## complexfunctions




Just as $z$ can be expressed by its real and imaginary parts, $z=x+i y$, we write $f(z)=w=u+i v$, where $u$ and $v$ are the real and imaginary parts of $w$, respectively. Doing so gives us the representation
$w=f(z)=f(x, y)=f(x+i y)=u+i v$.
Because $u$ and $v$ depend on $x$ and $y$, they can be considered to be real-valued functions of the real variables $x$ and $y$; that is,
$u=u(x, y)$ and $v=v(x, y)$.
Combining these ideas, we often write a complex function $f$ in the form
$f(z)=f(x+i y)=u(x, y)+i v(x, y)$.
Figure 2.1 illustrates the notion of a function (mapping) using these symbols.
EXAMPLE 2.1 Write $f(z)=z^{4}$ in the form $f(z)=u(x, y)+i v(x, y)$.
Solution Using the binomial formula, we obtain

$$
\begin{aligned}
f(z) & =(x+i y)^{4}=x^{4}+4 x^{3} i y+6 x^{2}(i y)^{2}+4 x(i y)^{3}+(i y)^{4} \\
\text { rnin } & =\left(x^{4}-6 x^{2} y^{2}+y^{4}\right)+i\left(4 x^{3} y-4 x y^{3}\right),
\end{aligned}
$$

so that $u(x, y)=x^{4}-6 x^{2} y^{2}+y^{4}$ and $v(x, y)=4 x^{3} y-4 x y^{3}$.

EXAMPLE 2.2 Express the function $f(z)=\bar{z} \operatorname{Re}(z)+z^{2}+\operatorname{Im}(z)$ in the form $f(z)=u(x, y)+i v(x, y)$.
Solution Using the elementary properties of complex numbers, it follows that $f(z)=(x-i y) x+\left(x^{2}-y^{2}+i 2 x y\right)+y=\left(2 x^{2}-y^{2}+y\right)+i(x y)$, so that $u(x, y)=2 x^{2}-y^{2}+y$ and $v(x, y)=x y$.

Examples 2.1 and 2.2 show how to find $u(x, y)$ and $v(x, y)$ when a rule for computing $f$ is given. Conversely, if $u(x, y)$ and $v(x, y)$ are two real-valued functions of the real variables $x$ and $y$, they determine a complex-valued function $f(x, y)=u(x, y)+i v(x, y)$, and we can use the formulas
$x=\frac{z+\bar{z}}{2} \quad$ and $\quad y=\frac{z-\bar{z}}{2 i}$
to find a formula for $f$ involving the variables $z$ and $\bar{z}$.
EXAMPLE 2.3 Express $f(z)=4 x^{2}+i 4 y^{2}$ by a formula involving the variables $z$ and $\bar{z}$.

Solution Calculation reveals that

$$
\begin{aligned}
f(z) & =4\left(\frac{z+\bar{z}}{2}\right)^{2}+i 4\left(\frac{z-\bar{z}}{2 i}\right)^{2} \\
& =z^{2}+2 z \bar{z}+\bar{z}^{2}-i\left(z^{2}-2 z \bar{z}+\bar{z}^{2}\right) \\
& =(1-i) z^{2}+(2+2 i) z \bar{z}+(1-i) \bar{z}^{2} .
\end{aligned}
$$

Using $z=r e^{i \theta}$ in the expression of a complex function $f$ may be convenient. It gives us the polar representation
$f(z)=f\left(r e^{i \theta}\right)=u(r, \theta)+i v(r, \theta)$,
where $u$ and $v$ are real functions of the real variables $r$ and $\theta$.
Remark 2.1 For a given function $f$, the functions $u$ and $v$ defined here are different from those defined by Equation (2-1) because Equation (2-1) involves Cartesian coordinates and Equation (2-2) involves polar coordinates.

EXAMPLE 2.4 Express $f(z)=z^{2}$ in both Cartesian and polar form.
Solution For the Cartesian form, a simple calculation gives
$f(z)=f(x+i y)=(x+i y)^{2}=\left(x^{2}-y^{2}\right)+i(2 x y)=u(x, y)+i v(x, y)$
so that
$u(x, y)=x^{2}-y^{2}$, and $v(x, y)=2 x y$.
For the polar form, we refer to Equation (1-39) to get
$f\left(r e^{i \theta}\right)=\left(r e^{i \theta}\right)^{2}=r^{2} e^{i 2 \theta}=r^{2} \cos 2 \theta+i r^{2} \sin 2 \theta=U(r, \theta)+i V(r, \theta)$,
so that
$U(r, \theta)=r^{2} \cos 2 \theta, \quad$ and $\quad V(r, \theta)=r^{2} \sin 2 \theta$.

Once we have defined $u$ and $v$ for a function $f$ in Cartesian form, we must use different symbols if we want to express $f$ in polar form. As is clear here, the functions $u$ and $U$ are quite different, as are $v$ and $V$. Of course, if we are working only in one context, we can use any symbols we choose.

EXAMPLE 2.5 Express $f(z)=z^{5}+4 z^{2}-6$ in polar form.
Solution Again, using Equation (1-39) we obtain

$$
\begin{aligned}
f(z) & =f\left(r e^{i \theta}\right)=r^{5}(\cos 5 \theta+i \sin 5 \theta)+4 r^{2}(\cos 2 \theta+i \sin 2 \theta)-6 \\
\text { rnin } & =\left(r^{5} \cos 5 \theta+4 r^{2} \cos 2 \theta-6\right)+i\left(r^{5} \sin 5 \theta+4 r^{2} \sin 2 \theta\right) \\
\text { STR } & =u(r, \theta)+i v(r, \theta) .
\end{aligned}
$$

We now look at the geometric interpretation of a complex function. If $D$ is the domain of real-valued functions $u(x, y)$ and $v(x, y)$, the equations
$u=u(x, y) \quad$ and $\quad v=v(x, y)$
describe a transformation (or mapping) from $D$ in the $x y$ plane into the $u v$ plane, also called the $w$ plane. Therefore, we can also consider the function
$w=f(z)=u(x, y)+i v(x, y)$
to be a transformation (or mapping) from the set $D$ in the $z$ plane onto the range $R$ in the $w$ plane. This idea was illustrated in Figure 2.1. In the following paragraphs we present some additional key ideas. They are staples for any kind of function, and you should memorize all the terms in bold.

If $A$ is a subset of the domain $D$ of $f$, the set $B=\{f(z): z \in A\}$ is called the image of the set $A$, and $f$ is said to map $A$ onto $B$. The image of a single point is a single point, and the image of the entire domain, $D$, is the range, $R$. The mapping $w=f(z)$ is said to be from $A$ into $S$ if the image of $A$ is contained in $S$. Mathematicians use the notation $f: A \rightarrow S$ to indicate that a function maps $A$ into $S$.

Figure 2.2 illustrates a function $f$ whose domain is $D$ and whose range is $R$. The shaded areas depict that the function maps $A$ onto $B$. The function also maps $A$ into $R$, and, of course, it maps $D$ onto $R$.

The inverse image of a point $w$ is the set of all points $z$ in $D$ such that $w=f(z)$. The inverse image of a point may be one point, several points, or nothing at all. If the last case occurs then the point $w$ is not in the range of $f$. For example, if $w=f(z)=i z$, the inverse image of the point -1 is the single point $i$, because $f(i)=i(i)=-1$, and $i$ is the only point that maps to -1 . In the case of $w=f(z)=z^{2}$, the inverse image of the point -1 is the set $\{i,-i\}$.


Figure $2.2 \quad f$ maps $A$ onto $B ; f$ maps $A$ into $R$.

You will learn in Chapter 5 that if $w=f(z)=e^{z}$, the inverse image of the point 0 is the empty set-there is no complex number $z$ such that $e^{z}=0$.

The inverse image of a set of points, $S$, is the collection of all points in the domain that map into $S$. If $f$ maps $D$ onto $R$, it is possible for the inverse image of $R$ to be a function as well, but the original function must have a special property: A function $f$ is said to be one-to-one if it maps distinct points $z_{1} \neq z_{2}$ onto distinct points $f\left(z_{1}\right) \neq f\left(z_{2}\right)$. Many times an easy way to prove that a function $f$ is one-to-one is to suppose $f\left(z_{1}\right)=f\left(z_{2}\right)$, and from this assumption deduce that $z_{1}$ must equal $z_{2}$. Thus, $f(z)=i z$ is one-to-one because if $f\left(z_{1}\right)=f\left(z_{2}\right)$, then $i z_{1}=i z_{2}$. Dividing both sides of the last equation by $i$ gives $z_{1}=z_{2}$. Figure 2.3 illustrates the idea of a one-to-one function: Distinct points get mapped to distinct points.

The function $f(z)=z^{2}$ is not one-to-one because - $i \neq i$, but $f(i)=$ $f(-i)=-1$. Figure 2.4 depicts this situation: At least two different points get mapped to the same point.

In the exercises we ask you to demonstrate that one-to-one functions give rise to inverses that are functions. Loosely speaking, if $w=f(z)$ maps the set $A$ one-to-one and onto the set $B$, then for each $w$ in $B$ there exists exactly one point $z$ in $A$ such that $w=f(z)$. For any such value of $z$ we can take the




Figure 2.4 The function $f(z)=z^{2}$ is not one-to-one.
equation $w=f(z)$ and "solve" for $z$ as a function of $w$. Doing so produces an inverse function $z=g(w)$ where the following equations hold:
$g(f(z))=z \quad$ for all $z \in A, \quad$ and
$f(g(w))=w \quad$ for all $w \in B$.
Conversely, if $w=f(z)$ and $z=g(w)$ are functions that map $A$ into $B$ and $B$ into $A$, respectively, and Equations (2-3) hold, then $f$ maps the set $A$ one-to-one and onto the set $B$.

Further, if $f$ is a one-to-one mapping from $D$ onto $T$ and if $A$ is a subset of $D$, then $f$ is a one-to-one mapping from $A$ onto its image $B$. We can also show that if $\zeta=f(z)$ is a one-to-one mapping from $A$ onto $B$ and $w=g(\zeta)$ is a one-to-one mapping from $B$ onto $S$, then the composite mapping $w=g(f(z))$ is a one-to-one mapping from $A$ onto $S$.

We usually indicate the inverse of $f$ by the symbol $f^{-1}$. If the domains of $f$ and $f^{-1}$ are $A$ and $B$, respectively, we can rewrite Equations (2-3) as
$f^{-1}(f(z))=z \quad$ for all $z \in A, \quad$ and $f\left(f^{-1}(w)\right)=w \quad$ for all $w \in B$.

Also, for $z_{0} \in A$ and $w_{0} \in B$, $w_{0}=f\left(z_{0}\right) \quad$ iff $f^{-1}\left(w_{0}\right)=z_{0}$.

EXAMPLE 2.6 If $w=f(z)=i z$ for any complex number $z$, find $f^{-1}(w)$.
Solution We can easily show $f$ is one-to-one and onto the entire complex plane. We solve for $z$, given $w=f(z)=i z$, to get $z=\frac{w}{i}=-i w$. By Equations (2-5), this result implies that $f^{-1}(w)=-i w$ for all complex numbers $w$.

Remark 2.2 Once we have specified $f^{-1}(w)=-i w$ for all complex numbers $w$, we note that there is nothing magical about the symbol $w$. We could just as easily write $f^{-1}(z)=-i z$ for all complex numbers $z$.

We now show how to find the image $B$ of a specified set $A$ under a given mapping $u+i v=w=f(z)$. The set $A$ is usually described with an equation or inequality involving $x$ and $y$. Using inverse functions, we can construct a chain of equivalent statements leading to a description of the set $B$ in terms of an equation or an inequality involving $u$ and $v$.

EXAMPLE 2.7 Show that the function $f(z)=i z$ maps the line $y=x+1$ in the $x y$ plane onto the line $v=-u-1$ in the $w$ plane.

Solution (Method 1): With $A=\{(x, y): y=x+1\}$, we want to describe $B=f(A)$. We let $z=x+i y \in A$ and use Equations (2-5) and Example 2.6 to get

$$
\begin{aligned}
u+i v=w=f(z) \in B & \Longleftrightarrow f^{-1}(w)=z=x+i y \in A \\
& \Longleftrightarrow-i w \in A \text { STRIBUTION FORSAL } \\
& \Longleftrightarrow v-i u \in A \\
& \Longleftrightarrow(v,-u) \in A \\
& \Longleftrightarrow-u=v+1 \\
& \Longleftrightarrow v=-u-1, \text { LLC }
\end{aligned}
$$

where $\Longleftrightarrow$ means if and only if (iff).
Note what this result says: $u+i v=w \in B \Longleftrightarrow v=-u-1$. The image of $A$ under $f$, therefore, is the set $B=\{(u, v): v=-u-1\}$.
(Method 2): We write $u+i v=w=f(z)=i(x+i y)=-y+i x$ and note that the transformation can be given by the equations $u=-y$ and $v=x$. Because $A$ is described by $A=\{x+i y: y=x+1\}$, we can substitute $u=-y$ and $v=x$ into the equation $y=x+1$ to obtain $-u=v+1$, which we can rewrite as $v=-u-1$. If you use this method, be sure to pay careful attention to domains and ranges.

We now look at some elementary mappings. If we let $B=a+i b$ denote a fixed complex constant, the transformation
$w=T(z)=z+B=x+a+i(y+b)$
is a one-to-one mapping of the $z$ plane onto the $w$ plane and is called a translation. This transformation can be visualized as a rigid translation whereby the point $z$ is displaced through the vector $B=a+i b$ to its new position $w=T(z)$. The inverse mapping is given by
$z=T^{-1}(w)=w-B=u-a+i(v-b)$
and shows that $T$ is a one-to-one mapping from the $z$ plane onto the $w$ plane. The effect of a translation is depicted in Figure 2.5.


Figure 2.5 The translation $w=T(z)=z+B=x+a+i(y+b)$.



Figure 2.6 The rotation $w=R(z)=r e^{i(\theta+\alpha)}$.

If we let $\alpha$ be a fixed real number, then for $z=r e^{i \theta}$, the transformation
$w=R(z)=z e^{i \alpha}=r e^{i \theta} e^{i \alpha}=r e^{i(\theta+\alpha)}$
is a one-to-one mapping of the $z$ plane onto the $w$ plane and is called a rotation. It can be visualized as a rigid rotation whereby the point $z$ is rotated about the origin through an angle $\alpha$ to its new position $w=R(z)$. If we use polar coordinates and designate $w=\rho^{i \phi}$ in the $w$ plane, then the inverse mapping is $z=R^{-1}(w)=w e^{-i \alpha}=\rho e^{i \phi} e^{-i \alpha}=\rho e^{i(\phi-\alpha)}$.

This analysis shows that $R$ is a one-to-one mapping of the $z$ plane onto the $w$ plane. The effect of rotation is depicted in Figure 2.6.

EXAMPLE 2.8 The ellipse centered at the origin with a horizontal major axis of four units and vertical minor axis of two units can be represented by the parametric equation
$s(t)=2 \cos t+i \sin t=(2 \cos t, \sin t), \quad$ for $0 \leq t \leq 2 \pi$.


Figure 2.7 (a) Plot of the original ellipse; (b) plot of the rotated ellipse.

Suppose that we wanted to rotate the ellipse by an angle of $\frac{\pi}{6}$ radians and shift the center of the ellipse 2 units to the right and 1 unit up. Using complex arithmetic, we can easily generate a parametric equation $r(t)$ that does so:

$$
\begin{aligned}
r(t) & =s(t) e^{i \frac{\pi}{6}}+(2+i) \\
& =(2 \cos t+i \sin t)\left(\cos \frac{\pi}{6}+i \sin \frac{\pi}{6}\right)+(2+i) \\
& =\left(2 \cos t \cos \frac{\pi}{6}-\sin t \sin \frac{\pi}{6}\right)+i\left(2 \cos t \sin \frac{\pi}{6}+\sin t \cos \frac{\pi}{6}\right)+(2+i) \\
& =\left(\sqrt{3} \cos t-\frac{1}{2} \sin t+2\right)+i\left(\cos t+\frac{\sqrt{3}}{2} \sin t+1\right) \\
\text { BA } & =\left(\sqrt{3} \cos t-\frac{1}{2} \sin t+2, \cos t+\frac{\sqrt{3}}{2} \sin t+1\right), \text { for } 0 \leq t \leq 2 \pi . \text { artle }
\end{aligned}
$$

Figure 2.7 shows parametric plots of these ellipses, using the software program Maple.

If we let $K>0$ be a fixed positive real number, then the transformation $w=S(z)=K z=K x+i K y$
is a one-to-one mapping of the $z$ plane onto the $w$ plane and is called a magnification. If $K>1$, it has the effect of stretching the distance between points by the factor $K$. If $K<1$, then it reduces the distance between points by the factor $K$. The inverse transformation is given by
$z=S^{-1}(w)=\frac{1}{K} w=\frac{1}{K} u+i \frac{1}{K} v$
and shows that $S$ is one-to-one mapping from the $z$ plane onto the $w$ plane. The effect of magnification is shown in Figure 2.8.


Figure 2.8 The magnification $w=S(z)=K z=K x+i K y$.

Finally, if we let $A=K e^{i \alpha}$ and $B=a+i b$, where $K>0$ is a positive real number, then the transformation
$w=L(z)=A z+B$
is a one-to-one mapping of the $z$ plane onto the $w$ plane and is called a linear transformation. It can be considered as the composition of a rotation, a magnification, and a translation. It has the effect of rotating the plane through an angle given by $\alpha=\operatorname{Arg} A$, followed by a magnification by the factor $K=|A|$, followed by a translation by the vector $B=a+i b$. The inverse mapping is given by $z=L^{-1}(w)=\frac{1}{A} w-\frac{B}{A}$ and shows that $L$ is a one-to-one mapping from the $z$ plane onto the $w$ plane.

EXAMPLE 2.9 Show that the linear transformation $w=i z+i$ maps the right half-plane $\operatorname{Re}(z) \geq 1$ onto the upper half-plane $\operatorname{Im}(w) \geq 2$.

Solution (Method 1): Let $A=\{(x, y): x \geq 1\}$. To describe $B=f(A)$, we solve $w=i z+i$ for $z$ to get $z=\frac{w-i}{i}=-i w-1=f^{-1}(w)$. Using Equations (2-5) and the method of Example 2.7 we have

$$
\begin{aligned}
u+i v=w=f(z) \in B & \Longleftrightarrow f^{-1}(w)=z \in A \\
& \Longleftrightarrow-i w-1 \in A L L \\
\text { NOTFORSALE } & \Longleftrightarrow v-1-i u \in A \\
& \Longleftrightarrow(v-1,-u) \in A \\
& \Longleftrightarrow v-1 \geq 1 \\
& \Longleftrightarrow v \geq 2 .
\end{aligned}
$$

Thus, $B=\{w=u+i v: v \geq 2\}$, which is the same as saying $\operatorname{Im}(w) \geq 2$.
(Method 2): When we write $w=f(z)$ in Cartesian form as $w=u+i v=i(x+i y)+i=-y+i(x+1)$,
we see that the transformation can be given by the equations $u=-y$ and $v=x+1$. Substituting $x=v-1$ in the inequality $\operatorname{Re}(z)=x \geq 1$ gives $v-1 \geq 1$, or $v \geq 2$, which is the upper half-plane $\operatorname{Im}(w) \geq 2$.
(Method 3): The effect of the transformation $w=f(z)$ is a rotation of the plane through the angle $\alpha=\frac{\pi}{2}$ (when $z$ is multiplied by $i$ ) followed by a translation by the vector $B=i$. The first operation yields the set $\operatorname{Im}(w) \geq 1$. The second shifts this set up 1 unit, resulting in the set $\operatorname{Im}(w) \geq 2$.

We illustrate this result in Figure 2.9.



Figure 2.9 The linear transformation $w=f(z)=i z+i$.

Translations and rotations preserve angles. First, magnifications rescale distance by a factor $K$, so it follows that triangles are mapped onto similar triangles, preserving angles. Then, because a linear transformation can be considered to be a composition of a rotation, a magnification, and a translation, it follows that linear transformations preserve angles. Consequently, any geometric object is mapped onto an object that is similar to the original object; hence linear transformations can be called similarity mappings.

EXAMPLE 2.10 Show that the image of $D_{1}(-1-i)=\{z:|z+1+i|<1\}$ under the transformation $w=(3-4 i) z+6+2 i$ is the open disk $D_{5}(-1+3 i)=$ $\{w:|w+1-3 i|<5\}$.

Solution The inverse transformation is $z=\frac{w-6-2 i}{3-4 i}$, so if we designate the range of $f$ as $B$, then

$$
\begin{aligned}
w=f(z) \in B & \Longleftrightarrow f^{-1}(w)=z \in D_{1}(-1-i) \\
& \Longleftrightarrow \frac{w-6-2 i}{3-4 i} \in D_{1}(-1-i) \\
\text { SALE OR DIST } & \left.\Longleftrightarrow \frac{w-6-2 i}{3-4 i}+1+i \right\rvert\,<1 \\
& \Longleftrightarrow\left|\frac{w-6-2 i}{3-4 i}+1+i\right||3-4 i|<1 \cdot|3-4 i| \\
& \Longleftrightarrow|w-6-2 i+(1+i)(3-4 i)|<5 \\
& \Longleftrightarrow|w+1-3 i|<5 .
\end{aligned}
$$

Hence the disk with center $-1-i$ and radius 1 is mapped one-to-one and onto the disk with center $-1+3 i$ and radius 5 as shown in Figure 2.10.


Figure 2.10 The mapping $w=S(z)=(3-4 i) z+6+2 i$.

EXAMPLE 2.11 Show that the image of the right half-plane $\operatorname{Re}(z) \geq 1$ under the linear transformation $w=(-1+i) z-2+3 i$ is the half-plane $v \geq u+7$.

Solution The inverse transformation is given by
$z=\frac{w+2-3 i}{N_{-1}+i R}=\frac{u+2+i(v-3)}{\text { AL- }-1+i D I S}$,
which we write as
$x+i y=\frac{-u+v-5}{2}+i \frac{-u-v+1}{2}$.
Substituting $x=\frac{(-u+v-5)}{2}$ into $\operatorname{Re}(z)=x \geq 1$ gives $\frac{(-u+v-5)}{2} \geq 1$, which simplifies to $v \geq u+7$. Figure 2.11 illustrates the mapping.


Figure 2.11 The mapping $w=f(z)=(-1+i) z-2+3 i$.

## --------EXERCISES FOR SECTION 2.1

1. Find $f(1+i)$ for the following functions.
(a) $f(z)=z+z^{-2}+5$.
(b) $f(z)=\frac{1}{z^{2}+1}$.
(c) $f(z)=f(x+i y)=x+y+i\left(x^{3} y-y^{2}\right)$.
(d) $f(z)=z^{2}+4 z \bar{z}-5 \operatorname{Re}(z)+\operatorname{Im}(z)$.
2. Let $f(z)=z^{21}-5 z^{7}+9 z^{4}$. Use polar coordinates to find
(a) $f(-1+i)$.
(b) $f(1+i \sqrt{3})$.
3. Express the following functions in the form $u(x, y)+i v(x, y)$.
(a) $f(z)=z^{3}$.
(b) $f(z)=\bar{z}^{2}+(2-3 i) z$.
(c) $f(z)=\frac{1}{z^{2}}$.
4. Express the following functions in the polar coordinate form $u(r, \theta)+i v(r, \theta)$.
(a) $f(z)=z^{5}+\bar{z}^{5}$.
(b) $f(z)=z^{5}+\bar{z}^{3}$.
(c) For what values of $z$ are the above expressions valid? Why?
5. Let $f(z)=f(x+i y)=e^{x} \cos y+i e^{x} \sin y$. Find
(a) $f(0)$.
(b) $f(i \pi)$.
(c) $f\left(i \frac{2 \pi}{3}\right)$.
(d) $f(2+i \pi)$.
(e) $f(3 \pi i)$.
(f) Is $f$ a one-to-one function? Why or why not?
6. For $z \neq 0$, let $f(z)=f(x+i y)=\frac{1}{2} \ln \left(x^{2}+y^{2}\right)+i \arctan \frac{y}{x}$. Find
(a) $f(1)$.
(b) $f(\sqrt{3}+i)$.
(c) $f(1+i \sqrt{3})$.
(d) $f(3+4 i)$.
(e) Is $f$ a one-to-one function? Why or why not?
7. For $z \neq 0$, let $f(z)=\ln r+i \theta$, where $r=|z|$, and $\theta=\operatorname{Arg} z$. Find
(a) $f(1)$.
(b) $f(-2)$.
(c) $f(1+i)$.
(d) $f(-\sqrt{3}+i)$.
(e) Is $f$ a one-to-one function? Why or why not?
8. Suppose that $f$ maps $A$ into $B, g$ maps $B$ into $A$, and that Equations (2-3) hold.
(a) Show that $f$ is one-to-one.
(b) Show that $f$ maps $A$ onto $B$.
9. Suppose $f$ is a one-to-one mapping from $D$ onto $T$ and that $A$ is a subset of $D$.
(a) Show that $f$ is one-to-one from $A$ onto $B$, where $B=\{f(z): z \in A\}$.
(b) Show, additionally, that if $g$ is one-to-one from $B$ onto $S$, then $h(z)$ is one-to-one from $A$ onto $S$, where $h(z)=g(f(z))$.
10. Let $w=f(z)=(3+4 i) z-2+i$.
(a) Find the image of the disk $|z-1|<1$.
(b) Find the image of the line $x=t, y=1-2 t \quad$ for $-\infty<t<\infty$.
(c) Find the image of the half-plane $\operatorname{Im}(z)>1$.
(d) For parts (a) and (b), and (c), sketch the mapping. Identify three points of your choice and their corresponding images.
11. Let $w=(2+i) z-2 i$. Find the triangle onto which the triangle with vertices $z_{1}=-2+i, z_{2}=-2+2 i$, and $z_{3}=2+i$ is mapped.
12. Let $S(z)=K z$, where $K>0$ is a positive real constant. Show that the equation $\left|S\left(z_{1}\right)-S\left(z_{2}\right)\right|=K\left|z_{1}-z_{2}\right|$ holds and interpret this result geometrically.
13. Find the linear transformations $w=f(z)$ that satisfy the following conditions.
(a) The points $z_{1}=2$ and $z_{2}=-3 i$ map onto $w_{1}=1+i$ and $w_{2}=1$.
(b) The circle $|z|=1$ maps onto the circle $|w-3+2 i|=5$, and $f(-i)=$ $3+3 i$.
(c) The triangle with vertices $-4+2 i,-4+7 i$, and $1+2 i$ maps onto the triangle with vertices 1,0 , and $1+i$, respectively.
14. Give a proof that the image of a circle under a linear transformation is a circle. Hint: Let the circle have the parametrization $z=z_{0}+\operatorname{Re}^{i t}, 0 \leq t \leq 2 \pi$.
15. Prove that the composition of two linear transformations is a linear transformation.
16. Show that a linear transformation that maps the circle $\left|z-z_{0}\right|=R_{1}$ onto the circle $\left|w-w_{0}\right|=R_{2}$ can be expressed in the form $A\left(w-w_{0}\right) R_{1}=\left(z-z_{0}\right) R_{2}, \quad$ where $|A|=1$.

### 2.2 THE MAPPINGS $w=z^{n}$ and $w=z^{\frac{1}{n}}$

In this section we turn our attention to power functions.
For $z=r e^{i \theta} \neq 0$, we can express the function $w=f(z)=z^{2}$ in polar coordinates as
$w=f(z)=z^{2}=r^{2} e^{i 2 \theta}$.
If we also use polar coordinates for $w=\rho e^{i \phi}$ in the $w$ plane, we can express this mapping by the system of equations
$\rho=r^{2} \quad$ and $\quad \phi=2 \theta$.
Because an argument of the product $(z)(z)$ is twice an argument of $z$, we say that $f$ doubles angles at the origin. Points that lie on the ray $r>0, \theta=\alpha$ are mapped onto points that lie on the ray $\rho>0, \phi=2 \alpha$. If we now restrict the domain of $w=f(z)=z^{2}$ to the region

$$
\begin{equation*}
A=\left\{r e^{i \theta}: r>0 \quad \text { and } \quad \frac{-\pi}{2}<\theta \leq \frac{\pi}{2}\right\} \tag{2-6}
\end{equation*}
$$

then the image of $A$ under the mapping $w=z^{2}$ can be described by the set
$B=\left\{\rho e^{i \phi}: \rho>0 \quad\right.$ and $\left.\quad-\pi<\phi \leq \pi\right\}$,
which consists of all points in the $w$ plane except the point $w=0$.

The inverse mapping of $f$, which we denote $g$, is then $z=g(w)=w^{\frac{1}{2}}=\rho^{\frac{1}{2}} e^{i \frac{\phi}{2}}$,
where $w \in B$. That is,
$z=g(w)=w^{\frac{1}{2}}=|w|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(w)}{2}}$,
where $w \neq 0$. The function $g$ is so important that we call special attention to it with a formal definition.

Definition 2.1: Principal square root
The function
$g(w)=w^{\frac{1}{2}}=|w|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(w)}{2}}, \quad$ for $w \neq 0$,
is called the principal square root function.

We leave as an exercise to show that $f$ and $g$ satisfy Equations (2-3) and thus are inverses of each other that map the set $A$ one-to-one and onto the set $B$ and the set $B$ one-to-one and onto the set $A$, respectively. Figure 2.12 illustrates this relationship.

What are the images of rectangles under the mapping $w=z^{2}$ ? To find out, we use the Cartesian form
$w=u+i v=f(z)=z^{2}=x^{2}-y^{2}+i 2 x y=\left(x^{2}-y^{2}, 2 x y\right)=(u, v)$
and the resulting system of equations
$u=x^{2}-y^{2} \quad$ and $\quad v=2 x y$.


Figure 2.12 The mappings $w=z^{2}$ and $z=w^{\frac{1}{2}}$.

EXAMPLE 2.12 Show that the transformation $w=f(z)=z^{2}$, for $z \neq 0$, usually maps vertical and horizontal lines onto parabolas and use this fact to find the image of the rectangle $\{(x, y): 0<x<a, 0<y<b\}$.
Solution Using Equations (2-9), we determine that the vertical line $x=a$ is mapped onto the set of points given by the equations $u=a^{2}-y^{2}$ and $v=2 a y$. If $a \neq 0$, then $y=\frac{v}{2 a}$ and
$u=a^{2}-\frac{v^{2}}{4 a^{2}}$.
Equation (2-10) represents a parabola with vertex at $a^{2}$, oriented horizontally, and opening to the left. If $a>0$, the set $\left\{(u, v): u=a^{2}-y^{2}, v=2 a y\right\}$ has $v>0$ precisely when $y>0$, so the part of the line $x=a$ lying above the $x$-axis is mapped to the top half of the parabola.

The horizontal line $y=b$ is mapped onto the parabola given by the equations $u=x^{2}-b^{2}$ and $v=2 x b$. If $b \neq 0$, then as before we get
$u=-b^{2}+\frac{v^{2}}{4 b^{2}}$.
Equation (2-11) represents a parabola with vertex at $-b^{2}$, oriented horizontally and opening to the right. If $b>0$, the part of the line $y=b$ to the right of the $y$-axis is mapped to the top half of the parabola because the set $\left\{(u, v): u=x^{2}-b^{2}, v=2 b x\right\}$ has $v>0$ precisely when $x>0$.

Quadrant I is mapped onto quadrants I and II by $w=z^{2}$, so the rectangle $0<x<a, 0<y<b$ is mapped onto the region bounded by the top halves of the parabolas given by Equations $(2-10)$ and $(2-11)$ and the $u$-axis. The vertices $0, a, a+i b$, and $i b$ of the rectangle are mapped onto the four points $0, a^{2}$, $a^{2}-b^{2}+i 2 a b$, and $-b^{2}$, respectively, as indicated in Figure 2.13.

Finally, we can easily verify that the vertical line $x=0, y \neq 0$ is mapped to the set $\left\{\left(-y^{2}, 0\right): y \neq 0\right\}$. This is simply the set of negative real numbers. Similarly, the horizontal line $y=0, x \neq 0$ is mapped to the set $\left\{\left(x^{2}, 0\right): x \neq 0\right\}$, which is the set of positive real numbers.

What happens to images of regions under the mapping
$w=f(z)=|z|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(z)}{2}}=r^{\frac{1}{2}} e^{i \frac{\theta}{2}}$ for $z=r e^{i \theta} \neq 0$,
where $-\pi<\theta \leq \pi$ ? If we use polar coordinates for $w=\rho e^{i \phi}$ in the $w$ plane, we can represent this mapping by the system
$\rho=r^{\frac{1}{2}}$ and $\phi=\frac{\theta}{2}$.
Equations (2-12) indicate that the argument of $f(z)$ is half the argument of $z$ and that the modulus of $f(z)$ is the square root of the modulus of $z$. Points


Figure 2.13 The transformation $w=z^{2}$.
OR SALE OR DISTRIBUTION NOT FOR SALE OR DISTRIBUTION


Figure 2.14 The mapping $w=z^{\frac{1}{2}}$.
that lie on the ray $r>0, \theta=\alpha$ are mapped onto the ray $\rho>0, \phi=\frac{\alpha}{2}$. The image of the $z$ plane (with the point $z=0$ deleted) consists of the right halfplane $\operatorname{Re}(w)>0$ together with the positive $v$-axis. The mapping is shown in Figure 2.14.

We can use knowledge of the inverse mapping $z=w^{2}$ to get further insight into how the mapping $w=z^{\frac{1}{2}}$ acts on rectangles. If we let $z=x+i y \neq 0$, then $z=w^{2} \equiv u^{2}-v^{2}+i 2 u v$,
and we note that the point $z=x+i y$ in the $z$ plane is related to the point $w=u+i v=z^{\frac{1}{2}}$ in the $w$ plane by the system of equations
$x=u^{2}-v^{2}$ and $y=2 u v$.

EXAMPLE 2.13 Show that the transformation $w=f(z)=z^{\frac{1}{2}}$ usually maps vertical and horizontal lines onto portions of hyperbolas.

Solution Let $a>0$. Equations (2-13) map the right half-plane given by $\operatorname{Re}(z)>a$ (i.e., $x>a$ ) onto the region in the right half-plane satisfying $u^{2}-v^{2}>$ $a$ and lying to the right of the hyperbola $u^{2}-v^{2}=a$. If $b>0$, Equations (2-13) map the upper half-plane $\operatorname{Im}(z)>b$ (i.e., $y>b$ ) onto the region in quadrant I satisfying $2 u v>b$ and lying above the hyperbola $2 u v=b$. This situation is illustrated in Figure 2.15. We leave as an exercise the investigation of what happens when $a=0$ or $b=0$.


Figure 2.15 The mapping $w=z^{\frac{1}{2}}$.

We can easily extend what we've done to integer powers greater than 2 . We begin by letting $n$ be a positive integer, considering the function $w=f(z)=z^{n}$, for $z=r e^{i \theta} \neq 0$, and then expressing it in the polar coordinate form
$w=f(z)=z^{n}=r^{n} e^{i n \theta}$.
If we use polar coordinates $w=\rho e^{i \phi}$ in the $w$ plane, the mapping defined by Equation (2-14) can be given by the system of equations
$\rho=r^{n} \quad$ and $\quad \phi=n \theta$.
The image of the ray $r>0, \theta=\alpha$ is the ray $\rho>0, \phi=n \alpha$, and the angles at the origin are increased by the factor $n$. The functions $\cos n \theta$ and $\sin n \theta$ are periodic with period $\frac{2 \pi}{n}$, so $f$ is in general an $n$-to-one function; that is, $n$ points in the $z$ plane are mapped onto each nonzero point in the $w$ plane.

If we now restrict the domain of $w=f(z)=z^{n}$ to the region
$E=\left\{r e^{i \theta}: r>0\right.$ and $\left.\frac{-\pi}{n}<\theta \leq \frac{\pi}{n}\right\}$,
then the image of $E$ under the mapping $w=z^{n}$ can be described by the set
$F=\left\{\rho e^{i \phi}: \rho>0 \quad\right.$ and $\left.\quad-\pi<\phi \leq \pi\right\}$,
which consists of all points in the $w$ plane except the point $w=0$. The inverse mapping of $f$, which we denote $g$, is then
$z=g(w)=w^{\frac{1}{n}}=\rho^{\frac{1}{n}} e^{i \frac{\phi}{n}}$,
where $w \in F$. That is,
$z=g(w)=w^{\frac{1}{n}}=|w|^{\frac{1}{n}} e^{i \frac{\operatorname{Arg}(w)}{n}}$,
where $w \neq 0$. As with the principal square root function, we make an analogous definition for $n$th roots.

Definition 2.2: Principal $n$th root The function
$g(w)=w^{\frac{1}{n}}=|w|^{\frac{1}{n}} e^{i \frac{\operatorname{Arg}(w)}{n}}, \quad$ for $w \neq 0$,
is called the principal $n$th root function.

We leave as an exercise to show that $f$ and $g$ are inverses of each other that map the set $E$ one-to-one and onto the set $F$ and the set $F$ one-to-one and onto the set $E$, respectively. Figure 2.16 illustrates this relationship.


Figure 2.16 e The mappings $w=z^{n}$ and $z=w^{\frac{1}{n}}$.

## The Quadratic Formula

We are now able to present a familiar result. It's proof, which is left as an exercise, depends on the ideas of this section, and Section 1.5.

DTheorem 2.1 (The Quadratic Formula) The solutions to the equation $a z^{2}+b z+c=0$ are OR DISTRIBUTION

$$
z=\frac{-b+\sqrt{b^{2}-4 a c}}{2 a} \text { and } z=\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}
$$

where the principal square root, Equation (2-8), is used in each case.

EXAMPLE 2.14 Find all solutions to the equation $z^{2}+z+i z+5 i=0$.
Solution First, rewrite the equation as $z^{2}+(1+i) z+5 i=0$. The quadratic formula then gives

$$
z=\frac{-(1+i) \pm \sqrt{(1+i)^{2}-4(1)(5 i)}}{2(1)}=\frac{-(1+i) \pm \sqrt{-18 i}}{2} .
$$

Now, $\operatorname{Arg}(-18 i)=-\frac{\pi}{2}$, and $|-18 i|=18$, so by Theorem 2.1 and Equation (2-8) the solutions are
$z=\frac{-(1+i) \pm 18^{\frac{1}{2}} e^{-i \frac{\pi}{4}}}{2}=\frac{-(1+i) \pm 3 \sqrt{2} e^{-i \frac{\pi}{4}}}{2}=\frac{-(1+i) \pm 3 \sqrt{2}\left(\frac{\sqrt{2}}{2}-i \frac{\sqrt{2}}{2}\right)}{2}$.
Simplifying the last expression gives $z=1-2 i$ and $z=-2+i$.
(b) $3 z^{2}-10 z+3$ (useful for Exercise 6, Section 8.2).
(c) $z^{2}+2 z+5=0$ (useful for Exercise 4a, Section 12.3).
(d) $2 z^{2}+2 z+1=0$ (useful for Exercise 5a, Section 12.3).
4. Prove Theorem 2.1, the quadratic formula.
5. Use your knowledge of the principal square root function to explain the fallacy in the following logic: $1=\sqrt{(-1)(-1)}=\sqrt{(-1)} \sqrt{(-1)}=(i)(i)=-1$.
6. Show that the functions $f(z)=z^{2}$ and $g(w)=w^{\frac{1}{2}}=\left\lvert\, w^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(w)}{2}}\right.$ with domains given by Equations (2-6) and (2-7), respectively, satisfy Equations (2-3) of Section 2.1. Thus, $f$ and $g$ are inverses of each other that map the shaded regions in Figure 2.14 one-to-one and onto each other.
7. Sketch the set of points satisfying the following relations.
(a) $\operatorname{Re}\left(z^{2}\right)>4$.
(b) $\operatorname{Im}\left(z^{2}\right)>6$.
8. Find and illustrate the images of the following sets under the mapping $w=z^{\frac{1}{2}}$.
(a) $\left\{r e^{i \theta}: r>1\right.$ and $\left.\frac{\pi}{3}<\theta<\frac{\pi}{2}\right\}$.
(b) $\left\{r e^{i \theta}: 1<r<9\right.$ and $\left.0<\theta<\frac{2 \pi}{3}\right\}$.
(c) $\left\{r e^{i \theta}: r<4\right.$ and $\left.-\pi<\theta<\frac{\pi}{2}\right\}$.
(d) The vertical line $\{(x, y): x=4\}$.
(e) The infinite strip $\{(x, y): 2<y<6\}$.
(f) The region to the right of the parabola $x=4-\frac{y^{2}}{16}$.

Hint: Use the inverse mapping $z=w^{2}$ to show that the answer is the right half-plane $\operatorname{Re}(w)>2$.
9. Find the image of the right half-plane $\operatorname{Re}(z)>1$ under the mapping $w=z^{2}+2 z+1$.
10. Find the image of the following sets under the mapping $w=z^{3}$.
(a) $\left\{r e^{i \theta}: 1<r<2\right.$ and $\left.\frac{\pi}{4}<\theta<\frac{\pi}{3}\right\}$.
(b) $\left\{r e^{i \theta}: r>3\right.$ and $\left.\frac{2 \pi}{3}<\theta<\frac{3 \pi}{4}\right\}$.
11. Find the image of $\left\{r e^{i \theta}: r>2\right.$, and $\left.\frac{\pi}{4}<\theta<\frac{\pi}{3}\right\}$ under the following mappings.
(a) $w=z^{3}$.
(b) $w=z^{4}$.
(c) $w=z^{6}$.
12. Find the image of the sector $r>0,-\pi<\theta<\frac{2 \pi}{3}$ under the following mappings.
(a) $w=z^{\frac{1}{2}}$.
(b) $w=z^{\frac{1}{3}}$.
(c) $w=z^{\frac{1}{4}}$.
13. Show what happens when $a=0$ and $b=0$ in Example 2.13
14. Establish the result referred to in the comment after Definition 2.2.

### 2.3 LIMITS AND CONTINUITY

Let $u=u(x, y)$ be a real-valued function of the two real variables $x$ and $y$. Recall that $u$ has the limit $u_{0}$ as $(x, y)$ approaches $\left(x_{0}, y_{0}\right)$ provided the value of $u(x, y)$ can be made to get as close as we want to the value $u_{0}$ by taking $(x, y)$ to be sufficiently close to $\left(x_{0}, y_{0}\right)$. When this happens we write
$\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} u(x, y)=u_{0}$.
In more technical language, $u$ has the limit $u_{0}$ as $(x, y)$ approaches $\left(x_{0}, y_{0}\right)$ iff $\left|u(x, y)-u_{0}\right|$ can be made arbitrarily small by making both $\left|x-x_{0}\right|$ and $\left|y-y_{0}\right|$ small. This condition is like the definition of a limit for functions of one variable. The point $(x, y)$ is in the $x y$ plane, and the distance between $(x, y)$ and $\left(x_{0}, y_{0}\right)$ is $\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}$. With this perspective we can now give a precise definition of a limit.

Definition 2.3: Limit of $u(x, y)$
The expression $\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} u(x, y)=u_{0}$ means that for each number $\varepsilon>0$, there is a corresponding number $\delta>0$ such that

$$
\begin{equation*}
\left|u(x, y)-u_{0}\right|<\varepsilon \quad \text { whenever } 0<\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}<\delta . \tag{2-15}
\end{equation*}
$$

EXAMPLE 2.15 Show, if $u(x, y)=\frac{2 x^{3}}{\left(x^{2}+y^{2}\right)}$, then $\lim _{(x, y) \rightarrow(0,0)} u(x, y)=0$.
Solution If $x=r \cos \theta$ and $y=r \sin \theta$, then
$u(x, y)=\frac{2 r^{3} \cos ^{3} \theta}{r^{2} \cos ^{2} \theta+r^{2} \sin ^{2} \theta}=2 r \cos ^{3} \theta$.
Because $\sqrt{(x-0)^{2}+(y-0)^{2}}=r$ and because $\left|\cos ^{3} \theta\right|<1$, $|u(x, y)-0|=2 r\left|\cos ^{3} \theta\right|<\varepsilon \quad$ whenever $\quad 0<\sqrt{x^{2}+y^{2}}=r<\frac{\varepsilon}{2}$.

Hence, for any $\varepsilon>0$, Inequality (2-15) is satisfied for $\delta=\frac{\varepsilon}{2}$; that is, $u(x, y)$ has the limit $u_{0}=0$ as $(x, y)$ approaches $(0,0)$.

The value $u_{0}$ of the limit must not depend on how $(x, y)$ approaches $\left(x_{0}, y_{0}\right)$, so $u(x, y)$ must approach the value $u_{0}$ when $(x, y)$ approaches $\left(x_{0}, y_{0}\right)$ along any curve that ends at the point $\left(x_{0}, y_{0}\right)$. Conversely, if we can find two curves $C_{1}$ and $C_{2}$ that end at $\left(x_{0}, y_{0}\right)$ along which $u(x, y)$ approaches two distinct values $u_{1}$ and $u_{2}$, then $u(x, y)$ does not have a limit as $(x, y)$ approaches $\left(x_{0}, y_{0}\right)$.

EXAMPLE 2.16 Show that the function $u(x, y)=\frac{x y}{x^{2}+y^{2}}$ does not have a limit as $(x, y)$ approaches ( 0,0 ).

Solution If we let $(x, y)$ approach $(0,0)$ along the $x$-axis, then
$\lim _{(x, 0) \rightarrow(0,0)} u(x, 0)=\lim _{(x, 0) \rightarrow(0,0)} \frac{(x)(0)}{x^{2}+0^{2}}=0$.
But if we let $(x, y)$ approach $(0,0)$ along the line $y=x$, then
$\lim _{(x, x) \rightarrow(0,0)} u(x, x)=\lim _{(x, x) \rightarrow(0,0)} \frac{(x)(x)}{x^{2}+x^{2}}=\frac{1}{2}$.
Because the value of the limit differs depending on how $(x, y)$ approaches $(0,0)$, we conclude that $u(x, y)$ does not have a limit as $(x, y)$ approaches $(0,0)$.

Let $f(z)$ be a complex function of the complex variable $z$ that is defined for all values of $z$ in some neighborhood of $z_{0}$, except perhaps at the point $z_{0}$. We say that $f$ has the limit $w_{0}$ as $z$ approaches $z_{0}$, provided the value $f(z)$ can be made as close as we want to the value $w_{0}$ by taking $z$ to be sufficiently close to $z_{0}$. When this happens we write
$\lim _{z \rightarrow z_{0}} f(z)=w_{0}$.
The distance between the points $z$ and $z_{0}$ can be expressed by $\left|z-z_{0}\right|$, so we can give a precise definition similar to the one for a function of two variables.

Definition 2.4: Limit of $f(z)$
The expression $\lim _{z \rightarrow z_{0}} f(z)=w_{0}$ means that for each real number $\varepsilon>0$, there exists a real number $\delta>0$ such that
$\left|f(z)-w_{0}\right|<\varepsilon \quad$ whenever $\quad 0<\left|z-z_{0}\right|<\delta$.


Figure 2.17 The limit $f(z) \rightarrow w_{0}$ as $z \rightarrow z_{0}$.

Using Equations (1-49) and (1-51), we can also express the last relationship as
$f(z) \in D_{\varepsilon}\left(w_{0}\right) \quad$ whenever $\quad z \in D_{\delta}^{*}\left(z_{0}\right)$.

The formulation of limits in terms of open disks provides a good context for looking at this definition. It says that for each disk of radius $\varepsilon$ about the point $w_{0}$ (represented by $D_{\varepsilon}\left(w_{0}\right)$ ) there is a punctured disk of radius $\delta$ about the point $z_{0}$ (represented by $\left.D_{\delta}^{*}\left(z_{0}\right)\right)$ such that the image of each point in the punctured $\delta$ disk lies in the $\varepsilon$ disk. The image of the $\delta$ disk does not have to fill up the entire $\varepsilon$ disk; but if $z$ approaches $z_{0}$ along a curve that ends at $z_{0}$, then $w=f(z)$ approaches $w_{0}$. The situation is illustrated in Figure 2.17.

EXAMPLE 2.17 Show that if $f(z)=\bar{z}$, then $\lim _{z \rightarrow z_{0}} f(z)=\overline{z_{0}}$, where $z_{0}$ is any complex number.

Solution As $f$ merely reflects points about the $y$-axis, we suspect that any $\varepsilon$ disk about the point $\overline{z_{0}}$ would contain the image of the punctured $\delta$ disk about $z_{0}$ if $\delta=\varepsilon$. To confirm this conjecture, we let $\varepsilon$ be any positive number and set $\delta=\varepsilon$. Then we suppose that $z \in D_{\delta}^{*}\left(z_{0}\right)=D_{\varepsilon}^{*}\left(z_{0}\right)$, which means that $0<\left|z-z_{0}\right|<\varepsilon$. The modulus of a conjugate is the same as the modulus of the number itself, so the last inequality implies that $0<\left|\overline{z-z_{0}}\right|<\varepsilon$. This is the same as $0<\left|\bar{z}-\overline{z_{0}}\right|<\varepsilon$. Since $f(z)=\bar{z}$ and $w_{0}=\overline{z_{0}}$, this is the same as $0<\left|f(z)-w_{0}\right|<\varepsilon$, or $f(z) \in D_{\varepsilon}\left(w_{0}\right)$, which is what we needed to show.

If we consider $w=f(z)$ as a mapping from the $z$ plane into the $w$ plane and think about the previous geometric interpretation of a limit, then we are led to conclude that the limit of a function $f$ should be determined by the limits of its real and imaginary parts, $u$ and $v$. This conclusion also gives us a tool for computing limits.

DTheorem 2.2 Let $f(z)=u(x, y)+i v(x, y)$ be a complex function that is defined in some neighborhood of $z_{0}$, except perhaps at $z_{0}=x_{0}+i y_{0}$. Then
$\lim _{z \rightarrow z_{0}} f(z)=w_{0}=u_{0}+i v_{0}$
iff
$\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} u(x, y)=u_{0} \mid$ and $\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} v(x, y)=v_{0}$.
Proof We first assume that Statement (2-16) is true and show that Statement (2-17) is true. According to the definition of a limit, for each $\varepsilon>0$, there is a corresponding $\delta>0$ such that
that is,
$\left|f(z)-w_{0}\right|<\varepsilon \quad$ whenever $0<\left|z-z_{0}\right|<\delta$.
Because $f(z)-w_{0}=u(x, y)-u_{0}+i\left(v(x, y)-v_{0}\right)$, we can use Inequalities (1-21) to conclude that
$\left|u(x, y)-u_{0}\right| \leq\left|f(z)-w_{0}\right| \quad$ and $\quad\left|v(x, y)-v_{0}\right| \leq\left|f(z)-w_{0}\right|$.
It now follows that $\left|u(x, y)-u_{0}\right|<\varepsilon$ and $\left|v(x, y)-v_{0}\right|<\varepsilon$ whenever $0<\left|z-z_{0}\right|<\delta$, and so Statement (2-17) is true. AL Conversely, we now assume that Statement (2-17) is true. Then for each $\varepsilon>0$, there exists $\delta_{1}>0$ and $\delta_{2}>0$ so that
$\left|u(x y)-u_{0}\right|<\frac{\varepsilon}{2} \quad$ whenever $\quad 0<\left|z-z_{0}\right|<\delta_{1} \quad$ and
$\left|v(x y)-v_{0}\right|<\frac{\varepsilon}{2} \quad$ whenever $\quad 0<\left|z-z_{0}\right|<\delta_{2}$.
We choose $\delta$ to be the minimum of the two values $\delta_{1}$ and $\delta_{2}$. Then we can use the triangle inequality
$\left|f(z)-w_{0}\right| \leq\left|u(x, y)-u_{0}\right|+\left|v(x, y)-v_{0}\right|$
to conclude that
,
$\left|f(z)-w_{0}\right|<\varepsilon$ whever $0<\left|z-z_{0}\right|<\delta$

Bartlett Lea
$\left|f(z)-w_{0}\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad$ whenever $\quad 0<\left|z-z_{0}\right|<\delta ;$
that is,
$f(z) \in D_{\varepsilon}\left(w_{0}\right)$ whenever $z \in D_{\delta}^{*}\left(z_{0}\right)$.
Hence the truth of Statement (2-17) implies the truth of Statement (2-16), and the proof of the theorem is complete.

EXAMPLE 2.18 Show that $\lim _{z \rightarrow 1+i}\left(z^{2}-2 z+1\right)=-1$.
Solution We let
$f(z)=z^{2}-2 z+1=x^{2}-y^{2}-2 x+1+i(2 x y-2 y)$.
Computing the limits for $u$ and $v$, we obtain

$$
\begin{aligned}
& \lim _{(x, y) \rightarrow(1,1)} u(x, y)=1-1-2+1=-1 \text { and } \\
& \lim _{(x, y) \rightarrow(1,1)} v(x, y)=2-2=0
\end{aligned}
$$

so our previous theorem implies that $\lim _{z \rightarrow 1+i} f(z)=-1$.

Limits of complex functions are formally the same as those of real functions, and the sum, difference, product, and quotient of functions have limits given by the sum, difference, product, and quotient of the respective limits. We state this result as a theorem and leave the proof as an exercise.

D Theorem 2.3 Suppose that $\lim _{z \rightarrow z_{0}} f(z)=A$ and $\lim _{z \rightarrow z_{0}} g(z)=B$. Then Bartlett Learning, LLC
$\lim _{z \rightarrow z_{0}}[f(z) \pm g(z)]=A \pm B$,
$\lim _{z \rightarrow z_{0}} f(z) g(z)=A B, \quad$ and
$\lim _{z \rightarrow z_{0}} \frac{f(z)}{g(z)}=\frac{A}{B}, \quad$ where $B \neq 0$

Definition 2.5: Continuity of $u(x, y)$
Let $u(x, y)$ be a real-valued function of the two real variables $x$ and $y$. We say that $u$ is continuous at the point $\left(x_{0}, y_{0}\right)$ if three conditions are satisfied:

$$
\begin{equation*}
\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} u(x, y) \text { exists; } \tag{2-21}
\end{equation*}
$$

$u\left(x_{0}, y_{0}\right)$ exists; and
$\lim _{(x, y) \rightarrow\left(x_{0}, y_{0}\right)} u(x, y)=u\left(x_{0}, y_{0}\right)$.

Condition (2-23) actually implies Conditions (2-21) and (2-22) because the existence of the quantity on each side of Equation (2-23) is implicitly understood to exist. For example, if $u(x, y)=\frac{x^{3}}{x^{2}+y^{2}}$ when $(x, y) \neq(0,0)$ and if $u(0,0)=0$, then $u(x, y) \rightarrow 0$ as $(x, y) \rightarrow(0,0)$ so that Conditions (2-21), (2-22), and (2-23) are satisfied. Hence $u(x, y)$ is continuous at $(0,0)$.

There is a similar definition for complex-valued functions.

Definition 2.6: Continuity of $f(z)$
Let $f(z)$ be a complex function of the complex variable $z$ that is defined for all values of $z$ in some neighborhood of $z_{0}$. We say that $f$ is continuous at $z_{0}$ if three conditions are satisfied:

$$
\begin{equation*}
\lim _{z \rightarrow z_{0}} f(z) \text { exists; } \tag{2-24}
\end{equation*}
$$

$f\left(z_{0}\right)$ exists;

$$
\begin{equation*}
\lim _{z \rightarrow z_{0}} f(z)=f\left(z_{0}\right) . \tag{2-26}
\end{equation*}
$$

Remark 2.3 Example 2.17 shows that the function $f(z)=\bar{z}$ is continuous.

A complex function $f$ is continuous iff its real and imaginary parts, $u$ and $v$, are continuous. The proof of this fact is an immediate consequence of Theorem 2.2. Continuity of complex functions is formally the same as that of real functions, and sums, differences, and products of continuous functions are continuous; their quotient is continuous at points where the denominator is not zero. These results are summarized by the following theorems. We leave the proofs as exercises.

D Theorem 2.4 Let $f(z)=u(x, y)+i v(x, y)$ be defined in some neighborhood of $z_{0}$. Then $f$ is continuous at $z_{0}=x_{0}+i y_{0}$ iff $u$ and $v$ are continuous at $\left(x_{0}, y_{0}\right)$.

DTheorem 2.5 Suppose that $f$ and $g$ are continuous at the point $z_{0}$. Then the following functions are continuous at $z_{0}$ :

- the sum $f+g$, where $(f+g)(z)=f(z)+g(z)$;
- the difference $f-g$, where $(f-g)(z)=f(z)-g(z)$;
- the product $f g$, where $(f g)(z)=f(z) g(z)$;
- the quotient $\frac{f}{g}$, where $\frac{f}{g}(z)=\frac{f(z)}{g(z)}$, provided $g\left(z_{0}\right) \neq 0$; and
- the composition $f \circ g$, where $(f \circ g)(z)=f(g(z))$, provided $f$ is continuous in a neighborhood of $g\left(z_{0}\right)$.

EXAMPLE 2.19 Show that the polynomial function given by
$w=P(z)=a_{0}+a_{1} z+a_{2} z^{2}+\cdots+a_{n} z^{n}$
is continuous at each point $z_{0}$ in the complex plane.
Solution $F$ If $a_{0}$ is the constant function, then $\lim _{z \rightarrow z_{0}} a_{0}=a_{0}$; and if $a_{1} \neq 0$, then we can use Definition 2.3 with $f(z)=a_{1} z$ and the choice $\delta=\frac{\varepsilon}{\left|a_{1}\right|}$ to prove that $\lim _{z \rightarrow z_{0}}\left(a_{1} z\right)=a_{1} z_{0}$. Using Property (2-19) and mathematical induction, we obtain
$\lim _{z \rightarrow z_{0}}\left(a_{k} z^{k}\right)=a_{k} z_{0}^{k}$ RIB for $k=0,1,2, \ldots, n$.
We can extend Property (2-18) to a finite sum of terms and use the result of Equation (2-27) to get
$\lim _{z \rightarrow z_{0}} P(z)=\lim _{z \rightarrow z_{0}}\left(\sum_{k=0}^{n} a_{k} z^{k}\right)=\sum_{k=0}^{n} a_{k} z_{0}^{k}=P\left(z_{0}\right)$.
Conditions (2-24), (2-25), and (2-26) are satisfied, so we conclude that $P$ is continuous at $z_{0}$.

One technique for computing limits is to apply Theorem 2.5 to quotients. If we let $P$ and $Q$ be polynomials and if $Q\left(z_{0}\right) \neq 0$, then
$\lim _{z \rightarrow z_{0}} \frac{P(z)}{Q(z)}=\frac{P\left(z_{0}\right)}{Q\left(z_{0}\right)}$.

Another technique involves factoring polynomials. If both $P\left(z_{0}\right)=0$ and $Q\left(z_{0}\right)=0$, then $P$ and $Q$ can be factored as $P(z)=\left(z-z_{0}\right) P_{1}(z)$ and $Q(z)=$ $\left(z-z_{0}\right) Q_{1}(z)$. If $Q_{1}\left(z_{0}\right) \neq 0$, then the limit is
$\lim _{z \rightarrow z_{0}} \frac{P(z)}{Q(z)}=\lim _{z \rightarrow z_{0}} \frac{\left(z-z_{0}\right) P_{1}(z)}{\left(z-z_{0}\right) Q_{1}(z)}=\frac{P_{1}\left(z_{0}\right)}{Q_{1}\left(z_{0}\right)}$.

EXAMPLE 2.20 Show that $\lim _{z \rightarrow 1+i} \frac{z^{2}-2 i}{z^{2}-2 z+2}=1-i$.
Solution Here $P$ and $Q$ can be factored in the form

$$
P(z)=(z+1-i)(z+1+i) \text { and } Q(z)=(z-1-i)(z-1+i) \text { RIBUTION }
$$

so that the limit is obtained by the calculation

$$
\begin{aligned}
& \lim _{z \rightarrow 1+i} \frac{z^{2}-2 i}{z^{2}-2 z+2}=\lim _{z \rightarrow 1+i} \frac{(z-1-i)(z+1+i)}{(z-1-i)(z-1+i)} \\
& \text { NOTFORSS}=\lim _{z \rightarrow 1+i} \frac{z+1+i}{z-1+i} \\
&=\frac{(1+i)+1+i}{(1+i)-1+i} \\
&=\frac{2+2 i}{2 i} \mathrm{CUTION} \\
& \text { 8. Bartlett Learn }
\end{aligned}
$$



1. Find the following limits.
(a) $\lim _{z \rightarrow 2+i}\left(z^{2}-4 z+2+5 i\right)$.
(b) $\lim _{z \rightarrow i} \frac{z^{2}+4 z+2}{z+1}$.
(c) $\lim _{z \rightarrow i} \frac{z^{4}-1}{z-i}$.
(d) $\lim _{z \rightarrow 1+i} \frac{z^{2}+z-2+i}{z^{2}-2 z+1}$.
(e) $\lim _{z \rightarrow 1+i} \frac{z^{2}+z-1-3 i}{z^{2}-2 z+2}$ by factoring.
2. Determine where the following functions are continuous.
(a) $z^{4}-9 z^{2}+i z-2$.
(b) $\frac{z+1}{z^{2}+1}$.
(c) $\frac{z^{2}+6 z+5}{z^{2}+3 z+2}$.
(d) $\frac{z^{4}+1}{z^{2}+2 z+2}$.
(e) $\frac{x+i y}{x-1}$.
(f) $\frac{x+i y}{|z|-1}$.
3. State why $\lim _{z \rightarrow z_{0}}\left(e^{x} \cos y+i x^{2} y\right)=e^{x_{0}} \cos y_{0}+i x_{0}^{2} y_{0}$.
4. State why $\lim _{z \rightarrow z_{0}}\left[\ln \left(x^{2}+y^{2}\right)+i y\right]=\ln \left(x_{0}^{2}+y_{0}^{2}\right)+i y_{0}$, provided $\left|z_{0}\right| \neq 0$.
5. Show that
(a) $\lim _{z \rightarrow 0} \frac{|z|^{2}}{z}=0$.
(b) $\lim _{z \rightarrow 0} \frac{x^{2}}{z}=0$.
6. Let $f(z)=\frac{z \operatorname{Re}(z)}{|z|}$ when $z \neq 0$, and let $f(0)=0$. Show that $f(z)$ is continuous for all values of $z$.
7. Let $f(z)=\frac{z^{2}}{|z|^{2}}=\frac{x^{2}-y^{2}+i 2 x y}{x^{2}+y^{2}}$.
(a) Find $\lim _{z \rightarrow 0} f(z)$ as $z \rightarrow 0$ along the line $y=x$.
(b) Find $\lim _{z \rightarrow 0} f(z)$ as $z \rightarrow 0$ along the line $y=2 x$.
(c) Find $\lim _{z \rightarrow 0} f(z)$ as $z \rightarrow 0$ along the parabola $y=x^{2}$.
(d) What can you conclude about the limit of $f(z)$ as $z \rightarrow 0$ ? Why?
8. Let $f(z)=f(x, y)=\frac{x y^{3}}{x^{2}+2 y^{6}}+i \frac{x^{3} y}{5 x^{6}+y^{2}}$ when $z \neq 0$, and let $f(0)=0$.
(a) Show that $\lim _{z \rightarrow 0} f(z)=f(0)=0$ if $z$ approaches zero along any straight line that passes through the origin.
(b) Show that $f$ is not continuous at the point 0 .
9. For $z \neq 0$, let $f(z)=\frac{\bar{z}}{z}$. Does $f(z)$ have a limit as $z \rightarrow 0$ ?
10. Does $\lim _{z \rightarrow-4} \operatorname{Arg} z$ exist? Why? Hint: Use polar coordinates and let $z$ approach -4 from the upper and lower half-planes.
11. Let $f(z)=z^{\frac{1}{2}}=r^{\frac{1}{2}}\left(\cos \frac{\theta}{2}+i \sin \frac{\theta}{2}\right)$, where $z=r e^{i \theta}, r>0$, and $-\pi<\theta \leq \pi$. Use the polar form of $z$ and show that
(a) $f(z) \rightarrow i$ as $z \hookrightarrow-1$ along the upper semicircle $r=1,0<\theta \leq \pi$.
(b) $f(z) \rightarrow-i$ as $z \rightarrow-1$ along the lower semicircle $r=1,-\pi<\theta<0$.
12. Let $f(z)=\frac{x^{2}+i y^{2}}{|z|^{2}}$ when $z \neq 0$, and let $f(0)=1$. Show that $f(z)$ is not continuous at $z_{0}=0$.
13. Let $f(z)=x e^{y}+i y^{2} e^{-x}$. Show that $f(z)$ is continuous for all values of $z$.
14. Use the definition of the limit to show that $\lim _{z \rightarrow 3+4 i} z^{2}=-7+24 i$.
15. Let $f(z)=\frac{\mathrm{Re}(z)}{|z|}$ when $z \neq 0$, and let $f(0)=1$. Is $f(z)$ continuous at the origin?
16. Let $f(z)=\left\lvert\, \frac{[\operatorname{Re}(z)]^{2}}{|z|}\right.$ when $z \neq 0$, and let $f(0)=0$. Is $f(z)$ continuous at the origin?
17. Let $f(z)=z^{\frac{1}{2}}=|z|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(z)}{2}}$, where $z \neq 0$. Show that $f(z)$ is discontinuous at each point along the negative $x$-axis.
18. Let $f(z)=\ln |z|+i \operatorname{Arg} z$, where $-\pi<\operatorname{Arg} z \leq \pi$. Show that $f(z)$ is discontinuous at $z_{0}=0$ and at each point along the negative $x$-axis.
19. Let $|g(z)|<M$ and $\lim _{z \rightarrow z_{0}} f(z)=0$. Show that $\lim _{z \rightarrow z_{0}} f(z) g(z)=0$. Note: Theorem 2.3 is of no use here because you don't know whether $\lim _{z \rightarrow z_{0}} g(z)$ exists. Give an $\boldsymbol{\varepsilon}, \delta$ argument.
20. Let $\Delta z=z=z_{0}$. Show that $\lim _{z \rightarrow z_{0}} f(z)=w_{0}$ iff $\lim _{\Delta z \rightarrow 0} f\left(z_{0}+\Delta z\right)=w_{0}$.
21. Let $f(z)$ be continuous for all values of $z$.
(a) Show that $g(z)=f(\bar{z})$ is continuous for all $z$.
(b) Show that $g(z)=\overline{f(z)}$ is continuous for all $z$.
22. Verify the identities
(a) (2-18).
(b) (2-19).
(c) $(2-20)$.
23. Verify the results of Theorem 2.5 .
24. Show that the principal branch of the argument, $\operatorname{Arg} z$, is discontinuous at 0 and all points along the negative real axis.

### 2.4 BRANCHES OF FUNCTIONS

In Section 2.2 we defined the principal square root function and investigated some of its properties. We left unanswered some questions concerning the choices of square roots. We now look at these questions because they are similar to situations involving other elementary functions.

In our definition of a function in Section 2.1, we specified that each value of the independent variable in the domain is mapped onto one and only one value in the range. As a result, we often talk about a single-valued function, which
emphasizes the "only one" part of the definition and allows us to distinguish such functions from multiple-valued functions, which we now introduce.

Let $w=f(z)$ denote a function whose domain is the set $D$ and whose range is the set $R$. If $w$ is a value in the range, then there is an associated inverse relation $z=g(w)$ that assigns to each value $w$ the value (or values) of $z$ in $D$ for which the equation $f(z)=w$ holds. But unless $f$ takes on the value $w$ at most once in $D$, then the inverse relation $g$ is necessarily many valued, and we say that $g$ is a multivalued function. For example, the inverse of the function $w=f(z)=z^{2}$ is the square root function $z=g(w)=w^{\frac{1}{2}}$. For each value $z$ other than $z=0$, then, the two points $z$ and $-z$ are mapped onto the same point $w=f(z)$; hence $g$ is, in general, a two-valued function.

The study of limits, continuity, and derivatives loses all meaning if an arbitrary or ambiguous assignment of function values is made. For this reason we did not allow multivalued functions to be considered when we defined these concepts. When working with inverse functions, you have to specify carefully one of the many possible inverse values when constructing an inverse function, as when you determine implicit functions in calculus. If the values of a function $f$ are determined by an equation that they satisfy rather than by an explicit formula, then we say that the function is defined implicitly or that $f$ is an implicit function. In the theory of complex variables we present a similar concept.

We now let $w=f(z)$ be a multiple-valued function. A branch of $f$ is any single-valued function $f_{0}$ that is continuous in some domain (except, perhaps, on the boundary). At each point $z$ in the domain, it assigns one of the values of $f(z)$.

EXAMPLE 2.21 We consider some branches of the two-valued square root function $f(z)=z^{\frac{1}{2}}(z \neq 0)$. Recall that the principal square root function is
$f_{1}(z)=|z|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(z)}{2}}=r^{\frac{1}{2}} e^{i \frac{\theta}{2}}=r^{\frac{1}{2}} \cos \frac{\theta}{2}+i r^{\frac{1}{2}} \sin \frac{\theta}{2}$,
where $r=|z|$ and $\theta=\operatorname{Arg}(z)$ so that $-\pi<\theta \leq \pi$. The function $f_{1}$ is a branch of $f$. Using the same notation, we can find other branches of the square root function. For example, if we let
$f_{2}(z)=|z|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg}(z)+2 \pi}{2}}=r^{\frac{1}{2}} e^{i \frac{\theta+2 \pi}{2}}=r^{\frac{1}{2}} \cos \left(\frac{\theta+2 \pi}{2}\right)+i r^{\frac{1}{2}} \sin \left(\frac{\theta+2 \pi}{2}\right)$,
then

$$
f_{2}(z)=r^{\frac{1}{2}} e^{i \frac{\theta+2 \pi}{2}}=r^{\frac{1}{2}} e^{i \frac{\theta}{2}} e^{i \pi}=-r^{\frac{1}{2}} e^{i \frac{\theta}{2}}=-f_{1}(z)
$$

so $f_{1}$ and $f_{2}$ can be thought of as "plus" and "minus" square root functions. The negative real axis is called a branch cut for the functions $f_{1}$ and $f_{2}$. Each point on the branch cut is a point of discontinuity for both functions $f_{1}$ and $f_{2}$.

EXAMPLE 2.22 Show that the function $f_{1}$ is discontinuous along the negative real axis.

Solution Let $z_{0}=r_{0} e^{i \pi}$ denote a negative real number. We compute the limit as $z$ approaches $z_{0}$ through the upper half-plane $\{z: \operatorname{Im}(z)>0\}$ and the limit as $z$ approaches $z_{0}$ through the lower half-plane $\{z: \operatorname{Im}(z)<0\}$. In polar coordinates these limits are given by

$$
\begin{aligned}
\lim _{(r, \theta) \rightarrow\left(r_{0}, \pi\right)} f_{1}\left(r e^{i \theta}\right) & =\lim _{(r, \theta) \rightarrow\left(r_{0}, \pi\right)} r^{\frac{1}{2}}\left(\cos \frac{\theta}{2}+i \sin \frac{\theta}{2}\right)=i r_{0}^{\frac{1}{2}}, \quad \text { and } \\
\lim _{(r, \theta) \rightarrow\left(r_{0},-\pi\right)} f_{1}\left(r e^{i \theta}\right) & =\lim _{(r, \theta) \rightarrow\left(r_{0},-\pi\right)} r^{\frac{1}{2}}\left(\cos \frac{\theta}{2}+i \sin \frac{\theta}{2}\right)=-i r_{0}^{\frac{1}{2}}
\end{aligned}
$$

The two limits are distinct, so the function $f_{1}$ is discontinuous at $z_{0}$.

Remark 2.4 Likewise, $f_{2}$ is discontinuous at $z_{0}$. The mappings $w=f_{1}(z)$, $w=f_{2}(z)$, and the branch cut are illustrated in Figure 2.18.

We can construct other branches of the square root function by specifying that an argument of $z$ given by $\theta=\arg z$ is to lie in the interval $\alpha<\theta \leq \alpha+2 \pi$. The corresponding branch, denoted $f_{\alpha}$, is

$$
\begin{equation*}
f_{\alpha}(z)=r^{\frac{1}{2}} \cos \frac{\theta}{2}+i r^{\frac{1}{2}} \sin \frac{\theta}{2} \tag{2-30}
\end{equation*}
$$

where $z=r e^{i \theta} \neq 0$ and $\alpha<\theta \leq \alpha+2 \pi$.


Figure 2.18 The branches $f_{1}$ and $f_{2}$ of $f(z)=z^{\frac{1}{2}}$.


Figure 2.19 The branch $f_{\alpha}$ of $f(z)=z^{\frac{1}{2}}$.

The branch cut for $f_{\alpha}$ is the ray $r \geq 0, \theta=\alpha$, which includes the origin. The point $z=0$, common to all branch cuts for the multivalued square root function, is called a branch point. The mapping $w=f_{\alpha}(z)$ and its branch cut are illustrated in Figure 2.19.

### 2.4.1 The Riemann Surface for $w=z^{\frac{1}{2}}$

A Riemann surface is a construct useful for visualizing a multivalued function. It was introduced by G. F. B. Riemann (1826-1866) in 1851. The idea is ingen-ious-a geometric construction that permits surfaces to be the domain or range of a multivalued function. Riemann surfaces depend on the function being investigated. We now give a nontechnical formulation of the Riemann surface for the multivalued square root function.

Consider $w=f(z)=z^{\frac{1}{2}}$, which has two values for any $z \neq 0$. Each function $f_{1}$ and $f_{2}$ in Figure 2.18 is single-valued on the domain formed by cutting the $z$ plane along the negative $x$-axis. Let $D_{1}$ and $D_{2}$ be the domains of $f_{1}$ and $f_{2}$, respectively. The range set for $f_{1}$ is the set $H_{1}$ consisting of the right half-plane, and the positive $v$-axis; the range set for $f_{2}$ is the set $H_{2}$ consisting of the left half-plane and the negative $v$-axis. The sets $H_{1}$ and $H_{2}$ are "glued together" along the positive $v$-axis and the negative $v$-axis to form the $w$ plane with the origin deleted.

We stack $D_{1}$ directly above $D_{2}$. The edge of $D_{1}$ in the upper half-plane is joined to the edge of $D_{2}$ in the lower half-plane, and the edge of $D_{1}$ in the lower half-plane is joined to the edge of $D_{2}$ in the upper half-plane. When these domains are glued together in this manner, they form $R$, which is a Riemann surface domain for the mapping $w=f(z)=z^{\frac{1}{2}}$. The portions of $D_{1}, D_{2}$, and $R$ that lie in $\{z:|z|<1\}$ are shown in Figure 2.20.

The beauty of this structure is that it makes this "full square root function" continuous for all $z \neq 0$. Normally, the principal square root function would be discontinuous along the negative real axis, as points near -1 but above that axis would get mapped to points close to $i$, and points near -1 but below

(a) A portion of $D_{1}$ and its image under $w=f_{1}$.

(b) A portion of $D_{2}$ and its image under $w=f_{2}$.

(c) A portion of $R$ and its image under $w=z^{\frac{1}{2}}$.

Figure 2.20 Formation of the Riemann surface for $w=z^{\frac{1}{2}}$ : (a) a portion of $D_{1}$ and its image under $w=z^{\frac{1}{2}}$; (b) a portion of $D_{2}$ and its image under $w=z^{\frac{1}{2}}$; (c) a portion of $R$ and its image under $w=z^{\frac{1}{2}}$.
the axis would get mapped to points close to $-i$. As Figure 2.20(c) indicates, however, between the point $A$ and the point $B$, the domain switches from the edge of $D_{1}$ in the upper half-plane to the edge of $D_{2}$ in the lower half-plane. The corresponding mapped points $A^{\prime}$ and $B^{\prime}$ are exactly where they should be. The surface works in such a way that going directly between the edges of $D_{1}$ in the upper and lower half-planes is impossible (likewise for $D_{2}$ ). Going counterclockwise, the only way to get from the point $A$ to the point $C$, for example, is to follow the path indicated by the arrows in Figure 2.20(c).

## --------EXERCISES FOR SECTION 2.4

1. Let $f_{1}(z)$ and $f_{2}(z)$ be the two branches of the square root function given by Equations (2-28) and (2-29), respectively. Use the polar coordinate formulas in Section 2.2 to find the image of
(a) quadrant II, $x<0$ and $y>0$, under the mapping $w=f_{1}(z)$.
(b) quadrant II, $x<0$ and $y>0$, under the mapping $w=f_{2}(z)$.
(c) the right half-plane $\operatorname{Re}(z)>0$ under the mapping $w=f_{1}(z)$.
(d) the right half-plane $\operatorname{Re}(z)>0$ under the mapping $w=f_{2}(z)$.
2. Let $\alpha=0$ in Equation (2-30). Find the range of the function $w=f_{\alpha}(z)$.
3. Let $\alpha=2 \pi$ in Equation (2-30). Find the range of the function $w=f_{\alpha}(z)$.
4. Find a branch of the square root that is continuous along the negative $x$-axis.
5. Let $f_{1}(z)=|z|^{\frac{1}{3}} e^{i \frac{\operatorname{Arg}(z)}{3}}=r^{\frac{1}{3}} \cos \frac{\theta}{3}+i r^{\frac{1}{3}} \sin \frac{\theta}{3}$, where $|z|=r \neq 0$, and $\theta=\operatorname{Arg}(z)$. $f_{1}$ denotes the principal cube root function.
(a) Show that $f_{1}$ is a branch of the multivalued cube root $f(z)=z^{\frac{1}{3}}$.
(b) What is the range of $f_{1}$ ?
(c) Where is $f_{1}$ continuous?
6. Let $f_{2}(z)=r^{\frac{1}{3}} \cos \left(\frac{\theta+2 \pi}{3}\right)+i r^{\frac{1}{3}} \sin \left(\frac{\theta+2 \pi}{3}\right)$, where $r>0$ and $-\pi<\theta \leq \pi$.
(a) Show that $f_{2}$ is a branch of the multivalued cube root $f(z)=z^{\frac{1}{3}}$.
(b) What is the range of $f_{2}$ ?
(c) Where is $f_{2}$ continuous?
(d) What is the branch point associated with $f$ ?
7. Find a branch of the multivalued cube root function that is different from those in Exercises 5 and 6. State the domain and range of the branch you find.
8. Let $f(z)=z^{\frac{1}{n}}$ denote the multivalued $n$th root, where $n$ is a positive integer.
(a) Show that $f$ is, in general, an $n$-valued function.
(b) Write the principal $n$th root function.
(c) Write a branch of the multivalued $n$th root function that is different from the one in part (b).
9. Describe a Riemann surface for the domain of definition of the multivalued function
(a) $w=f(z)=z^{\frac{1}{3}}$.
(b) $w=f(z)=z^{\frac{1}{4}}$.
10. Discuss how Riemann surfaces should be used for both the domain and the range to help describe the behavior of the multivalued function $w=f(z)=z^{\frac{2}{3}}$.

### 2.5 THE RECIPROCAL TRANSFORMATION $w=\frac{1}{z}$

The mapping $w=f(z)=\frac{1}{z}$ is called the reciprocal transformation and maps the $z$ plane one-to-one and onto the $w$ plane except for the point $z=0$, which has no image, and the point $w=0$, which has no preimage or inverse image. Using exponential notation $w=\rho e^{i \phi}$, if $z=r e^{i \theta} \neq 0$, we have
$w=\rho e^{i \phi}=\frac{1}{z}=\frac{1}{r} e^{-i \theta}$.
The geometric description of the reciprocal transformation is now evident. It is an inversion (that is, the modulus of $\frac{1}{z}$ is the reciprocal of the modulus of $z$ ) followed by a reflection through the $x$-axis. The ray $r>0, \theta=\alpha$, is mapped one-to-one and onto the ray $\rho>0, \phi=-\alpha$. Points that lie inside the unit circle $C_{1}(0)=\{z:|z|=1\}$ are mapped onto points that lie outside the unit circle, and vice versa. The situation is illustrated in Figure 2.21.

We can extend the system of complex numbers by joining to it an "ideal" point denoted by $\infty$ and called the point at infinity. This new set is called the


Figure 2.21 The reciprocal transformation $w=\frac{1}{z}$.
extended complex plane. You will see shortly that the point $\infty$ has the property, loosely speaking, that $\lim _{n \rightarrow \infty} z=\infty$ iff $\lim _{n \rightarrow \infty}|z|=\infty$.

An $\varepsilon$ neighborhood of the point at infinity is the set $\left\{z:|z|>\frac{1}{\varepsilon}\right\}$. The usual way to visualize the point at infinity is by using what we call the stereographic projection, which is attributed to Riemann. Let $\Omega$ be a sphere of diameter 1 that is centered at the point $\left(0,0, \frac{1}{2}\right)$ in three-dimensional space where coordinates are specified by the triple of real numbers $(x, y, \xi)$. Here the complex number $z=x+i y$ is associated with the point $z=(x, y, 0)$.

The point $\mathbb{N}=(0,0,1)$ on $\Omega$ is called the north pole of $\Omega$. If we let $z$ be a complex number and consider the line segment $L$ in three-dimensional space that joins $z$ to the north pole $\mathbb{N}=(0,0,1)$, then $L$ intersects $\Omega$ in exactly one point $\mathbb{Z}$. The correspondence $z \leftrightarrow \mathbb{Z}$ is called the stereographic projection of the complex $z$ plane onto the Riemann sphere $\Omega$.

A point $z=x+i y=(x, y, 0)$ of unit modulus will correspond with $\mathbb{Z}=$ $\left(\frac{x}{2}, \frac{y}{2}, \frac{1}{2}\right)$. If $z$ has modulus greater than 1 , then $\mathbb{Z}$ will lie in the upper hemisphere where for points $\mathbb{Z}=(x, y, \xi)$ we have $\xi>\frac{1}{2}$. If $z$ has modulus less than 1 , then $\mathbb{Z}$ will lie in the lower hemisphere where for points $\mathbb{Z}=(x, y, \xi)$ we have $\xi<\frac{1}{2}$. The complex number $z=0=0+0 i$ corresponds with the south pole, $\mathbb{S}=(0,0,0)$. Now you can see that indeed $z \rightarrow \infty$ iff $|z| \rightarrow \infty$ iff $\mathbb{Z} \rightarrow \mathbb{N}$. Hence $\mathbb{N}$ corresponds with the "ideal" point at infinity. The situation is shown in Figure 2.22.

Let's reconsider the mapping $w=\frac{1}{z}$ by assigning the images $w=\infty$ and $w=0$ to the points $z=0$ and $z=\infty$, respectively. We now write the reciprocal transformation as
$w=f(z)= \begin{cases}\frac{1}{z} & \text { when } z \neq 0 \text { and } z \neq \infty ; \\ 0 & \text { when } z=\infty ; \\ \infty & \text { when } z=0 .\end{cases}$
Note that the transformation $w=f(z)$ is a one-to-one mapping of the extended complex $z$ plane onto the extended complex $w$ plane. Further, $f$ is a continuous mapping from the extended $z$ plane onto the extended $w$ plane. We leave the details to you.


Figure 2.22 The Riemann sphere.

EXAMPLE 2.23 Show that the image of the half-plane $A=\left\{z: \operatorname{Re}(z) \geq \frac{1}{2}\right\}$ under the mapping $w=\frac{1}{z}$ is the closed disk $\bar{D}_{1}(1)=\{w:|w-1| \leq 1\}$.

Solution Proceeding as we did in Example 2.7, we get the inverse mapping of $u+i v=w=f(z)=\frac{1}{z}$ as $z=f^{-1}(w)=\frac{1}{w}$. Then

$$
\begin{align*}
u+i v=w \in B & \Longleftrightarrow f^{-1}(w)=z=x+i y \in A \\
& \Longleftrightarrow \frac{1}{u+i v}=x+i y \in A \\
& \Longleftrightarrow \frac{u}{u^{2}+v^{2}}+i \frac{-v}{u^{2}+v^{2}}=x+i y \in A \\
\text { Ining, LLC } & \Longleftrightarrow \frac{u}{u^{2}+v^{2}}=x=\operatorname{Re}(x+i y) \geq \frac{1}{2} \\
& \Longleftrightarrow \frac{u}{u^{2}+v^{2}} \geq \frac{1}{2}  \tag{2-33}\\
& \Longleftrightarrow u^{2}-2 u+1+v^{2} \leq 1  \tag{2-34}\\
& \Longleftrightarrow(u-1)^{2}+(v-0)^{2} \leq 1,
\end{align*}
$$

which describes the disk $\bar{D}_{1}(1)$. As the reciprocal transformation is one-toone, preimages of the points in the disk $\bar{D}_{1}(1)$ will lie in the right half-plane $\operatorname{Re}(z) \geq \frac{1}{2}$. Figure 2.23 illustrates this result.



Figure 2.23 e The image of $\operatorname{Re}(z) \geq \frac{1}{2}$ under the mapping $w=\frac{1}{z}$.

Remark 2.5 Alas, there is a fly in the ointment here. As our notation indicates, Equations (2-33) and (2-34) are not equivalent. The former implies the latter, but not conversely. That is, Equation (2-34) makes sense when $(u, v)=(0,0)$, whereas Equation (2-33) does not. Yet Figure 2.23 seems to indicate that $f$ maps $\operatorname{Re}(z) \geq \frac{1}{2}$ onto the entire disk $\bar{D}_{1}(0)$, including the point $(0,0)$. Actually, it
does not, because $(0,0)$ has no preimage in the complex plane. The way out of this dilemma is to use the complex point at infinity. It is that quantity that gets mapped to the point $(u, v)=(0,0)$, for as we have already indicated in Equation (2-32), the preimage of 0 under the mapping $\frac{1}{z}$ is indeed $\infty$.

■ EXAMPLE 2.24 For the transformation $\frac{1}{z}$, find the image of the portion of the half-plane $\operatorname{Re}(z) \geq \frac{1}{2}$ that is inside the closed disk $\bar{D}_{1}\left(\frac{1}{2}\right)=\left\{z:\left|z-\frac{1}{2}\right|\right.$ $\leq 1\}$.

Solution Using the result of Example 2.23, we need only find the image of the disk $\bar{D}_{1}\left(\frac{1}{2}\right)$ and intersect it with the closed disk $\bar{D}_{1}(1)$. To begin, we note that
$\bar{D}_{1}\left(\frac{1}{2}\right)=\left\{(x, y): x^{2}+y^{2}-x \leq \frac{3}{4}\right\}$.
Because $z=f^{-1}(w)=\frac{1}{w}$, we have, as before,

$$
\begin{aligned}
u+i v=w \in f\left(\bar{D}_{1}\left(\frac{1}{2}\right)\right) & \Longleftrightarrow f^{-1}(w) \in \bar{D}_{1}\left(\frac{1}{2}\right) \\
& \Longleftrightarrow \frac{1}{w} \in \bar{D}_{1}\left(\frac{1}{2}\right)
\end{aligned}
$$

$$
\Longleftrightarrow \frac{u}{u^{2}+v^{2}}+i \frac{-v}{u^{2}+v^{2}} \in \bar{D}_{1}\left(\frac{1}{2}\right)
$$

$$
\Longleftrightarrow\left(\frac{u}{u^{2}+v^{2}}\right)^{2}+\left(\frac{-v}{u^{2}+v^{2}}\right)^{2}-\frac{u}{u^{2}+v^{2}} \leq \frac{3}{4}
$$

$$
\Longleftrightarrow \frac{1}{u^{2}+v^{2}}-\frac{u}{u^{2}+v^{2}} \leq \frac{3}{4}
$$

$$
\Longleftrightarrow\left(u+\frac{2}{3}\right)^{2}+v^{2} \geq\left(\frac{4}{3}\right)^{2}
$$

which is an inequality that determines the set of points in the $w$ plane that lie on and outside the circle $C_{\frac{4}{3}}\left(-\frac{2}{3}\right)=\left\{w:\left|w+\frac{2}{3}\right|=\frac{4}{3}\right\}$. Note that we do not have to deal with the point at infinity this time, as the last inequality is not satisfied when $(u, v)=(0,0)$.

When we intersect this set with $\bar{D}_{1}(1)$, we get the crescent-shaped region shown in Figure 2.24.

To study images of "generalized circles," we consider the equation

$$
A\left(x^{2}+y^{2}\right)+B x+C y+D=0
$$



Figure 2.24 The mapping $w=\frac{1}{z}$ discussed in Example 2.24.
where $A, B, C$, and $D$ are real numbers. This equation represents either a circle or a line, depending on whether $A \neq 0$ or $A=0$, respectively. Transforming the equation to polar coordinates gives
$A r^{2}+r(B \cos \theta+C \sin \theta)+D=0$.
Using the polar coordinate form of the reciprocal transformation given in Equation (2-31), we can express the image of the curve in the preceding equation as
$A+\rho(B \cos \phi-C \sin \phi)+D \rho^{2}=0$,
which represents either a circle or a line, depending on whether $D \neq 0$ or $D=0$, respectively. Therefore, we have shown that the reciprocal transformation $w=\frac{1}{z}$ carries the class of lines and circles onto itself.

EXAMPLE 2.25 Find the images of the vertical lines $x=a$ and the horizontal lines $y=b$ under the mapping $w=\frac{1}{z}$.
Solution Taking into account the point at infinity, we see that the image of the line $x=0$ is the line $u=0$; that is, the $y$-axis is mapped onto the $v$-axis. Similarly, the $x$-axis is mapped onto the $u$-axis. Again, the inverse mapping is $z=\frac{1}{w}=\frac{u}{u^{2}+v^{2}}+i \frac{-v}{u^{2}+v^{2}}$, so if $a \neq 0$, the vertical line $x=a$ is mapped onto the set of $(u, v)$ points satisfying $\frac{u}{u^{2}+v^{2}}=\bar{a}$. For $(u, v) \neq(0,0)$, this outcome is equivalent to
$u^{2}-\frac{1}{a} u+\frac{1}{4 a^{2}}+v^{2}=\left(u-\frac{1}{2 a}\right)^{2}+v^{2}=\left(\frac{1}{2 a}\right)^{2}$,
which is the equation of a circle in the $w$ plane with center $w_{0}=\frac{1}{2 a}$ and radius $\left|\frac{1}{2 a}\right|$. The point at infinity is mapped to $(u, v)=(0,0)$.



Figure 2.25 The images of horizontal and vertical lines under the reciprocal transformation.

Similarly, the horizontal line $y=b$ is mapped onto the circle
$u^{2}+v^{2}+\frac{1}{b} v+\frac{1}{4 b^{2}}=u^{2}+\left(v+\frac{1}{2 b}\right)^{2}=\left(\frac{1}{2 b}\right)^{2}$,
which has center $w_{0}=-\frac{i}{2 b}$ and radius $\left|\frac{1}{2 b}\right|$.
Figure 2.25 illustrates the images of several lines.
--------EXERCISES FOR SECTION 2.5
For Exercises 1-8, find the image of the given circle or line under the reciprocal transformation $w=\frac{1}{z}$.

1. The horizontal $\operatorname{line} \operatorname{Im}(z)=\frac{1}{5}$.
2. The circle $C_{\frac{1}{2}}\left(-\frac{i}{2}\right)=\left\{z:\left|z+\frac{i}{2}\right|=\frac{1}{2}\right\}$.
3. The vertical line $\operatorname{Re} z=-3$.
4. The circle $C_{1}(-2)=\{z:|z+2|=1\}$.
5. The line $2 x+2 y=1$.
6. The circle $C_{1}\left(\frac{i}{2}\right)=\left\{z:\left|z-\frac{i}{2}\right|=1\right\}$.
7. The circle $C_{1}\left(\frac{3}{2}\right)=\left\{z:\left|z-\frac{3}{2}\right|=1\right\}$.
8. The circle $C_{2}(-1+i)=\{z:|z+1-i|=2\}$.
9. Limits involving $\infty$. The function $f(z)$ is said to have the limit $L$ as $z$ approaches $\infty$, and we write $\lim _{z \rightarrow \infty} f(z)=L$ iff for every $\varepsilon>0$ there exists an $R>0$ such that $f(z) \in D_{\varepsilon}(L)$ (i.e., $|f(z)-L|<\varepsilon$ ) whenever $|z|>R$. Likewise, $\lim _{z \rightarrow z_{0}} f(z)=\infty$
iff for every $R>0$ there exists $\delta>0$ such that $|f(z)|>R$ whenever $z \in D_{\delta}^{*}\left(z_{0}\right)$ (i.e., $0<\left|z-z_{0}\right|<\delta$ ). Use this definition to
(a) show that $\lim _{z \rightarrow \infty} \frac{1}{z}=0$.
(b) show that $\lim _{z \rightarrow 0} \frac{1}{z}=\infty$.
10. A line that carries a charge of $\frac{q}{2}$ coulombs per unit length is perpendicular to the $z$ plane and passes through the point $z_{0}$. The electric field intensity $\mathbf{E}(z)$ at the point $z$ varies inversely as the distance from $z_{0}$ and is directed along the line from $z_{0}$ to $z$. Show that $\mathbf{E}(z)=\frac{k}{\bar{z} \bar{z}_{0}}$, where $k$ is some constant. (In Section 11.11 we show that, in fact, $k=q$ so that actually $\mathbf{E}(z)=\frac{q}{\bar{z}-\bar{z}_{0}}$.)
11. Use the result of Exercise 10 to find the points $z$ where the electric field intensity $\mathbf{E}(z)=0$ given the following conditions.
(a) Three positively charged rods carry a charge of $\frac{q}{2}$ coulombs per unit length and pass through the points $0,1-i$, and $1+i$.
(b) A positively charged rod carrying a charge of $\frac{q}{2}$ coulombs per unit length passes through the point 0 , and positively charged rods carrying a charge of $q$ coulombs per unit length pass through the points $2+i$ and $-2+i$.
12. Show that the reciprocal transformation $w=\frac{1}{z}$ maps the vertical strip given by $0<x<\frac{1}{2}$ onto the region in the right half-plane $\operatorname{Re}(w)>0$ that lies outside the disk $D_{1}(1)=\{w:|w-1|<1\}$.
13. Find the image of the disk $D_{\frac{4}{3}}\left(-\frac{2 i}{3}\right)=\left\{z:\left|z+\frac{2 i}{3}\right|<\frac{4}{3}\right\}$ under $f(z)=\frac{1}{z}$.
14. Show that the reciprocal transformation maps the disk $|z-1|<2$ onto the region that lies exterior to the circle $\left\{w:\left|w+\frac{1}{3}\right|=\frac{2}{3}\right\}$.
15. Find the image of the half-plane $y>\frac{1}{2}-x$ under the mapping $w=\frac{1}{z}$.
16. Show that the half-plane $y<x-\frac{1}{2}$ is mapped onto the disk $|w-1-i|<\sqrt{2}$ by the reciprocal transformation.
17. Find the image of the quadrant $x>1, y>1$ under the mapping $w=\frac{1}{z}$.
18. Show that the transformation $w=\frac{2}{z}$ maps the disk $|\bar{z}-i|<1$ onto the lower half-plane $\operatorname{Im}(w)<-1$.
19. Show that the transformation $w=\frac{2-z}{z}=-1+\frac{2}{z}$ maps the disk $|z-1|<1$ onto the right half-plane $\operatorname{Re}(w)>0$.
20. Show that the parabola $2 x=1-y^{2}$ is mapped onto the cardioid $\rho=1+\cos \phi$ by the reciprocal transformation.
21. Use the definition in Exercise 9 to prove that $\lim _{z \rightarrow \infty} \frac{z+1}{z-1}=1$.
22. Show that $z=x+i y$ is mapped onto the point $\left(\frac{x}{x^{2}+y^{2}+1}, \frac{y}{x^{2}+y^{2}+1}, \frac{x^{2}+y^{2}}{x^{2}+y^{2}+1}\right)$ on the Riemann sphere.
23. Explain how the quantities $+\infty,-\infty$, and $\infty$ differ. How are they similar?

Mathematics Stack Exchange is a question and answer site for people
studying math at any level and professionals in related fields. It only takes a minute to sign up.

Sign up to join this community

Anybody can ask a question

Anybody can answer

The best answers are voted up and rise to the top

## MATHEMATICS

## Find the image of a circle under $w=f(z)=1 / z$

Asked 6 years, 11 months ago Active 4 years, 2 months ago Viewed 9 k times

In my introductory complex analysis class, we've gone over a fair number of examples using the reciprocal function to map a line to a circle. However, we've worked only one example that used a circle as its domain, and my professor left a good portion of the problem undone. To top things off, my book focuses on, wouldn't you know, domains consisting of a line.

In this problem, the domain of $f$ is $C_{1 / 2}(-i / 2)=\{z:|z+i / 2|=1 / 2\}$. Here's how I attempted to sort through it all:

$$
\begin{aligned}
x^{2}+(y+i / 2)^{2} & =\frac{1}{4} \\
\left(\frac{u}{u^{2}+v^{2}}\right)^{2}+\left(\frac{i}{2}-\frac{v}{u^{2}+v^{2}}\right)^{2} & =\frac{1}{4} \\
u^{2}+\left[(i / 2)\left(u^{2}+v^{2}\right)-v\right]^{2} & =\frac{1}{4}\left(u^{2}+v^{2}\right)^{2} \\
u^{2}+v^{2}-v i\left(u^{2}+v^{2}\right)-\frac{1}{4}\left(u^{2}+v^{2}\right)^{2} & =\frac{1}{4}\left(u^{2}+v^{2}\right)^{2} \\
(1-v i)\left(u^{2}+v^{2}\right) & =\frac{1}{2}\left(u^{2}+v^{2}\right)^{2} \\
1-v i & =\frac{1}{2}\left(u^{2}+v^{2}\right) \\
2 & =u^{2}+v^{2}+2 v i \\
1 & =u^{2}+(v+i)^{2}
\end{aligned}
$$

And so the image is itself a circle, centered at $(0,-1)$, with radius 1 .

Does this look okay? Have I been sloppy with any of the notation? I feel like I'm still figuring a lot of this out, so please nitpick.

```
complex-analysis solution-verification
```

Share Cite Improve this question Follow
asked Feb 14 '14 at 3:23


## 3 Answers

| Active | Oldest | Votes |
| :--- | :--- | :--- |

There are various ways to do this. As usual, the more theory you know, the less hard work you have to do. Here are hints for a lo-tech method, followed by a hi-tech method.

First a comment. The method you are attempting is to write a complex variable as two real variables. This being the case,

$$
x^{2}+\left(y+\frac{i}{2}\right)^{2}=\frac{1}{4}
$$

cannot possibly be right, since it is supposed to be a real equation yet it includes $i$. It should be

$$
x^{2}+\left(y+\frac{1}{2}\right)^{2}=\frac{1}{4}
$$

If you make this change and then do more or less what you already tried (but there is also some careless algebra which you will have to fix) it should work.

For the "hi-tech/lo-algebra" method you will have to know the following theorem: the image of a circle through the origin under the map $w=1 / z$ is a line. So, take two points on your circle, say

$$
z=-i \quad \text { and } \quad z=\frac{1}{2}-\frac{1}{2} i
$$

(Not $z=0$ as then $w=\infty$, or to put it in finite terms, $w$ does not exist.) Find $w$ for both of these and then determine the line through the two $w$ values. Good luck!

Share Cite Improve this answer edited Feb 14 '14 at 4:17 Follow
answered Feb 14 '14 at 3:44
David
$\begin{array}{llll}\text { 74.3k } & 7 & 78 & 143\end{array}$

Thanks, David. I started over without the complex numbers and ended up with $(1-v)\left(u^{2}+v^{2}\right)=0$. The latter of these factors can equal 0 only when $u=v=0$, but that can't be, considering how $x$ and $y$ are defined. Therefore, $1-v=0$, which means $v=1$. That leaves us with a horizontal line, i.e., $\{w: \Im(w)=1\} .-$ dmk Feb 14 '14 at 4:12

Also: I tried checking my answer using your hi-tech method, but I had one question: Did you mean


$$
\text { Yes, } z=-i \text {, I'll fix it. Thanks!! - David Feb } 14 \text { '14 at 4:16 }
$$

If the domain consists of all points in the complex plane whose distance from $-i / 2$ is $1 / 2$, then $z=0$ is one such point: $|0-i / 2|=1 / 2$. In such a case, inversion through the unit circle under the mapping $w=f(z)=1 / z$ gives the point at infinity, so your algebra cannot possibly be correct. The image is necessarily a line; moreover, it is a horizontal line that passes through the intersection point(s) of the unit circle and the circle centered at $-i / 2$ with radius $1 / 2$.

Share Cite Improve this answer Follow
answered Feb 14 '14 at 3:28


I noticed the $1 / 0$ thing the other day in class but had forgotten; thanks for reminding me. I had assumed that meant, e.g., if we go out far enough on the line (i.e., the domain), we could get as close as we want to 0 on the circle. Another point of confusion was that my professor had said $w=1 / z$ "maps lines to lines and circles to circles", which was pretty much negated by the first example, when a line got
mapped to a circle. I gradually deduced than either mapped to either. (Correctly? We saw an example mapping a circle to a circle, but you write the image is "necessarily" a line.) - dmk Feb 14 '14 at 3:44

Beyond that, could you please comment on where I went awry in my algebra? Thanks for your feedback.

- dmk Feb 14 '14 at 3:45

On the Riemann sphere $\mathbb{C} \cup\{\infty\}$, a line is a circle. It is a circle with infinite radius. Your professor's statement erroneously implies a distinction where there is none: it should simply read, " $w=1 / z$ maps circles to circles," where 'circle' also includes lines. - heropup Feb 14 '14 at 3:49

Regarding where you calculations went awry, refer to David's reply: once you express the image of the mapping in $\mathbb{R}^{2}$, there should be no more complex numbers in your expression: the circle should be $(x, y)$ such that $x^{2}+(y+1 / 2)^{2}=1 / 4$. - heropup Feb 14 '14 at 3:53

The typical domain point is

$$
z=-\frac{i}{2}+\frac{1}{2} e^{i \theta}
$$

The image of this point is

$$
\frac{1}{-\frac{i}{2}+\frac{1}{2} e^{i \theta}}=\frac{2}{-i+e^{i \theta}}=\frac{2}{\cos \theta-i(1-\sin \theta)} \cdot \frac{\cos \theta+i(1-\sin \theta)}{\cos \theta+i(1-\sin \theta)}
$$

This simplifies to

$$
\frac{\cos \theta}{1-\sin \theta}+i
$$

Verify that the real part of this expression ranges between $\pm \infty$ as $\theta$ moves through $[0,2 \pi]$.

> So the image is the horizontal line through $i$.

Share Cite Improve this answer edited Feb 14 '14 at 4:37 Follow
answered Feb 14 '14 at 4:32
\& ${ }_{38.1}{ }^{\text {MPW }}$
38.1k 126

This looks a lot different from how we'd been doing these problems, so I admit, it took me a while to have a close look at it. I'd forgotten that that's often the precise reason I do like an explanation. So anyway, yeah, I like this; it makes a lot of sense. Thank you! - dmk Feb 24 '14 at 1:58
pythonGuru
Asked 2 yrs ago.

Show that under the transformation $w=1 / z$, the images of the lines $y=x-1$ and $y=0$ are the circle $u^{2}+v^{2}-u-v=0$ and the line $v=0$, respectively. Sketch all four curves, determine corresponding directions along them, and verify the conformality of the mapping at the point $z 0=1$.

1
The equation $f(z)=\frac{1}{z}$, can be written as

$$
\begin{aligned}
u+i v & =\frac{1}{x+i y} \\
x+y i & =\frac{1}{u+i v} \frac{u-i v}{u-i v} \\
x+y i & =\frac{u-i v}{u^{2}+v^{2}} \\
x+y i & =\frac{u}{u^{2}+v^{2}}-i \frac{v}{u^{2}+v^{2}}
\end{aligned}
$$

Compare both sides

$$
x=\frac{u}{u^{2}+v^{2}}, \quad \text { and } \quad y=-\frac{v}{u^{2}+v^{2}}
$$

Substitute the value of $x$ and $y$ in the equation $y=x-1$.

$$
\begin{aligned}
-\frac{v}{u^{2}+v^{2}} & =\frac{u}{u^{2}+v^{2}}-1 \\
-\frac{v}{u^{2}+v^{2}} & =\frac{u-u^{2}-v^{2}}{u^{2}+v^{2}} \\
-v & =u-u^{2}-v^{2} \\
u^{2}+v^{2}-u-v & =0
\end{aligned}
$$

The graph in the $z$-plane and the $w$-plane are shown below, respectively.


3


4
Substitute the value of $y$ in the equation $y=0$.

$$
\begin{aligned}
-\frac{v}{u^{2}+v^{2}} & =0 \\
v & =0
\end{aligned}
$$

The graph in the $z$-plane and the $w$-plane are shown below, respectively.


6


7
The point $z_{0}=1$ is the point of intersection of the equation $y=x-1$ and $y=0$ which is mapped to the point $w_{0}=1$, the point of intersection of the equation $u^{2}+v^{2}-u-v=0$ and $v=0$. The angle from $y=x-1$ to $y=0$ at the point $z_{0}=1$ is $\frac{\pi}{4}$.


The equation $f(z)=\frac{1}{z}$, can be written as

$$
\begin{aligned}
& u+i v=\frac{1}{x+y i} \\
& u+i v=\frac{1}{x+i y} \frac{x-i y}{x-i y} \\
& u+i v=\frac{x-i y}{x^{2}+y^{2}}
\end{aligned}
$$

Substitute the value of $y=x-1$ in the above equation.

$$
\begin{aligned}
& u+i v=\frac{x-i(x-1)}{x^{2}+(x-1)^{2}} \\
& u+i v=\frac{x}{x^{2}+(x-1)^{2}}-i \frac{x-1}{x^{2}+(x-1)^{2}}
\end{aligned}
$$

Compare both sides

$$
u=\frac{x}{x^{2}+(x-1)^{2}}, \quad \text { and } \quad v=-\frac{x-1}{x^{2}+(x-1)^{2}}
$$

Now,

$$
\begin{aligned}
\frac{\mathrm{d} v}{\mathrm{~d} u} & =\frac{\frac{\mathrm{d} v}{\mathrm{~d} x}}{\frac{\mathrm{~d} u}{\mathrm{~d} x}} \\
& =\frac{\frac{\mathrm{d}}{\mathrm{~d} x}\left[-\frac{x-1}{x^{2}+(x-1)^{2}}\right]}{\frac{\mathrm{d}}{\mathrm{~d} x}\left[\frac{x}{x^{2}+(x-1)^{2}}\right]} \\
& =\frac{-\frac{-2 x^{2}+4 x-1}{\left(2 x^{2}-2 x+1\right)^{2}}}{\frac{-2 x^{2}+1}{\left(2 x^{2}-2 x+1\right)^{2}}} \\
& =-\frac{-2 x^{2}+4 x-1}{-2 x^{2}+1}
\end{aligned}
$$

At the point $\binom{x, y}{1,0}$ is

$$
\frac{\mathrm{d} v}{\mathrm{~d} u}=-\frac{-2 \cdot 0+4 \cdot 0-1}{-2 \cdot 0+1}=1
$$

9
So, the angle from $u^{2}+v^{2}-u-v=0$ to $v=0$ at the point $w_{0}=1$ is

$$
\begin{aligned}
\tan \theta & =1 \\
\theta & =\tan ^{-1}(1) \\
\theta & =\frac{\pi}{4}
\end{aligned}
$$

Hence, the angle is preserved from the $z$-plane to the $w$-plane at the point $z_{0}=1$.

$$
\frac{\mathrm{d} v}{\mathrm{~d} u}=1
$$

## Related questions:

Find the image of the region $x>1, y>0$ under the transformation $w=1 / z$.
2 answers

Find the image of the semi-infinite strip $x>0,0<y<1$ when $w=i / z$. Sketch the strip and its image.

2 answers

Find the image of the semi-infinite strip $x \geq 0,0 \leq y \leq \pi$ under the transformation $w$ $=\exp z$, and label corresponding portions of the boundaries.

2 answers

## Get step-by-step explanations for your tough textbook problems <br>  <br> Use Chegg Study

## About

## Shop

## Careers

Meet Our Experts

Academic Integrity

Help Center

Terms of Use

Privacy Policy

Inappropriate Content Policy

Do Not Sell My Info

Advertise

Site Map

### 18.04 Practice problems exam 1, Spring 2018

## Problem 1. Complex arithmetic

(a) Find the real and imaginary part of $\frac{z+2}{z-1}$.
(b) Solve $z^{4}-i=0$.
(c) Find all possible values of $\sqrt{\sqrt{i}}$.
(d) Express $\cos (4 x)$ in terms of $\cos (x)$ and $\sin (x)$.
(e) When does equality hold in the triangle inequality $\left|z_{1}+z_{2}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|$ ?
(f) Draw a picture illustrating the polar coordinates of $z$ and $1 / z$.

Problem 2. Functions
(a) Show that $\sinh (z)=-i \sin (i z)$.
(b) Give the real and imaginary part of $\cos (z)$ in terms of $x$ and $y$ using regular and hyperbolic sin and cos.
(c) Is it true that $\left|a^{b}\right|=|a|^{|b|}$ ?

Problem 3. Mappings
(a) Show that the function $f(z)=\frac{z-i}{z+i}$ maps the upper half plane to the unit disk.
(i) Show it maps the real axis to the unit circle.
(ii) Show it maps $i$ to 0 .
(iii) Conclude that the upper half plane is mapped to the unit disk.
(b) Show that the function $f(z)=\frac{z+2}{z-1}$ maps the unit circle to the line $x=-1 / 2$.

Problem 4. Analytic functions
(a) Show that $f(z)=\mathrm{e}^{z}$ is analytic using the Cauchy Riemann equations.
(b) Show that $f(z)=\bar{z}$ is not analytic.
(c) Give a region in the $z$-plane for which $w=z^{3}$ is a one-to-one map onto the entire $w$-plane.
(d) Choose a branch of $z^{1 / 3}$ and a region of the $z$-plane where this branch is analytic. Do this so that the image under $z^{1 / 3}$ is contained in your region from part (c).

Problem 5. Line integrals
(a) Compute $\int_{C} x d z$, where $C$ is the unit square.
(b) Compute $\int_{C} \frac{1}{|z|} d z$, where $C$ is the unit circle.
(c) Compute $\int_{C} z \cos \left(z^{2}\right) d z$, where $C$ is the unit circle.
(d) Draw the region $\mathbf{C}-\{x+i \sin (x)$ for $x \geq 0\}$. Is this region simply connected? Could you define a branch of $\log$ on this region?
(e) Compute $\int_{C} \frac{z^{2}}{z^{4}-1}$ over the circle of radius 3 with center 0 .
(f) Does $\int_{C} \frac{\mathrm{e}^{z}}{z^{2}} d z=0$ ?. Here $C$ is a simple closed curve.
(g) Compute $\int_{-\infty}^{\infty} \frac{1}{x^{4}+16} d x$.

## Problem 6.

Suppose $f(z)$ is entire and $|f(z)|>1$ for all $z$. Show that $f$ is a constant.

## Problem 7.

Suppose $f(z)$ is analytic and $|f|$ is constant on the disk $\left|z-z_{0}\right| \leq r$. Show that $f$ is constant on the disk.

## Extra problems from pset 4

Problem 8. (a) Let $f(z)=\mathrm{e}^{\cos (z)} z^{2}$. Let $A$ be the disk $|z-5| \leq 2$. Show that $f(z)$ attains both its maximum and minimum modulus in $A$ on the circle $|z-5|=2$.
Hint: Consider $1 / f(z)$.
(b) Suppose $f(z)$ is entire. Show that if $f^{(4)}(z)$ is bounded in the whole plane then $f(z)$ is a polynomial of degree at most 4.
(c) The function $f(z)=1 / z^{2}$ goes to 0 as $z \rightarrow \infty$, but it is not constant. Does this contradict Liouville's theorem?

## Problem 9.

Show $\int_{0}^{\pi} \mathrm{e}^{\cos \theta} \cos (\sin (\theta)) d \theta=\pi$. Hint, consider $\mathrm{e}^{z} / z$ over the unit circle.

## Problem 10.

(a) Suppose $f(z)$ is analytic on a simply connected region $A$ and $\gamma$ is a simple closed curve in $A$.. Fix $z_{0}$ in $A$, but not on $\gamma$. Use the Cauchy integral formulas to show that

$$
\int_{\gamma} \frac{f^{\prime}(z)}{z-z_{0}} d z=\int_{\gamma} \frac{f(z)}{\left(z-z_{0}\right)^{2}} d z
$$

(b) Challenge: Redo part (a), but drop the assumption that $A$ is simply connected.

Problem 11.
(a) Compute $\int_{C} \frac{\cos (z)}{z} d z$, where $C$ is the unit circle.
(b) Compute $\int_{C} \frac{\sin (z)}{z} d z$, where $C$ is the unit circle.
(c) Compute $\int_{C} \frac{z^{2}}{z-1} d z$, where $C$ is the circle $|z|=2$.
(d) Compute $\int_{C} \frac{\mathrm{e}^{z}}{z^{2}} d z$, where $C$ is the circle $|z|=1$.
(e) Compute $\int_{C} \frac{z^{2}-1}{z^{2}+1} d z$, where $C$ is the circle $|z|=2$.
(f) Compute $\int_{C} \frac{1}{z^{2}+z+1} d z$ where $C$ is the circle $|z|=2$.

Problem 12.
Suppose $f(z)$ is entire and $\lim _{z \rightarrow \infty} \frac{f(z)}{z}=0$. Show that $f(z)$ is constant.

You may use Morera's theorem: if $g(z)$ is analytic on $A-\left\{z_{0}\right\}$ and continuous on $A$, then $f$ is analytic on $A$.

MIT OpenCourseWare
https://ocw.mit.edu

### 18.04 Complex Variables with Applications

Spring 2018

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.

### 18.04 Practice problems exam 1, Spring 2018 Solutions

Problem 1. Complex arithmetic
(a) Find the real and imaginary part of $\frac{z+2}{z-1}$.
(b) Solve $z^{4}-i=0$.
(c) Find all possible values of $\sqrt{\sqrt{ }}$.
(d) Express $\cos (4 x)$ in terms of $\cos (x)$ and $\sin (x)$.
(e) When does equality hold in the triangle inequality $\left|z_{1}+z_{2}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|$ ?
(f) Draw a picture illustrating the polar coordinates of $z$ and $1 / z$.

Answers. (a) $\frac{x+2+i y}{x-1+i y} \cdot \frac{x-1-i y}{x-1-i y}=\frac{(x+2)(x-1)+y^{2}}{(x-1)^{2}+y^{2}}+i \frac{-3 y}{(x-1)^{2}+y^{2}}$
(b) $i=\mathrm{e}^{i(\pi / 2+2 n \pi)}$. So $z=i^{1 / 4}=\mathrm{e}^{i(\pi / 8+n \pi / 2)}= \pm 1 \mathrm{e}^{i \pi / 8}, \pm i \mathrm{e}^{i \pi / 8}$.
(c) Same answer as part (b).
(d) Euler:

$$
\begin{aligned}
\cos (4 x)+i \sin (4 x) & =\mathrm{e}^{i 4 x}=(\cos (x)+i \sin (x))^{4} \\
& =\cos ^{4}(x)-6 \cos ^{2}(x) \sin ^{2}(x)+\sin ^{4}(x)+i\left(4 \cos ^{3}(x) \sin (x)-4 \cos (x) \sin ^{3}(x)\right) .
\end{aligned}
$$

Therefore, $\cos (4 x)=\cos ^{4}(x)-6 \cos ^{2}(x) \sin ^{2}(x)+\sin ^{4}(x)$.
(e) When $z_{1}$ and $z_{2}$ have the same argument, i.e. are on the same ray from the origin.
(f)


Problem 2. Functions
(a) Show that $\sinh (z)=-i \sin (i z)$.

Solution: $-i \sin (i z)=-i \frac{\mathrm{e}^{i \cdot i z}-\mathrm{e}^{-i \cdot i z}}{2 i}=-\frac{\mathrm{e}^{-z}-\mathrm{e}^{z}}{2}=\sinh (z)$. QED
(b) Give the real and imaginary part of $\cos (z)$ in terms of $x$ and $y$ using regular and hyperbolic sin and cos.

Solution: We calculate this using exponentials.

$$
\begin{aligned}
\cos (z) & =\frac{\mathrm{e}^{i z}+\mathrm{e}^{-i z}}{2}=\frac{\mathrm{e}^{-y+i x}+\mathrm{e}^{y-i x}}{2} \\
& =\frac{\mathrm{e}^{-y} \mathrm{e}^{i x}+\mathrm{e}^{y} \mathrm{e}^{-i x}}{2} \\
& =\frac{\mathrm{e}^{-y}(\cos (x)+i \sin (x))+\mathrm{e}^{y}(\cos (x)-i \sin (x))}{2} \\
& =\frac{\mathrm{e}^{-y}+\mathrm{e}^{y}}{2} \cos (x)+i \frac{\mathrm{e}^{-y}-\mathrm{e}^{y}}{2} \sin (x) \\
& =\cos (x) \cosh (y)-i \sin (x) \sinh (y)
\end{aligned}
$$

Alternatively using the cosine addition formula:

$$
\cos (z)=\cos (x+i y)=\cos (x) \cos (i y)-\sin (x) \sin (i y)=\cos (x) \cosh (y)-i \sin (x) \sinh (y) .
$$

(c) Is it true that $\left|a^{b}\right|=|a|^{|b|}$ ?

Solution: No: here's a counterexample: $\left|\mathrm{e}^{i}\right|=1$, but $|e|^{|i|}=\mathrm{e}^{1}=\mathrm{e}$.

Problem 3. Mappings
(a) Show that the function $f(z)=\frac{z-i}{z+i}$ maps the upper half plane to the unit disk.
(i) Show it maps the real axis to the unit circle.
(ii) Show it maps i to 0 .
(iii) Conclude that the upper half plane is mapped to the unit disk.

Solution: (i) If $z$ is real then $z-i=\overline{z+i}$, so numerator and denominator have the same norm, i.e. the fraction has norm 1. QED
(ii) Clearly $f(i)=0$.
(iii) The boundary of the half plane is mapped to the boundary of the disk and a point in the interior of the half plane is mapped to the interior of the disk. This is enough to conclude that the image of the half plane is inside the disk.
Since it's easy to invert $u=f(z): z=i \frac{1+u}{1-u}$. It is easy to see that the map is in fact one-to-one and onto the disk.
(b) Show that the function $f(z)=\frac{z+2}{z-1}$ maps the unit circle to the line $x=-1 / 2$.

Solution: We will learn good ways to manipulate expressions like this later in the course. Here we can do a direct calculation. Let $z=\mathrm{e}^{i \theta}=\cos (\theta)+i \sin (\theta)$ be a point on the unit circle. Then

$$
\begin{aligned}
f(z) & =\frac{\cos (\theta)+2+i \sin (\theta)}{\cos (\theta)-1+i \sin (\theta)} \cdot \frac{\cos (\theta)-1-i \sin (\theta)}{\cos (\theta)-1-i \sin (\theta)} \\
& =\frac{(\cos (\theta)+2)(\cos (\theta)-1)+\sin ^{2}(\theta)+i()}{(\cos (\theta)-1)^{2}+\sin ^{2}(\theta)} \\
& =\frac{-1+\cos (\theta)+i(\cdots)}{2-2 \cos (\theta)} \\
& =-\frac{1}{2}+i \frac{(\cdots)}{2-2 \cos (\theta)}
\end{aligned}
$$

Here we left the imaginary part uncomputed because the question is to show that the real part is $-1 / 2$. Which we did!

Problem 4. Analytic functions
(a) Show that $f(z)=\mathrm{e}^{z}$ is analytic using the Cauchy Riemann equations.

Solution: $\mathrm{e}^{z}=\mathrm{e}^{x} \cos (y)+i \mathrm{e}^{x} \sin (y)$. Call the real and imaginary parts $u$ and $v$ respectively. Putting the partials in a matrix we have

$$
\left(\begin{array}{cc}
u_{x} & u_{y} \\
v_{x} & v_{y}
\end{array}\right)=\left(\begin{array}{cc}
\mathrm{e}^{x} \cos (y) & -\mathrm{e}^{x} \sin (y) \\
\mathrm{e}^{x} \sin (y) & \mathrm{e}^{x} \cos (y)
\end{array}\right) .
$$

We see that $u_{x}=v_{y}$ and $u_{y}=-v_{x}$. Thus we have verified the Cauchy Riemann equations. So, $f(z)$ is analytic.
(b) Show that $f(z)=\bar{z}$ is not analytic.

Solution: $f(z)=x-i y=u+i v$, where $u=x$ and $v=-y$. Taking partials

$$
\left(\begin{array}{ll}
u_{x} & u_{y} \\
v_{x} & v_{y}
\end{array}\right)=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

We see that $u_{x} \neq v_{y}$. So the Cauchy Riemann equations are not satisfied and so, $f(z)$ is not analytic.
(c) Give a region in the $z$-plane for which $w=z^{3}$ is a one-to-one map onto the entire $w$-plane.

Solution: Since $z^{3}$ triples arguments, we divide the plane into thirds and pick one third. We've chosen the shaded region in the figure below.


The region includes the the positive $x$-axis but not the dashed line.
(d) Choose a branch of $z^{1 / 3}$ and a region of the $z$-plane where this branch is analytic. Do this so that the image under $z^{1 / 3}$ is contained in your region from part (c).
Solution: We choose the branch of $\arg$ with $0<\arg (z)<2 \pi$. So, the plane has a branch cut along the nonnegative real axis. Under $w=z^{1 / 3}$ the image points all have $0<\arg (w)<2 \pi / 3$, as required by the problem.


Problem 5. Line integrals
(a) Compute $\int_{C} x d z$, where $C$ is the unit square.

Solution: First note that as a function $x$ means $\operatorname{Re}(z)$. We do the integral for each of the four sides separately.

$\gamma_{1}: \gamma_{1}(t)=t$, with $0 \leq t \leq 1$. So, $\int_{\gamma_{1}} x d z=\int_{0}^{1} t d t=1 / 2$.
$\gamma_{2}: \gamma_{2}(t)=1+i t$, with $0 \leq t \leq 1$. So, $\int_{\gamma_{2}} x d z=\int_{0}^{1} 1 i d t=i$.
$\gamma_{3}: \gamma_{3}(t)=1-t+i$, with $0 \leq t \leq 1$. So, $\int_{\gamma_{3}} x d z=\int_{0}^{1}(1-t)(-d t)=-1 / 2$.
$\gamma_{4}: \gamma_{3}(t)=(1-t) i$, with $0 \leq t \leq 1$. So, $\int_{\gamma_{4}} x d z=\int_{0}^{1} 0(-d t)=0$.
Addding the together: the integral over the square is $i$.
(b) Compute $\int_{C} \frac{1}{|z|} d z$, where $C$ is the unit circle.

Solution: Parametrize the circle, as usual, by $\gamma(\theta)=\mathrm{e}^{i \theta}$. Since $|\gamma(\theta)|=1$ the integral is

$$
\int_{C} \frac{1}{|z|} d z=\int_{0}^{2 \pi} i \mathrm{e}^{i \theta} d \theta=0
$$

(c) Compute $\int_{C} z \cos \left(z^{2}\right) d z$, where $C$ is the unit circle.

Solution: Since $z \cos \left(z^{2}\right)$ is entire, it is analytic on and inside the closed curve $C$. Therefore by Cauchy's theorem, the integral is 0 .
(d) Draw the region $\mathbf{C}-\{x+i \sin (x)$ for $x \geq 0\}$. Is this region simply connected? Could you define a branch of $\log$ on this region?

Solution: Yes, the region is simply connected. Yes, you can define a branch of $\log$ on this region: To define a branch of log you have to have a region where the argument is well defined and continuous. You can do this as long as the cut blocks any path that circles the origin. The figure below illustrates values of $\arg (z)$ at a few points in the region.

(e) Compute $\int_{C} \frac{z^{2}}{z^{4}-1}$ over the circle of radius 3 with center 0 .

Solution: The fourth roots of 1 are $\pm 1, \pm i$. Thus,

$$
f(z)=\frac{z^{2}}{z^{4}-1}=\frac{z^{2}}{(z-1)(z+1)(z-i)(z+i)} .
$$

Since the curve contains all four roots we need to write it as four loops each containing just one of the roots. Then we use Cauchy's formula to compute the integral over each loop.
Loop around 1: Let $g(z)=\frac{z^{2}}{(z+1)(z-i)(z+i)}$. The integral of $f$ over this loop equals $2 \pi i g(1)=$ $\pi i / 2$.
Loop around -1: The integral of $f$ over this loop is $-\pi i / 2$.
Loop around i: The integral of $f$ over this loop is $\pi / 2$.
Loop around -i: The integral of $f$ over this loop is $-\pi / 2$.
Summing all 4 contributions we get 0 .
(f) Does $\int_{C} \frac{\mathrm{e}^{z}}{z^{2}} d z=0$ ?. Here $C$ is a simple closed curve.

Solution: Not always. We know $f(z)=\mathrm{e}^{z}$ is entire. So, if $C$ goes around 0 then, by Cauchy's formula for derivatives

$$
\int \frac{f(z)}{z^{2}} d z=2 \pi i f^{\prime}(0)=2 \pi i
$$

If $C$ does not go around 0 then the integral is 0 .
(g) Compute $\int_{-\infty}^{\infty} \frac{1}{x^{4}+16} d x$.

Solution: Let $f(z)=1 /\left(z^{4}+16\right)$ and let $I$ be the integral we want to compute. The trick is to integrate $f$ over the closed contour $C_{1}+C_{R}$ shown, and then show that the contribution of $C_{R}$ to this integral vanishes as $R$ goes to $\infty$.


The 4 singularities of $f(z)$ are $2 \mathrm{e}^{i(\pi / 4+n \pi / 2)}= \pm \sqrt{2} \pm i \sqrt{2}$. The ones inside the contour are $2 \mathrm{e}^{i \pi / 4}=$ $\sqrt{2}+i \sqrt{2}, 2 \mathrm{e}^{3 i \pi / 4}=-\sqrt{2}+i \sqrt{2}$. As usual we break $C_{1}+C_{R}$ into two loops, one surrounding each singularity and use Cauchy's formula to compute the integral over each loop separately. Factoring, we have

$$
f(z)=\frac{1}{z^{4}+16}=\frac{1}{(z-(\sqrt{2}+i \sqrt{2}))(z-(\sqrt{2}-i \sqrt{2}))(z-(-\sqrt{2}+i \sqrt{2}))(z-(-\sqrt{2}-i \sqrt{2}))} .
$$

Loop around $\sqrt{2}+i \sqrt{2}$ : Let $f_{1}(z)=\frac{1}{(z-(\sqrt{2}-i \sqrt{2}))(z-(-\sqrt{2}+i \sqrt{2}))(z-(-\sqrt{2}-i \sqrt{2}))}$. By Cauchy's integral formula the integral is $2 \pi i f_{1}(\sqrt{2}+i \sqrt{2})=\frac{\sqrt{2} \pi(1-i)}{32}$.
Loop around $-1+i$ : the integral is $\frac{\sqrt{2} \pi(1+i)}{32}$.
Summing, the integral around $C_{1}+C_{R}$ is $\sqrt{2} \pi / 16$.
Now we'll look at $C_{1}$ and $C_{r}$ separately:
Parametrize $C_{1}$ by $\gamma(x)=x$, with $-R \leq x \leq R$. So

$$
\int_{C_{1}} f(z) d z=\int_{-R}^{R} \frac{1}{x^{4}+16} d x .
$$

This goes to the $I$ as $R \rightarrow \infty$.
We parametrize $C_{R}$ by $\gamma(\theta)=R \mathrm{e}^{i \theta}$, with $0 \leq \theta \leq \pi$. So

$$
\int_{C_{R}} f(z) d z=\int_{0}^{\pi} \frac{1}{R^{4} \mathrm{e}^{4 i \theta}+16} i \mathrm{Re}^{i \theta} d \theta /
$$

By the triangle inequality, if $R>1$

$$
\left|\int_{C_{R}} f(z) d z\right| \leq \int_{0}^{\pi} \frac{R}{R^{4}-16} d \theta=\frac{\pi R}{R^{4}-16}
$$

Clearly this goes to 0 as $R$ goes to infinity.
Thus, the integral over the contour $C_{1}+C_{R}$ goes to $I$ as $R$ gets large. But this integral always has the same value $\sqrt{2} \pi / 16$. We have shown that $I=\sqrt{2} \pi / 16$.
As a sanity check, we note that our answer is real and positive as it needs to be.

## Problem 6.

Suppose $f(z)$ is entire and $|f(z)|>1$ for all $z$. Show that $f$ is a constant.
Answer. Since $|f(z)|>1$ we know $f$ is never 0 . Therefore $1 / f(z)$ is entire and $|1 / f(z)|<1$. Being entire and bounded it is constant by Liouville's theorem.

## Problem 7.

Suppose $f(z)$ is analytic and $|f|$ is constant on the disk $\left|z-z_{0}\right| \leq r$. Show that $f$ is constant on the disk.
Answer. This follows from the maximum modulus principle. Since $|f|$ is constant on the disk, its maximum modulus does not occur only on the boundary. Therefore it must be constant.

## Extra problems from pset 4

Problem 8. (a) Let $f(z)=\mathrm{e}^{\cos (z)} z^{2}$. Let $A$ be the disk $|z-5| \leq 2$. Show that $f(z)$ attains both its maximum and minimum modulus in $A$ on the circle $|z-5|=2$.
Hint: Consider $1 / f(z)$.
Solution: Since $f(z)$ is analytic on and inside the disk, the maximum modulus principle tells us it attains its maximum modulus on the boundary.
Since $\mathrm{e}^{w}$ is never 0 and $z^{2}$ is not zero anywhere in $A$ we know that $1 / f(z)$ is analytic on and inside the disk. Therefore it attains its maximum modulus on the boundary. But the point where $1 /|f(z)|$ is maximized is the point where $|f(z)|$ is minimized.
(b) Suppose $f(z)$ is entire. Show that if $f^{(4)}(z)$ is bounded in the whole plane then $f(z)$ is a polynomial of degree at most 4 .
Solution: By the maximum modulus principle $f^{(4)}(z)$ is a constant. Integrating a constant 4 times leads to a polynomial of degree a most 4.
(c) The function $f(z)=1 / z^{2}$ goes to 0 as $z \rightarrow \infty$, but it is not constant. Does this contradict Liouville's theorem?

Solution: No, Liouville's theorem requires the function be entire. $f(z)$ has a singularity at the origin, so it is not entire.

Problem 9 .
Show $\int_{0}^{\pi} \mathrm{e}^{\cos \theta} \cos (\sin (\theta)) d \theta=\pi$. Hint, consider $\mathrm{e}^{z} / z$ over the unit circle.
Solution: (Follow the hint.) Parametrize the unit circle as $\gamma(\theta)=\mathrm{e}^{i \theta}$, with $0 \leq \theta \leq 2 \pi$. So,

$$
\begin{aligned}
\int_{\gamma} \frac{\mathrm{e}^{z}}{z} d z & =\int_{0}^{2 \pi} \frac{\mathrm{e}^{\cos \theta+i \sin \theta}}{\mathrm{e}^{i \theta}} i \mathrm{e}^{i \theta} d \theta=i \int_{0}^{2 \pi} \mathrm{e}^{\cos \theta+i \sin } d \theta \\
& =i \int_{0}^{2 \pi} \mathrm{e}^{\cos \theta}(\cos (\sin \theta)+i \sin (\sin \theta)) d \theta=\int_{0}^{2 \pi} \mathrm{e}^{\cos \theta}(i \cos (\sin \theta)-\sin (\sin \theta)) d \theta .
\end{aligned}
$$

This is close to what we want. Let's use Cauchy's integral formula to evaluate it and then extract the
value we need. By Cauchy the integral is $2 \pi i e^{0}=2 \pi i$. So,

$$
\int_{0}^{2 \pi} \mathrm{e}^{\cos \theta}(i \cos (\sin \theta)-\sin (\sin \theta)) d \theta=2 \pi i
$$

Taking the imaginary part we have

$$
\int_{0}^{2 \pi} \mathrm{e}^{\cos \theta} \cos (\sin \theta) d \theta=2 \pi
$$

This integral is $2 \pi$, while our integral is supposed to be $\pi$. But, by symmetry ours is half the above. (It might be easier to see this if you use the limits $[-\pi, \pi]$ instead of $[0,2 \pi]$.)
So, we have shown that the integral is $\pi$.

## Problem 10.

(a) Suppose $f(z)$ is analytic on a simply connected region $A$ and $\gamma$ is a simple closed curve in A.. Fix $z_{0}$ in $A$, but not on $\gamma$. Use the Cauchy integral formulas to show that

$$
\int_{\gamma} \frac{f^{\prime}(z)}{z-z_{0}} d z=\int_{\gamma} \frac{f(z)}{\left(z-z_{0}\right)^{2}} d z .
$$

Since $A$ is simply connected we know $f$ and $f^{\prime}$ are analytic on and inside $\gamma$. Therefore we can use Cauchy's formulas.

$$
\begin{array}{rlr}
\int_{\gamma} \frac{f^{\prime}(z)}{z-z_{0}} d z & =2 \pi i f^{\prime}\left(z_{0}\right) & \quad \text { (by Cauchy's integral formula.) } \\
\int_{\gamma} \frac{f(z)}{\left(z-z_{0}\right)^{2}} d z . & =2 \pi i f^{\prime}\left(z_{0}\right) & \text { (by Cauchy's formula for derivatives.) }
\end{array}
$$

These are the same, so we are done.
(b) Challenge: Redo part (a), but drop the assumption that $A$ is simply connected.

Let $g(z)=\frac{f(z)}{z-z_{0}} . g$ is analytic on a neighborhood of $\gamma$. Note: $g^{\prime}(z)=\frac{f^{\prime}(z)}{z-z_{0}}-\frac{f(z)}{\left(z-z_{0}\right)^{2}}$. So,

$$
\int_{\gamma} \frac{f^{\prime}(z)}{z-z_{0}} d z-\int_{\gamma} \frac{f(z)}{\left(z-z_{0}\right)^{2}} d z=\int_{\gamma} g^{\prime}(z) d z=0 .
$$

It equals 0 because the integral of a derivative around a closed curve is 0 . So, the two integrals on the left side are equal.

## Problem 11.

(a) Compute $\int_{C} \frac{\cos (z)}{z} d z$, where $C$ is the unit circle.

Solution: $2 \pi i \cos (0)=2 \pi i$.
(b) Compute $\int_{C} \frac{\sin (z)}{z} d z$, where $C$ is the unit circle.

Solution: $2 \pi i \sin (0)=0$.
(c) Compute $\int_{C} \frac{z^{2}}{z-1} d z$, where $C$ is the circle $|z|=2$.

Solution: $\left.2 \pi i z^{2}\right|_{z=1}=2 \pi i$.
(d) Compute $\int_{C} \frac{\mathrm{e}^{z}}{z^{2}} d z$, where $C$ is the circle $|z|=1$.

Solution: $\left.2 \pi i \frac{d \mathrm{e}^{z}}{d z}\right|_{z=0}=2 \pi i$.
(e) Compute $\int_{C} \frac{z^{2}-1}{z^{2}+1} d z$, where $C$ is the circle $|z|=2$.

Solution: Singularities are at $\pm i$.

$$
\text { Integral }=2 \pi i \frac{-2}{2 i}+2 \pi i \frac{-2}{-2 i}=0 .
$$

(f) Compute $\int_{C} \frac{1}{z^{2}+z+1} d z$ where $C$ is the circle $|z|=2$.

Solution: There are two roots. Splitting the contour as we've done several times leads to a total integral of 0 .

## Problem 12.

Suppose $f(z)$ is entire and $\lim _{z \rightarrow \infty} \frac{f(z)}{z}=0$. Show that $f(z)$ is constant.
You may use Morera's theorem: if $g(z)$ is analytic on $A-\left\{z_{0}\right\}$ and continuous on $A$, then $f$ is analytic on $A$.
Solution: Let $g(z)=\frac{f(z)-f(0)}{z}$. Since $g(z)$ is analytic on $\mathbf{C}-\{0\}$ and continuous on $\mathbf{C}$ it is analytic on all of $\mathbf{C}$, by Morera's theorem

We claim $g(z) \equiv 0$.
Suppose not, then we can pick a point $z_{0}$ with $g\left(z_{0}\right) \neq 0$. Since $g(z)$ goes to 0 as $|z|$ gets large we can pick $R$ large enough that $|g(z)|<\left|g\left(z_{0}\right)\right|$ for all $|z|=R$. But this violates the maximum modulus theorem, which says that the maximum modulus of $g(z)$ on the disk $|z| \leq R$ occurs on the circle $|z|=R$. This disaster means our assumption that $g(z) \neq 0$ was wrong. We conclude $g(z) \equiv 0$ as claimed.

This means that $f(z)=f\left(z_{0}\right)$ for all $z$, i.e. $f(z)$ is constant.

MIT OpenCourseWare
https://ocw.mit.edu

### 18.04 Complex Variables with Applications

Spring 2018

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.

### 18.04 Practice problems exam 2, Spring 2018

Problem 1. Harmonic functions
(a) Show $u(x, y)=x^{3}-3 x y^{2}+3 x^{2}-3 y^{2}$ is harmonic and find a harmonic conjugate.
(b) Find all harmonic functions $u$ on the unit disk such that $u(1 / 2)=2$ and $u(z) \geq 2$ for all $z$ in the disk.
(c) The temperature of the boundary of the unit disk is maintained at $T=1$ in the first quadrant, $T=2$ in the second quadrant, $T=3$ in the third quadrant and $T=4$ in the fourth quadrant. What is the temperature at the center of the disk
(d) Show that if $u$ and $v$ are conjugate harmonic functions then $u v$ is harmonic.
(e) Show that if $u$ is harmonic then $u_{x}$ is harmonic.
(f) Show that if $u$ is harmonic and $u^{2}$ is harmonic the $u$ is constant.
(We always assume harmonic functions are real valued.)

## Problem 2.

Let $f(z)=\frac{1}{(z-1)(z-3)}$. Find Laurent series for $f$ on each of the 3 annular regions centered at $z=0$ where $f$ is analytic.

## Problem 3.

Find the first few terms of the Laurent series around 0 for the following.
(a) $f(z)=z^{2} \cos (1 / 3 z)$ for $0<|z|$.
(b) $f(z)=\frac{1}{\mathrm{e}^{z}-1}$ for $0<|z|<R$. What is $R$ ?

## Problem 4.

What is the annulus of convergence for $\sum_{n=-\infty}^{\infty} \frac{z^{n}}{2^{|n|}}$.

## Problem 5.

Find and classify the isolated singularities of each of the following. Compute the residue at each such singularity.
(a) $f_{1}(z)=\frac{z^{3}+1}{z^{2}(z+1)}$
(b) $f_{2}(z)=\frac{1}{\mathrm{e}^{z}-1}$
(c) $f_{3}(z)=\cos (1-1 / z)$

## Problem 6.

(a) Find a function $f$ that has a pole of order 2 at $z=1+i$ and essential singularies at $z=0$ and $z=1$.
(b) Find a function $f$ that has a removable singularity at $z=0$, a pole of order 6 at $z=1$ and an essential singularity at $z=i$.

## Problem 7.

True or false. If true give an argument. If false give a counterexample
(a) If $f$ and $g$ have a pole at $z_{0}$ then $f+g$ has a pole at $z_{0}$.
(b) If $f$ and $g$ have a pole at $z_{0}$ and both have nonzero residues the $f g$ has a pole at $z_{0}$ with a nonzero residue.
(c) If $f$ has an essential singularity at $z=0$ and $g$ has a pole of finite order at $z=0$ the $f+g$ has an essential singularity at $z=0$.
(d) If $f(z)$ has a pole of order $m$ at $z=0$ then $f\left(z^{2}\right)$ has a pole of order $2 m$

## Problem 8.

Find the Laurent series for each of the following.
(a) $1 / \mathrm{e}^{(1-z)}$ for $1<|z|$.

Problem 9.
Let $h(z)=\frac{1}{\sin (z)}-\frac{1}{z}+\frac{2 z}{z^{2}-\pi^{2}}$ in the disk $|z|<2 \pi$.
(a) Show that all the apparent singularities are removable.
(b) Find the first 4 terms of the Taylor series around $z=0$.

Problem 10.
Find the residue at $\infty$ of each of the following.
(a) $f(z)=\mathrm{e}^{z}$
(b) $f(z)=\frac{z-1}{z+1}$.

Problem 11.
Use the following steps to sketch the stream lines for the flow with complex potential $\Phi(z)=z+$ $\log (z-i)+\log (z+i)$
(i) Identify the components, i.e. sources, sinks, etc of the flow.
(ii) Find the stagnation points.
(iii) Sketch the flow near each of the sources.
(iv) Sketch the flow far from the sources.
(v) Tie the picture together.

Problem 12.
Compute the following definite integrals
(a) $\int_{-\pi}^{\pi} \frac{1}{1+\sin ^{2}(\theta)} d \theta$. (Solution: $\pi \sqrt{2}$ )
(b) $\int_{-\infty}^{\infty} \frac{x}{\left(x^{2}+4 x+13\right)^{2}} d x$. (Solution: $-\pi / 27$ )
(c) p.v. $\int_{-\infty}^{\infty} \frac{x \sin (x)}{1+x^{2}} d x$.
(d) p.v. $\int_{-\infty}^{\infty} \frac{\cos (x)}{x+i} d x$.
(e) $I=$ p.v. $\int_{-\infty}^{\infty} \frac{x \mathrm{e}^{2 i x}}{x^{2}-1} d x$.

MIT OpenCourseWare
https://ocw.mit.edu

### 18.04 Complex Variables with Applications

Spring 2018

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.

### 18.04 Practice problems exam 2, Spring 2018 Solutions

Problem 1. Harmonic functions
(a) Show $u(x, y)=x^{3}-3 x y^{2}+3 x^{2}-3 y^{2}$ is harmonic and find a harmonic conjugate.

It's easy to compute:

$$
\begin{array}{lr}
u_{x}=3 x^{2}-3 y^{2}+6 x, & u_{x x}=6 x+6 \\
u_{y}=-6 x y-6 y, & u_{y y}=-6 x-6
\end{array}
$$

It's clear that $\nabla^{2} u=u_{x x}+u_{y y}=0$, so $u$ is harmonic.
If $v$ is a conjugate harmonic function to $u$, then $u+i v$ is analytic and the Cauchy-Riemann equations tell us that $v_{x}=-u_{y}$ and $v_{y}=u_{x}$. Therefore, we can integrate $u_{x}$ and $u_{y}$ to find $v$.

$$
\begin{array}{lr}
v_{x}=-u_{y}=6 x y+6 y & \Rightarrow v=3 x^{2} y+6 x y+g(y) \\
v_{y}=u_{x}=3 x^{2}-3 y^{2}+6 x & \Rightarrow v=3 x^{2} y-y^{3}+6 x y+h(x)
\end{array}
$$

Comparing the two expressions for $v$ we see that $g(y)=-y^{3}+C$ and $h(x)=C$. So $v=3 x^{2} y-y^{3}+6 x y+C$.
(b) Find all harmonic functions $u$ on the unit disk such that $u(1 / 2)=2$ and $u(z) \geq 2$ for all $z$ in the disk.
Solution: The only possibility is the constant function $u(z) \equiv 2$. The maximum principle for harmonic functions says that if $u$ takes a relative maximum or minimum at an interior point then it is constant. (This is a consequence of the mean value theorem.)
(c) The temperature of the boundary of the unit disk is maintained at $T=1$ in the first quadrant, $T=2$ in the second quadrant, $T=3$ in the third quadrant and $T=4$ in the fourth quadrant. What is the temperature at the center of the disk
Solution: The mean value theorem says that $f(0)$ is the average over any circle centered at 0 . This is clearly the average of the (constant) values in each quadrant. So $f(0)=2.5$.
(d) Show that if $u$ and $v$ are conjugate harmonic functions then $u v$ is harmonic.

Solution: Easy method. We know $f=u+i v$ is analytic. Therefore $f^{2}=u^{2}-y^{2}+2 i u v$ is also analytic. $\operatorname{So}, \operatorname{Im}\left(f^{2}\right)=2 u v$ is harmonic. QED

## Calculation method.

$$
\begin{aligned}
& (u v)_{x x}=u_{x x} v+2 u_{x} v_{x}+u v_{x x} \\
& (u v)_{y y}=u_{y y} v+2 u_{y} v_{y}+u v_{y y}
\end{aligned}
$$

We know $u, v$ are harmonic and satisfy the Cauchy-Riemann equations $u_{x}=v_{y}, u_{y}=-v_{x}$. So adding the above equations we get

$$
(u v)_{x x}+(u v)_{y y}=\left(u_{x x}+u_{y y}\right) v+2\left(-u_{x} u_{y}+u_{x} u_{y}\right)+u\left(v_{x x}+v_{y y}\right)=0 .
$$

We have shown that $u v$ is harmonic.
(e) Show that if $u$ is harmonic then $u_{x}$ is harmonic.

Solution: Easy method. For some conjugate $v, f=u+i v$ is harmonic. Since $f^{\prime}=u_{x}+i v_{x}$, we know $\operatorname{Re}\left(f^{\prime}\right)=u_{x}$ is harmonic.
Direct calculation $\left(u_{x}\right)_{x x}+\left(u_{x}\right)_{y y}=\left(u_{x x}+u_{y y}\right)_{x}=0$.
(f) Show that if $u$ is harmonic and $u^{2}$ is harmonic the $u$ is constant.
(We always assume harmonic functions are real valued.)
Solution: We calculate this directly.

$$
\left(u^{2}\right)_{x x}=2\left(u_{x}\right)^{2}+2 u u_{x x}, \quad\left(u^{2}\right)_{y y}=2\left(u_{y}\right)^{2}+2 u u_{y y} .
$$

Assume that $u$ and $u^{2}$ are harmonic, then

$$
0=\left(u^{2}\right)_{x x}+\left(u^{2}\right)_{y y}=2\left(\left(u_{x}\right)^{2}+\left(u_{y}\right)^{2}\right)+2 u\left(u_{x x}+u_{y y}\right)=2\left(\left(u_{x}\right)^{2}+\left(u_{y}\right)^{2}\right) .
$$

As a sum of squares, $\left(u_{x}\right)^{2}+\left(u_{y}\right)^{2}=0$ implies $u_{x}=u_{y}=0$. This implies $u$ is constant. QED.

## Problem 2.

Let $f(z)=\frac{1}{(z-1)(z-3)}$. Find Laurent series for $f$ on each of the 3 annular regions centered at $z=0$ where $f$ is analytic.
Solution: The poles are at $z=1$ and $z=3$. This divides the plane into 3 annular regions, with $f$ analytic on each region:

$$
A_{1}:|z|<1, \quad A_{2}: 1<|z|<3, \quad A_{3}: 3<|z| .
$$



Using partial fractions we get $f(z)=-\frac{1}{2} \cdot \frac{1}{z-1}+\frac{1}{2} \cdot \frac{1}{z-3}$. We write each of these terms as geometric series in each region.

$$
\begin{array}{ll}
\frac{1}{z-1}=-\frac{1}{1-z}=-\left(1+z+z^{2}+\ldots\right) & (\text { converges for }|z|<1) \\
\frac{1}{z-1}=\frac{1}{z(1-1 / z}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^{2}}+\ldots\right) & (\text { converges for }|z|>1) \\
\frac{1}{z-3}=-\frac{1}{3(1-z / 3)}=-\left(1+(z / 3)+(z / 3)^{2}+\ldots\right) & (\text { converges for }|z|<3) \\
\frac{1}{z-3}=\frac{1}{z(1-3 / z}=\frac{1}{z}\left(1+\frac{3}{z}+\frac{3^{2}}{z^{2}}+\ldots\right) & (\text { converges for }|z|>3)
\end{array}
$$

On each region we can add the appropriate form of these series.
On $A_{1}:|z|<1$ :

$$
f(z)=\frac{1}{2}\left(1+z+z^{2}+\ldots\right)-\frac{1}{2} \cdot \frac{1}{3}\left(1+z / 3+(z / 3)^{2}+\ldots\right)=\frac{1}{2} \sum_{n=0}^{\infty}\left(1-\frac{1}{3^{n+1}}\right) z^{n} .
$$

On $A_{2}: 1<|z|<3$ :

$$
f(z)=-\frac{1}{2} \cdot \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^{2}}+\ldots\right)-\frac{1}{6}\left(1+z / 3+(z / 3)^{2}+\ldots\right)=-\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{z^{n}}-\frac{1}{6} \sum_{n=0}^{\infty}(z / 3)^{n} .
$$

On $A_{3}:|z|>3$ :

$$
f(z)=-\frac{1}{2} \cdot \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^{2}}+\ldots\right)+\frac{1}{2 z}\left(1+\frac{3}{z}+\frac{3^{3}}{z^{2}}+\ldots\right)=\frac{1}{2} \sum_{n=1}^{\infty} \frac{\left(-1+3^{n-1}\right)}{z^{n}} .
$$

## Problem 3.

Find the first few terms of the Laurent series around 0 for the following.
(a) $f(z)=z^{2} \cos (1 / 3 z)$ for $0<|z|$.

Solution: Using the known series for $\cos (z)$ we get

$$
f(z)=z^{2}\left(1-\frac{1}{2!\cdot 3^{2} z^{2}}+\frac{1}{4!\cdot 3^{4} z^{4}}-\ldots\right)=z^{2}-\frac{1}{2!\cdot 3^{2}}+\frac{1}{4!\cdot 3^{4} z^{2}}-\ldots
$$

(b) $f(z)=\frac{1}{\mathrm{e}^{z}-1}$ for $0<|z|<R$. What is $R$ ?

Solution: Writing out $\mathrm{e}^{z}$ as a power series we have

$$
f(z)=\frac{1}{z+z^{2} / 2!+z^{3} / 3!+\ldots}=\frac{1}{z} \cdot \frac{1}{1+z / 2!+z^{2} / 3!+\ldots}
$$

For $z$ near 0 the expression $z / 2!+z^{2} / 3!+\ldots$ is small so we can use the geometric series:

$$
f(z)=\frac{1}{z}\left(1-\left(z / 2!+z^{2} / 3!+\ldots\right)+\left(z / 2!+z^{2} / 3!+\ldots\right)^{2}-\left(z / 2!+z^{2} / 3!+\ldots\right)^{3}+\ldots\right) .
$$

It is hard to get a general expression for the terms of this series, but we can compute the first few explicitly.
$f(z)=\frac{1}{z}\left(1-\frac{z}{2}+z^{2}(-1 / 3!+1 / 4)+z^{3}(-1 / 4!+2 /(2!3!)-1 / 8)\right)=\frac{1}{z}\left(1-\frac{z}{2}+\frac{z^{2}}{12}-0 \cdot z^{3}+\ldots\right)$

## Problem 4.

What is the annulus of convergence for $\sum_{n=-\infty}^{\infty} \frac{z^{n}}{2^{|n|}}$.

Solution: We find region for singular and regular parts seperately.
Singular part: $\sum_{n=1}^{\infty} \frac{1}{2^{n} z^{n}}$. Either by recognizing this as a geometric series or using the ratio test we see it converges if $|z|>1 / 2$.
Regular part: $\sum_{n=0}^{\infty} \frac{z^{n}}{2^{n}}$. Either by recognizing this as a geometric series or using the ratio test we see it converges if $|z|<2$.
The annulus of convergence is $1 / 2<|z|<2$.

## Problem 5.

Find and classify the isolated singularities of each of the following. Compute the residue at each such singularity.
(a) $f_{1}(z)=\frac{z^{3}+1}{z^{2}(z+1)}$

Solution: $f_{1}$ has a pole of order 2 at $z=0$ and a apparently a simple pole at $z=-1$. (In fact we will see that $z=-1$ is a removable singularity.)
$\operatorname{Res}\left(f_{1}, 0\right)$ : Let $g(z)=z^{2} f_{1}(z)=\frac{z^{3}+1}{z+1}$. Clearly we want the coefficient of $z$ in the Taylor series for $g$. That is $\operatorname{Res}\left(f_{1}, 0\right)=g^{\prime}(0)=-1$. (Alternatively we could have written $1 /(z+1)$ as a geometric series and found the coefficient of $z$ from that.)
$\operatorname{Res}\left(f_{1},-1\right)$ : Let $g(z)=(z+1) f_{1}(z)=\frac{z^{3}+1}{z^{2}} . \operatorname{Res}\left(f_{1},-1\right)=g(-1)=0$. So the singularity is removable. In retrospect we could have seen this because $z^{+} 1=(z+1)\left(z^{2}-z+1\right)$.
(b) $f_{2}(z)=\frac{1}{\mathrm{e}^{z}-1}$

Solution: $f_{2}$ has poles whenever $\mathrm{e}^{z}-1=0$, i.e. when $z=2 n \pi i$ for any integer $n$. We'll show the poles are simple and compute their residues all at once by computing $\lim _{z \rightarrow 2 n \pi i}(z-2 n \pi i) f_{z}(z)$.

$$
\operatorname{Res}(f, 2 n \pi i)=\lim _{z \rightarrow 2 n \pi i}(z-2 n \pi i) f_{z}(z)=\lim _{z \rightarrow 2 n \pi i} \frac{z-2 n \pi i}{\mathrm{e}^{z}-1}=\frac{1}{\mathrm{e}^{2 n \pi i}}=1 .
$$

(The limit was computed using L'Hospital's rule.) Since the limit exists the pole is simple and the limit is the residue.
(c) $f_{3}(z)=\cos (1-1 / z)$

Solution: $f_{3}$ has exactly one singularity, which is at $z=0$. We'll find the residue by computing the first few terms of the Laurent expansion.

$$
\cos (1-1 / z)=\frac{\mathrm{e}^{i(1-1 / z)}+\mathrm{e}^{-i(1-1 / z)}}{2}=\frac{\mathrm{e}^{i} \mathrm{e}^{-i / z}+\mathrm{e}^{-i} \mathrm{e}^{i / z}}{2} .
$$

Using the power series for $\mathrm{e}^{w}$ we have

$$
\begin{aligned}
& \mathrm{e}^{i} \mathrm{e}^{-i / z}=\mathrm{e}^{i}\left(1-\frac{i}{z}-\frac{1}{2 z^{2}}+\ldots\right) \\
& \mathrm{e}^{-i} \mathrm{e}^{i / z}=\mathrm{e}^{-i}\left(1+\frac{i}{z}-\frac{1}{2 z^{2}}+\ldots\right)
\end{aligned}
$$

Looking at just the $1 / z$ terms we have

$$
\operatorname{Res}\left(f_{3}, 0\right)=\frac{-i \mathrm{e}^{i}+i \mathrm{e}^{-i}}{2}=\sin (1)
$$

Alternatively we could have used the trig identity $\cos (1-1 / z)=\cos (1) \cos (1 / z)+\sin (1) \sin (1 / z)$.

## Problem 6.

(a) Find a function $f$ that has a pole of order 2 at $z=1+i$ and essential singularies at $z=0$ and $z=1$.

Solution: It's easiest to write this as a sum.

$$
f(z)=\mathrm{e}^{1 / z}+\mathrm{e}^{1 /(z-1)}+\frac{1}{(z-1-i)^{2}} .
$$

The term $\mathrm{e}^{1 / z}$ has an essential singularty at $z=0$. Since the other two terms are analytic at $z=1$, $f$ has an essential singurity at $z=0$.

The singularities at 1 and $1+i$ can be analyzed in the same manner.
(b) Find a function $f$ that has a removable singularity at $z=0$, a pole of order 6 at $z=1$ and an essential singularity at $z=i$.
Solution: We'll do this in the same way as part (a).

$$
f(z)=\frac{z^{2}+8 z}{\sin (z)}+\frac{1}{(z-1)^{6}}+\mathrm{e}^{1 /(z-i)} .
$$

## Problem 7.

True or false. If true give an argument. If false give a counterexample
(a) If $f$ and $g$ have a pole at $z_{0}$ then $f+g$ has a pole at $z_{0}$.
(b) If $f$ and $g$ have a pole at $z_{0}$ and both have nonzero residues the $f g$ has a pole at $z_{0}$ with a nonzero residue.
(c) If $f$ has an essential singularity at $z=0$ and $g$ has a pole of finite order at $z=0$ the $f+g$ has an essential singularity at $z=0$.
(d) If $f(z)$ has a pole of order $m$ at $z=0$ then $f\left(z^{2}\right)$ has a pole of order $2 m$

Answers.(a) False. Counterexample: $f(z)=1 / z, g(z)=-1 / z$.
(b) False. Counterexample: $f(z)=1 / z, g(z)=1 / z$.
(c) True. When you add Laurent series you simply add the coefficients. The singular part of the series for $f$ has infinitely many nonzero coefficients. After a certain point, the singular part of $g$ has all zero coefficients. So after that point, the singular part of $f+g$ has the same coefficients as $f$. That is, it has infinitely many nonzero coefficients, so the singularity is essential.
(d) True. We know $f(z)=z^{-m} g(z)$, where $g(0) \neq 0$. So, $f(z)^{2}=z^{-2 m} g\left(z^{2}\right)$, where $g\left(0^{2}\right) \neq 0$. This shows, $f\left(z^{2}\right)$ has a 0 of order $2 m$.

## Problem 8.

Find the Laurent series for each of the following.
(a) $1 / \mathrm{e}^{(1-z)}$ for $1<|z|$.

Solution: $f(z)=1 / \mathrm{e}^{(1-z)}=\mathrm{e}^{z-1}$ is analytic on the entire plane. So,

$$
f(z)=\mathrm{e}^{-1} \mathrm{e}^{z}=\mathrm{e}^{-1}\left(1+z+\frac{z^{2}}{2!}+\frac{z^{3}}{3!}+\ldots\right)
$$

is the Taylor series for all $z$. Hence it is the Laurent series on $|z|>1$.

## Problem 9.

Let $h(z)=\frac{1}{\sin (z)}-\frac{1}{z}+\frac{2 z}{z^{2}-\pi^{2}}$ in the disk $|z|<2 \pi$.
(a) Show that all the apparent singularities are removable.
(b) Find the first 4 terms of the Taylor series around $z=0$.

Answers.(a) The apparent singularities of $h$ are at $0, \pm \pi$. There might be a slicker way to do this part, but here's one that's not too painful.
At $z=0: h(z)=\frac{z-\sin (z)}{z \sin (z)}+\frac{2 z}{z^{2}-\pi^{2}}$. The second term is analytic, so doesn't contribute to the singularity at 0 . Writing out the first term in terms of Taylor series we have

$$
\frac{z-\sin (z)}{z \sin (z)}=\frac{z^{3} / 3!-z^{5} / 5!+\ldots}{z^{2}-z^{4} / 3!+\ldots}
$$

The numerator has a zero of order 3 and the denominator one of order 2 , so the entire term has a 0 of order 1, i.e. the singularity is removable.
We can play the same game at $z=\pi$. To make things easier we use partial fractions

$$
h(z)=\frac{1}{\sin (z)}-\frac{1}{z}+\frac{1}{z-\pi}+\frac{1}{z+\pi}=\frac{(z-\pi)+\sin (z)}{(z-\pi) \sin (z)}-\frac{1}{z}+\frac{1}{z+\pi} .
$$

The second and third terms are analytic at $z=\pi$, so don't contribute to the singularity. The first term can be written as

$$
\frac{(z-\pi)+\sin (z)}{(z-\pi) \sin (z)}=\frac{(z-\pi)+\left(-(z-\pi)+(z-\pi)^{3} / 3!-(z-\pi)^{5} / 5!+\ldots\right)}{(z-\pi)\left(-(z-\pi)+(z-\pi)^{3} / 3!-(z-\pi)^{5} / 5!+\ldots\right)}=\frac{(z-\pi)^{3} / 3!}{(z-\pi)^{2}\left(-1+(z-\pi)^{2} / 2+\ldots\right)}
$$

As before, the numerator has a zero of order 3 and the denominator one of order 2 , so the singularity is removable.
The singularity at $z=-\pi$ is handled identically to $z=\pi$.
(b) For this part let's work on each term in the original expression of $h$.

$$
\begin{aligned}
\frac{1}{\sin (z)} & =\frac{1}{z} \frac{1}{1-\left(z^{2} / 3!-z^{4} / 5!+\ldots\right)} \\
& =\frac{1}{z}\left(1+\left(z^{2} / 3!-z^{4} / 5!+\ldots\right)+\left(z^{2} / 3!-z^{4} / 5!+\ldots\right)^{2}+\ldots\right) \\
& =\frac{1}{z}\left(1+z^{2} / 3!+z^{4}\left(-1 / 5!+1 /(3!)^{2}\right)+\ldots\right) \\
& =\frac{1}{z}\left(1+\frac{z^{2}}{6}+\frac{7 z^{4}}{360}+\ldots\right)
\end{aligned}
$$

$$
\frac{2 z}{z^{2}-\pi^{2}}=-\frac{2 z}{\pi^{2}\left(1-z^{2} / \pi^{2}\right)}=-\frac{2 z}{\pi^{2}}\left(1+z^{2} / \pi^{2}+z^{4} / \pi^{4}+\ldots\right)
$$

Combining all the parts we get

$$
\begin{aligned}
h(z) & =\frac{1}{z}+\frac{z}{6}+\frac{7 z^{3}}{360}+\ldots-\frac{1}{z}-\frac{2 z}{\pi^{2}}-\frac{2 z^{3}}{\pi^{4}}+\ldots \\
& =0+z\left(\frac{1}{6}-\frac{2}{\pi^{2}}\right)+z^{3}\left(\frac{7}{360}-\frac{2}{\pi^{4}}\right)+\ldots
\end{aligned}
$$

## Problem 10.

Find the residue at $\infty$ of each of the following.
(a) $f(z)=\mathrm{e}^{z}$
(b) $f(z)=\frac{z-1}{z+1}$.

Answers.(a) Easy method: $f(z)$ is entire so $\int_{C} f(z) d z=0$ for all closed $C$. Since the residue at infinity is minus the integral over a closed curve containing all the singularities we must have $\operatorname{Res}(f, \infty)=0$.
Method 2. Let $g(w)=\frac{1}{w^{2}} \mathrm{e}^{1 / w}$. The Laurent series for $g$ is

$$
g(w)=\frac{1}{w^{2}}\left(1+\frac{1}{w}+\ldots\right) .
$$

So $\operatorname{Res}(f, \infty)=-\operatorname{Res}(g, 0)=0$.
(b) Since Let $g(w)=\frac{1}{w^{2}} f(/ 1 / w)=\frac{1}{w^{2}} \frac{1 / w-1}{1 / w+1}$. Writing $1 /(w+1)$ as a geometric series we get

$$
g(w)=\frac{1}{w^{2}}(1-w)\left(1-w+w^{2}-w^{3}+\ldots\right)=\frac{1}{w^{2}}\left(1-2 w+2 w^{2}-\ldots\right)=\frac{1}{w^{2}}-\frac{2}{w}+2-\ldots
$$

Therefore $\operatorname{Res}(f, \infty)=-\operatorname{Res}(g, 0)=2$.

## Problem 11.

Use the following steps to sketch the stream lines for the flow with complex potential $\Phi(z)=z+$ $\log (z-i)+\log (z+i)$
(i) Identify the components, i.e. sources, sinks, etc of the flow.
(ii) Find the stagnation points.
(iii) Sketch the flow near each of the sources.
(iv) Sketch the flow far from the sources.
(v) Tie the picture together.

Solution: (i) The log terms with positive coefficients represent sources. The term $z$ represents a steady stream. So this is two sources in a steady stream.
(ii) Stagnation poihts are places where $\Phi^{\prime}(z)=0$. Computing:

$$
\Phi^{\prime}(z)=1+\frac{2 z}{z^{2}+1}=\frac{(z+1)^{2}}{z^{2}+1} .
$$

So there is a single stagnation point at $z=-1$.
(iii-v) Near the sources the flow looks like a source. Far away it looks like uniform flow to the right. By symmetry (or direct calculation) there are streamlines on the $x$-axis. We get the following picture. (I used Octave to draw draw the underlying vector field.)


## Problem 12.

Compute the following definite integrals
(a) $\int_{-\pi}^{\pi} \frac{1}{1+\sin ^{2}(\theta)} d \theta$. (Solution: $\pi \sqrt{2}$ )

Solution: On the unit circle $z=\mathrm{e}^{i \theta}, \sin (\theta)=\frac{z-1 / z}{2 i}=\frac{z^{2}-1}{21}$. So the integral becomes

$$
\int_{|z|=1} \frac{1}{1+\left(\left(z^{2}-1\right) / 2 i z\right)^{2}} \frac{d z}{i z}=\int_{|z|=1} \frac{-4 z}{i\left(z^{4}-6 z^{2}+1\right)} d z
$$

Let $f(z)=\frac{-4 z}{i\left(z^{4}-6 z^{2}+1\right)}$. So the integral is

$$
\int_{|z|=1} f(z) d z=2 \pi i \sum \text { residues of } f \text { inside the unit disk. }
$$

The poles of $f$ are at $z^{2}=3 \pm \sqrt{8}$. Of these, only $z^{2}=3-\sqrt{8}$ is inside the unit circle. So there are two poles inside the unit circle at $z_{1}=\sqrt{3-\sqrt{8}}$ and $z_{2}=-\sqrt{3-\sqrt{8}}$. These are simple poles and we can compute the residue using L'Hospital's rule.

$$
\operatorname{Res}\left(f, z_{1}\right)=\lim _{z \rightarrow z_{1}} \frac{\left(z-z_{1}\right)(-4 z)}{i\left(z^{4}-6 z^{2}+1\right)}=\frac{-4 z_{1}}{i\left(4 z_{1}^{3}-12 z_{1}\right)}=\frac{-1}{i\left(z_{1}^{2}-3\right)}=\frac{1}{i \sqrt{8}} .
$$

The residue at $z_{2}$ has the same value. So,

$$
\int_{|z|=1} f(z) d z=2 \pi i\left(\operatorname{Res}\left(f, z_{1}\right)+\operatorname{Res}\left(f, z_{2}\right)\right)=\frac{4 \pi}{\sqrt{8}}=\pi \sqrt{2} .
$$

(b) $\int_{-\infty}^{\infty} \frac{x}{\left(x^{2}+4 x+13\right)^{2}} d x$. (Solution: $-\pi / 27$ )

Solution: Call the integral in question $I$. Let $f(z)=z /\left(z^{2}+4 z+13\right)^{2}$. This decays faster than $1 / z^{2}$ so we can use path


We know $\lim _{R \rightarrow \infty} \int_{C_{R}} f(z) d z=0$, so, letting $R$ go to infinity in $\int_{C_{1}+C_{R}} f(z) d z$ we get

$$
I=2 \pi i \sum \text { residues of } f \text { in the upper half-plane. }
$$

The poles of $f$ are at $-2 \pm 3 i$. Only $z_{1}=-2+3 i$ is in the upper half-plane. All we have to do is compute the residue. Let $g(z)=\left(z-z_{1}\right)^{2} f(z)=\frac{z}{(z-(-2-3 i))^{2}}$. Since $g$ is analytic at $z_{1}$ we have

$$
\operatorname{Res}\left(f, z_{1}\right)=g^{\prime}\left(z_{1}\right)=\text { some algebra }=i / 54 .
$$

So $I==-\pi / 27$.
(c) p.v. $\int_{-\infty}^{\infty} \frac{x \sin (x)}{1+x^{2}} d x$.

Solution: Call the integral in question $I$. Replace $\sin (x)$ by $\mathrm{e}^{i x}$ and let

$$
\tilde{I}=\text { p.v. } \int_{-\infty}^{\infty} \frac{x \mathrm{e}^{i x}}{1+x^{2}} d x, \quad \text { so }, I=\operatorname{Im}(\tilde{I})
$$

Let $f(z)=\frac{z \mathrm{e}^{i z}}{1+z^{2}}$ and use the contour $C_{1}+C_{R}$.


The only pole of $f$ in the upper half-plane is at $z=i$. It is easy to compute $\operatorname{Res}(f, i)=i \mathrm{e}^{-1} 2 i=$ $\mathrm{e}^{-1} / 2$. So,

$$
\int_{C_{1}+C_{R}} f(z) d z=2 \pi i \operatorname{Res}(f, i)=\pi i \mathrm{e}^{-1} .
$$

Since $\left|z /\left(1+z^{2}\right)\right|<M /|z|$ for Large $z$ and the coefficient of $i x$ in the exponent of $f$ is positive, we know

$$
\lim _{R \rightarrow \infty} f(z) d z=0 .
$$

Also, $\lim _{R \rightarrow \infty} \int_{-R}^{R} f(z) d z=$ p.v. $\int_{-\infty}^{\infty} f(x) d x=\tilde{I}$.
In conclusion we have

$$
\tilde{I}=2 \pi i \operatorname{Res}(f, i)=\pi i \mathrm{e}^{-1}
$$

So $I=\operatorname{Im}(\tilde{I})=\pi \mathrm{e}^{-1}$.
(d) p.v. $\int_{-\infty}^{\infty} \frac{\cos (x)}{x+i} d x$.

Solution: Write $\cos (x)=\frac{\mathrm{e}^{i x}+\mathrm{e}^{-i x}}{2}$. So, Let $f_{1}(z)=\frac{\mathrm{e}^{i z}}{z+i}$ and $f_{2}(z)=\frac{\mathrm{e}^{-i z}}{z+i}$.

$$
\text { p.v. } \int_{-\infty}^{\infty} \frac{\cos (x)}{x+i} d x=\text { p.v. } \int_{-\infty}^{\infty} \frac{1}{2} f_{1}(x) d x+\text { p.v. } \int_{-\infty}^{\infty} \frac{1}{2} f_{2}(x) d x .
$$

We compute these integrals using two different contours



The reasoning is the same as in part (b). Both $f_{1}$ and $f_{2}$ have a single pole at $z-i$. So, using the contour $C_{1}+C_{R_{1}}$ we find

$$
\text { p.v. } \int_{-\infty}^{\infty} \frac{1}{2} f_{1}(x) d x=2 \pi i \text { Res } \frac{1}{2} f_{1} \text { in the upper half plane. }=0 .
$$

Likewise, using the contour $C_{1}-C_{R_{2}}$ we find

$$
\text { p.v. } \int_{-\infty}^{\infty} \frac{1}{2} f_{2}(x) d x=2 \pi i \operatorname{Res} \frac{1}{2} f_{2} \text { in the lower half plane. }=-2 \pi i \operatorname{Res}\left(f_{2} / 2,-i\right)=-\pi i \mathrm{e}^{-1} .
$$

(The minus sign is because $C_{1}-C_{R_{2}}$ is oriented in the clockwise direction.)
Answer to problem: the integral is $-\pi i \mathrm{e}^{-1}$.
(e) $I=$ p.v. $\int_{-\infty}^{\infty} \frac{x \mathrm{e}^{2 i x}}{x^{2}-1} d x$.

Solution: Since our integrand $f(z)=\frac{z \mathrm{e}^{2 i z}}{z^{2}-1}$ has poles on the real axis we will need to use an indented contour.


As usual, we chose the contour so that the integral over $C_{R}$ goes to 0 as $R$ goes to infinity. Since $f$ has no poles inside the contour we have

$$
\int_{C_{1}-C_{2}+C_{3}-C_{4}+C_{5}+C_{R}} f(z) d z=0 .
$$

The poles of $f$ at $\pm 1$ are simple. So, letting $R \rightarrow \infty$ and $r_{1}, r_{2} \rightarrow 0$ we get

$$
I=\pi i(\operatorname{Res}(f,-1)+\operatorname{Res}(f, 1)) .
$$

The residues are straightforward to compute.

$$
\operatorname{Res}(f,-1)=\mathrm{e}^{-2 i} / 2, \quad \operatorname{Res}(f, 1)=\mathrm{e}^{2 i} / 2
$$

So, $I=\pi i\left(\mathrm{e}^{2 i}+\mathrm{e}^{-2 i}\right) / 2=\pi i \cos (2)$.

MIT OpenCourseWare
https://ocw.mit.edu

### 18.04 Complex Variables with Applications

Spring 2018

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.

### 18.04 Practice problems for final exam, Spring 2018

On the final exam you will be given a copy of the Laplace table posted with these problems.

## Problem 1.

Which of the following are meromporphic in the whole plane.
(a) $z^{5}$
(b) $z^{5 / 2}$
(c) $\mathrm{e}^{1 / z}$
(d) $1 / \sin (z)$.

Problem 2.
(a) Let $f(z)=\frac{(z-2)^{2} z^{3}}{(z+5)^{3}(z+1)^{3}(z-1)^{4}}$. Compute $\int_{|z|=3} \frac{f^{\prime}(z)}{f(z)} d z$
(b) Find the number of roots of $g(z)=6 z^{4}+z^{3}-2 z^{2}+z-1=0$ in the unit disk.
(c) Suppose $f(z)$ is analytic on and inside the unit circle. Suppose also that $|f(z)|<1$ for $|z|=1$. Show that $f(z)$ has exactly one fixed point $f\left(z_{0}\right)=z_{0}$ inside the unit circle.
(d) True or false: Suppose $f(z)$ is analytic on and inside a simple closed curve $\gamma$. If $f$ has $n$ zeros inside $\gamma$ then $f^{\prime}(z)$ has $n-1$ zeros inside $\gamma$.

## Problem 3.

Let $A=\{z \mid 0 \leq \operatorname{Re}(z) \leq \pi / 2, \operatorname{Im}(z) \geq 0$.
Let $B=$ the first quadrant/
Show that $f(z)=\sin (z)$ maps $A$ conformally onto $B$

MIT OpenCourseWare
https://ocw.mit.edu

### 18.04 Complex Variables with Applications

Spring 2018

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.

### 18.04 Practice problems for final exam, Spring 2018 Solutions

On the final exam you will be given a copy of the Laplace table posted with these problems.

## Problem 1.

Which of the following are meromporphic in the whole plane.
(a) $z^{5}$
(b) $z^{5 / 2}$
(c) $\mathrm{e}^{1 / z}$
(d) $1 / \sin (z)$.
answers: Meromorphic means analytic except for poles of finite order.
(a) Yes, this is entire.
(b) No, this requires a branch cut in the plane to define a region where it's analytic.
(c) No, the singularity at $z=0$ is an essential singularity, not a finite pole.
(d) Yes, $\sin (z)$ has $\operatorname{simple}$ zeros at $n \pi$ for all integers $n$. So $1 / \sin (z)$ has simple poles at these points.

Problem 2.
(a) Let $f(z)=\frac{(z-2)^{2} z^{3}}{(z+5)^{3}(z+1)^{3}(z-1)^{4}}$. Compute $\int_{|z|=3} \frac{f^{\prime}(z)}{f(z)} d z$
(b) Find the number of roots of $g(z)=6 z^{4}+z^{3}-2 z^{2}+z-1=0$ in the unit disk.
(c) Suppose $f(z)$ is analytic on and inside the unit circle. Suppose also that $|f(z)|<1$ for $|z|=1$. Show that $f(z)$ has exactly one fixed point $f\left(z_{0}\right)=z_{0}$ inside the unit circle.
(d) True or false: Suppose $f(z)$ is analytic on and inside a simple closed curve $\gamma$. If $f$ has $n$ zeros inside $\gamma$ then $f^{\prime}(z)$ has $n-1$ zeros inside $\gamma$.
answers: (a) By the argument principle the $\int_{\gamma} \frac{f^{\prime}}{f} d z=2 \pi i\left(Z_{f, \gamma}-P_{f, \gamma}\right.$. In this case, the zeros of $f$ inside $\gamma$ are 2,0 of order 2 and 3 respectively. The poles inside $\gamma$ are -1 and 1 of order 3 and 4 respectively. So, the integral equals

$$
2 \pi i(2+3-3-4)=-4 \pi i
$$

(b) On the unit circle $\left|z^{3}-2 z^{2}+z-1\right|<5$ and $\left|6 z^{4}\right|=6$. Therefore by Rouche's theorem the number of zeros of $g(z)$ inside the unit circle is equal to the number of zeros of $6 z^{4}$, i.e. 4 .
(c) Let $g(z)=f(z)-z$. We want to show $g$ has exactly one root inside the unit circle. We know $|f(z)|<|-z|=1$ on the unit circle. So by Rouche's theorem $g(z)$ and $-z$ have the same number of zeros in the unit disk. That is, they both have exactly one such zero. QED.
(d) False. Consider $f(z)=\mathrm{e}^{z}-1$. This has 3 zeros inside the circle $|z|=3 \pi(0, \pm 2 \pi)$. But $f^{\prime}(z)=\mathrm{e}^{z}$ has no zeros.

## Problem 3.

Let $A=\{z \mid 0 \leq \operatorname{Re}(z) \leq \pi / 2, \operatorname{Im}(z) \geq 0$.

Let $B=$ the first quadrant/
Show that $f(z)=\sin (z)$ maps $A$ conformally onto $B$
answers: (a) You should supply a picture of the regions $A$ and $B$ and develop a picture tracking the argument we give. We see where $f$ maps the boundary of $A$. The boundary of $A$ has 3 pieces:

Piece 1: $z=i y$, with $y \geq 0$. On this piece

$$
\sin (z)=\frac{\mathrm{e}^{-y}-\mathrm{e}^{y}}{2 i}=\frac{\left(\mathrm{e}^{y}-\mathrm{e}^{-y}\right)}{2} i
$$

So, the image of piece 1 is the positive imaginary axis.
Piece 2: $z=x$, with $0 \leq x \leq \pi / 2$. On this piece $\sin (z)=\sin (x)$, so the image runs from 0 to 1 along the real axis.
Piece 3: $z=\pi / 2+i y$, with $y \geq 0$. On this piece

$$
\sin (z)=\frac{\mathrm{e}^{-y+\pi i / 2}-\mathrm{e}^{y-\pi i / 2}}{2 i}=\frac{\left(i \mathrm{e}^{-y}+i \mathrm{e}^{-y}\right)}{2 i}=\frac{\mathrm{e}^{-y}+\mathrm{e}^{y}}{2}=\cosh (y) .
$$

So, the image of piece 3 is the real axis greater than 1 .
We have shown that $f(z)$ maps the boundary of $A$ to the boundary of $B$.
To see that $A$ is mapped to $B$ it's enough to verify that one point inside $A$ is mapped to a point inside $B$. There are lots of ways to do this. Here's one. We know

$$
\sin (x+i y)=\frac{\mathrm{e}^{-y+i x}-\mathrm{e}^{y-i x}}{2 i}
$$

Pick $x=\pi / 4$ and $y$ so large that $\mathrm{e}^{-y}$ is very tiny. Then

$$
\sin (x+i y) \approx-\mathrm{e}^{y} \mathrm{e}^{-i x} 2 i=-\mathrm{e}^{y} \frac{\sqrt{2} / 2-i \sqrt{2} / 2}{2 i}=\mathrm{e}^{y} \frac{\sqrt{2}+i \sqrt{2}}{4}
$$

This last value is clearly in the first quadrant, i.e inside $B$.

MIT OpenCourseWare
https://ocw.mit.edu

### 18.04 Complex Variables with Applications

Spring 2018

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.

