beta, thus:

$$\frac{\Delta\Pi}{\Pi} = \alpha + \beta\frac{\Delta M}{M},$$

where, for convenience, α is introduced to denote

$$\alpha = r\Delta t(1-\beta).$$

Consider hedging the original portfolio of assets worth Π with a beta of β, by taking a short position of an amount h units of a future contract written on a sufficiently wide stock-index currently offering the future delivery at the end of the period Δt for a price F. We think of the future contract on the index as a proxy for the market, so we must have approximately

$$\frac{\Delta F}{F} = \frac{\Delta M}{M} - r\Delta t.$$

Hence the change in the value of the 'hedged portfolio' comprising the original portfolio and the future (or to use the jargon: its delta) is

$$\begin{aligned} & \Delta\Pi - h\Delta F \\ = \quad & (\alpha + \beta\frac{\Delta M}{M})\Pi - hF(\frac{\Delta M}{M} - r\Delta t) \\ = \quad & \alpha\Pi + (\beta\Pi - hF)\frac{\Delta M}{M} + rhF\Delta t. \end{aligned}$$

So taking the hedge ratio h to be

The beta hedge ratio.

$$h = \frac{\beta\Pi}{F},$$

we have a portfolio which is approximately independent of the uncertain change ΔM in the stock index. It should thus enjoy a return approximating to the risk-free rate, and indeed our calculations agree with this, since

$$\alpha\Pi + rhF\Delta t = r\Delta t(1 - \beta) + r\beta\Pi\Delta t = r\Delta t.$$

Learning activity 3.6

A portfolio with $\beta = 1.5$ is currently worth \$2.1M. The holder is fearful of market movements in the next quarter but does not want to sell off the portfolio in order to avoid transaction costs. It is proposed to hedge with futures of maturity 4 months. The index future stands at 300 and the unit contract is 500 times the index; in this case a single contract costs \$300×500. (The index is to be imagined as the price we have to pay to buy a unit of asset, and the contract is for 500 such units.) It is of course proposed that the futures contract be closed out after 3 months. What position should be taken and how many contracts should be bought?

3.15 Efficient market hypothesis

A famous assumption states that the operation of the markets is efficient, which is supposed to mean that the current share price reflects all the publicly available information. See Campbell, Lo and MacKinlay p. 22 for a careful explanation of several ways that this assumption may be formalized (which includes consideration also of private information). As a consequence, if the market is 'in equilibrium', market participants cannot know in advance (i.e. with certainty) that the price of a share is to rise (gain in value) by some amount tomorrow; for otherwise current demand (to take possession of the gain) would drive the price today immediately upwards by that same amount: the universal anticipation of change would lead to one-directional trading. Similarly, a known and sure tendency for a price to decrease would trigger a strong tendency to sell at the current

Various senses of efficiency.

better price; a continued selling volume would immediately bring the price down, perhaps dramatically, with the bulk of the sale playing presumably into the hands of players speculating on a comeback. Thus we see a self-fulfilling prophecy unfolding. (Such dramatic events can indeed take place, as they did on Black Monday October 19, 1987 – see Hull, Section 5.9 page 112 and see also Hull Chapter 32 in particular Section 32.2, page 736. However, that was an exceptional day, whereas our concern is 'with the rule' rather than the exception. But exceptions, though they might be held rare, do occur.) This lack of certainty in price direction signifies that we are only one step away from the language of probability: the expected price in the next instant is the same as the current price (after a due discounting adjustment to make money of different time-moments comparable). Gamblers viewing the game of speculating on the price of the share will say the game is a fair one. The fairness assumption is of course open to empirical verification. See 'Capital Ideas' for a riveting account of early empirical observations on this by Alfred Cowles in the 1920's (see Bernstein's *Capital ideas* p.30). To sum up: the findings were that all 'self-avowed expert opinion' on future price movements were in the long run no better than the toss of a coin. In consequence, the view that one cannot beat the market led to the growth of portfolio management with the intention of at least following the market by reference to measurements of price behaviour in the market as a whole, that is by way of indices. This is reminiscent of the establishment of the principle of conservation of energy in physics after periods of searching for the 'perpetuum mobile' - a perpetually working machine requiring no power.

Investment as a 'fair' game.

The economist Paul Samuelson, writing in 1965 (see 'Proof that properly anticipated prices fluctuate randomly', *Industrial Management Review,* 6(2) 1965, pp.41–49, sought to explain the 'fair game' phenomenon by reference to the behaviour of the 'futures price' $F_{T,t}$ viewing it as a random variable (more precisely random process). The futures price, as explained in Section 3.9, is the price contracted at date t for the delivery of a unit of a specified asset at date T. Suppose that the price of the asset at time t is modelled as having a probability of the following form.

$$\Pr[S_t \leq x | S_0, \ldots, S_{t-1}] = P(x, t; S_0, \ldots, S_{t-1})$$

and that the futures price is arrived by way of expectation as

$$F_{T,t} = E_P[S_T | S_0, \ldots, S_t] \text{ for } t = 0, 1, \ldots, T.$$

(Note that this includes the case $t = T$, for which $F_{T,T} = S_T$.) By assumption, the next period's future price will depend on past asset prices and on the next period's asset price, that is

$$F_{T,t+1} = E_P[S_T | S_0, \ldots, S_t, S_{t+1}].$$

Today's expectation of tomorrow's forward price, denoted

$$E_P[F_{T,t+1} | S_0, ..., S_t]$$

is thus equal to

$$E_P[E_P[S_T | S_0, \ldots, S_{t+1}] | S_0, ..., S_t].$$

By the Law of Iterated Expectation (see Activity 6.7) this equals

$$E_P[S_T | S_0, \ldots, S_t],$$

and, by assumption, this is just $F_{T,t}$. Thus we have

$$F_{T,t} = E_P[F_{T,t+1}].$$

Technicalities apart, the process $\{F_{T,t} : t = 1, 2, ..., T\}$ is said to be a **martingale** – meaning that any time in the ongoing process the expected value at the next instant is the currently observed value. We will discover, that the 'discounted' share is similarly a **martingale** – although under a different probability law, but derived from P.

Bachelier's model.

This brings us to the question of how to model the behaviour of share prices. The first mathematical attempt at the turn of the twentieth century (*Theory of Speculation*, 1900) by Louis Bachelier was unappreciated by contemporaries and went into apparent oblivion for more than half a century, partly due to lack of empirical research and partly due to the absence of appropriate computing power (see *Capital Ideas*, p.18). Bachelier's ideas thus waited for their rediscovery. See *Louis Bachelier's Theory of Speculation: the origin of modern finance.*

A reminder of your learning outcomes

At the end of this chapter and the relevant readings, you should be able to:

- understand what is meant by continuosly compounded interest
- compute the terminal value in a deposit account of an initial deposit and of a deposit stream
- compute the present value of a single future payment and of an income stream
- compute the expected value of a defaulting income stream.