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Minimizing the Probability of Ruin When Claims Follow Brownian Motion with Drift

S. David Promislow* and Virginia R. Young†

Abstract

We extend the work of Browne (1995) and Schmidli (2001), in which they minimize the probability of ruin of an insurer facing a claim process modeled by a Brownian motion with drift. We consider two controls to minimize the probability of ruin: (1) investing in a risky asset and (2) purchasing quota-share reinsurance. We obtain an analytic expression for the minimum probability of ruin and the corresponding optimal controls, and we demonstrate our results with numerical examples.

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1. Introduction

We find the optimal investment and quota-share reinsurance strategies to minimize the probability of ruin of an insurer who faces a claim process that follows Brownian motion with drift. Our financial market is composed of a riskless and a risky asset (see Section 2).

Browne (1995) minimized the probability of ruin by finding the optimal investment strategy in the financial market. He only briefly discussed the case for which the return on the riskless asset is positive, and we present that case in more detail in Section 3.1. In Section 3.2 we extend Browne's work by considering the constrained case of this particular problem, in which we constrain the amount invested in the risky asset to lie below current wealth; that is, we do not allow the individual to borrow in order to invest in the risky asset. Finally, in Section 3.3 we suppose that the insurer can borrow but at a rate higher than that earned on the riskless asset, and we find the optimal investment strategy to minimize the probability of ruin.

Schmidli (2001) found the optimal quota-share reinsurance strategy for an insurer facing a claim process that follows Brownian motion with drift but, otherwise, assumed that the surplus was not invested in any financial instrument. In Sections 4 and 5 we extend his work by allowing the insurer to invest in financial assets, in addition to purchasing quota-share reinsurance. In Section 4 the surplus is invested in a riskless asset that earns a positive return. In Section 5 we extend Section 4 by allowing the insurer to invest in both risky and riskless assets. In Section 5 we do not restrict the amount invested in the risky asset to lie below the current surplus. We leave that task for the interested researcher. In Section 6 we present numerical examples. Section 7 concludes the paper.

2. Insurance and Financial Markets

We model the claim process C according to Brownian motion with drift as

$$dC_t = a\,dt - b\,dB_t, \tag{2.1}$$

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in which a and b are positive constants and B is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Pr})$. Note that the Brownian motion model of the claim process is a limit of the classical compound Poisson model (Grandell 1991). In Section 6.2 we numerically compare the optimal investment strategy from the Brownian motion model with the one resulting from a compound Poisson model (Liu and Yang 2004). In actuarial practice one will want to use the model in equation (2.1) only when the ratio a/b is large enough—for example, at least 3—so that the probability of realizing negative claims in any one period is small.

We assume that the premium is paid continuously at the constant rate $c = (1 + \theta)a$ with $\theta > 0$. Therefore, before introducing investments and reinsurance, the surplus process U is given by the dynamics

$$\begin{cases} dU_t = cdt - dC_t \\ \qquad = \theta a dt + bdB_t, \\ U_0 = u. \end{cases} \tag{2.2}$$

In Sections 3 and 5 we allow the insurer to invest optimally in a risky asset whose price process S follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t^S, \tag{2.3}$$

in which B^S is a standard Brownian motion with correlation coeffcient ρ_S between B^S and B, the Brownian motion driving the claim process C. Let π_t denote the (dollar) amount that the insurer invests in the risky asset at time t. The remainder, $U_t - \pi_t$, is invested in a riskless asset that earns interest at a constant continuous rate $r \geq 0$; actuaries often refer to r as the *force of interest*. In Section 3.3 we consider the case for which the insurer can borrow at a rate $R \in (r, \mu)$, and we modify the model in this section appropriately.

In Sections 3 and 5 we minimize the probability of ruin of the insurer with respect to admissible investment strategies π. An investment strategy π is *admissible* if it is $\mathcal{F}_t$-progressively measurable (in which $\mathcal{F}_t$ is the augmentation of $\sigma(U_s : 0 \leq s \leq t)$) and if it satisfies the integrability condition $\int_0^t \pi_s^2 \, ds < \infty$, almost surely, for all $t \geq 0$.

In Sections 4 and 5 we allow the insurer to purchase quota-share reinsurance. We denote the proportion reinsured at time t by q_t, and we say that q is an *admissible* strategy if it is $\mathcal{F}_t$-progressively measurable and if q_t lies between 0 and 1. Reinsurance is available for a proportional loading of $\eta > \theta$.

Surplus subject to these two controls follows dynamics given by

$$\begin{cases} dU_t = (rU_t + (\mu - r)\pi_t + (\theta - \eta q_t)a)dt + \sigma\pi_t dB_t^S + (1 - q_t)bdB_t, \\ U_0 = u. \end{cases} \tag{2.4}$$

In Section 3, where we consider only optimal financial investment, we set $q_t \equiv 0$ in equation (2.4). In Section 4 where we invest only in the riskless asset, we set $\pi_t \equiv 0$. In Section 5 we consider the general optimization problem embodied in equation (2.4).

Throughout this paper we denote the time of ruin by $\tau = \inf\{t : U_t < 0\}$, which for a Brownian motion risk process equals $\inf\{t : U_t = 0\}$ with probability 1. The (minimum) probability of ruin is denoted by ψ and is defined by

$$\psi(u) = \inf_{(\pi,q)\in\mathcal{A}} \mathbf{Pr}(\tau < \infty | U_0 = u), \tag{2.5}$$

in which $\mathcal{A}$ is the set of admissible investment and reinsurance strategies. In the various special cases that we consider, we further constrain π and q as described in the paragraph following equation (2.4).

To help us solve for the minimum probability of ruin ψ and the corresponding optimal strategies, we have a verification theorem. We state and prove it for the general case—the one in Section 5—but it also holds for the case considered in Sections 3 and 4. First, for every $\alpha = (\pi, q) \in \mathbf{R} \times [0, 1]$, with π and q constant, we define a second-order differential operator $\mathcal{L}^\alpha$ as follows: For every open set $G \subset \mathbf{R}^+$ and for every $h \in C^2(G)$, we define the function $\mathcal{L}^\alpha h: G \to \mathbf{R}$ by

$$\mathcal{L}^\alpha h(u) = (ru + (\mu - r)\pi + (\theta - \eta q)a)h'(u) + \frac{1}{2}(\sigma^2\pi^2 + 2\sigma\pi b\rho_S(1 - q) + (1 - q)^2 b^2)h''(u). \tag{2.6}$$

Theorem 2.1

Suppose h is a decreasing function from $\mathbf{R}^+$ *to* $[0, 1]$ *and suppose* $\alpha^* = (\pi^*, q^*)$ *is a function from* $\mathbf{R}^+$ *to* $\mathbf{R} \times [0, 1]$, *with* π^* *and* q^* *admissible, such that*

1. $h \in C^2(\mathbf{R}^+)$
2. h' *is bounded on* $\mathbf{R}^+$
3. $\mathcal{L}^\alpha h(u) \geq 0$, *for* $\alpha \in \mathbf{R} \times [0, 1]$
4. $\mathcal{L}^{\alpha^*(u)} h(u) = 0$, *for* $u \in \mathbf{R}^+$
5. $h(0) = 1$, *and* $\lim_{u\to\infty} h(u) = 0$.

Under the above conditions, h is the minimum probability of ruin in ψ *in equation* (2.5), π^* *is the optimal investment strategy in the risky asset, and* q^* *is the optimal purchasing strategy of quota-share reinsurance.*

PROOF

Assume that we have h as specified in the statement of this theorem. Let $\alpha = (\pi, q)$ be some admissible function on $\mathbf{R}^+$ to $\mathbf{R} \times [0, 1]$. Let U^α denote the surplus process when we use α as the investment and quota-share reinsurance policy, and let $\alpha_s = \alpha(U_s^\alpha)$.

By applying Itô's lemma, we have

$$h(U_{t\wedge\tau}^\alpha) = h(u) + \int_0^{t\wedge\tau} \mathcal{L}^{\alpha_s} h(U_s^\alpha)ds + \int_0^{t\wedge\tau} \sigma\pi_s h'(U_s^\alpha)dB_s^S + \int_0^{t\wedge\tau} b(1 - q_s)h'(U_s^\alpha)dB_s. \tag{2.7}$$

Denote the conditional expectation given $U_0 = u$ by $\mathbf{E}^u$. The expectations of the third and fourth terms in equation (2.7) are 0 because

$$\begin{aligned} &\mathbf{E}^u\left[\int_0^{t\wedge\tau} (h'(U_s^\alpha))^2\ (\sigma^2\pi_s^2 + 2\sigma b\rho_S\pi_s(1 - q_s) + b^2(1 - q_s)^2)ds\right] \\ &\quad < \max_{u\geq 0}(h'(u))^2\left(\sigma^2\mathbf{E}^u\left[\int_0^{t\wedge\tau} \pi_s^2 ds\right] + 2\sigma b\rho_S\mathbf{E}^u\left[\int_0^{t\wedge\tau} \pi_s ds\right] + b^2 t\right) < \infty \end{aligned} \tag{2.8}$$

(see, e.g., Arnold 1974, Theorem 5.1.1). The second inequality in expression (2.8) holds because π and q are admissible and because $h'(u)$ is bounded on $\mathbf{R}^+$ by assumption (2).

It follows that

$$\mathbf{E}^u[h(U_{t\wedge\tau}^\alpha)] = h(u) + \mathbf{E}^u\left[\int_0^{t\wedge\tau} \mathcal{L}^{\alpha_s} h(U_s^\alpha)ds\right] \geq h(u), \tag{2.9}$$

where the inequality follows from assumption 3 of the theorem. The expression in equation (2.9) shows that $\{h(U_{t\wedge\tau}^\alpha)\}_{t\geq 0}$ is a submartingale.

Now, since $h(0) = 1$, we have $h(U_\tau^\alpha) = \mathbf{1}_{\{\tau<\infty\}}$. By taking the expectation of this expression, we have

$$\mathbf{E}^u[h(U_\tau^\alpha)] = \mathbf{Pr}^u(\tau^\alpha < \infty) \geq h(u). \tag{2.10}$$

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$\mathbf{Pr}^u$ denotes the conditional probability given $U_0 = u$. We write τ^α to emphasize the dependence of τ on the strategy α. The inequality in equation (2.10) follows from an application of an optional sampling theorem since $\{h(U^\alpha_{t\wedge\tau})\}_{t\geq 0}$ is a submartingale. Indeed, because $h(u) \in [0, 1]$ for all $u \geq 0$, we can apply an optional sampling theorem (Karatzas and Shreve 1991, Theorem 1.3.15). Therefore,

$$\inf_{\alpha\in\mathcal{A}} \mathbf{Pr}^u\ (\tau^\alpha < \infty) \geq h(u). \tag{2.11}$$

Let α^* be as specified in the statement of this theorem; that is, α^* is the minimizer of $\mathcal{L}^\alpha h$. It follows that $\{h(U^{\alpha^*}_{t\wedge\tau})\}_{t\geq 0}$ is a martingale. Therefore, by an argument similar to the one leading to (2.9), we have

$$\mathbf{E}^u[h(U^{\alpha^*}_\tau)] = \mathbf{Pr}^u(\tau^{\alpha^*} < \infty) = h(u). \tag{2.12}$$

But ψ is the minimum probability of ruin, so we have equality in expression (2.11). Thus, we have demonstrated that h as described in the theorem is, indeed, the minimum probability ψ as defined in equation (2.5), and we have demonstrated that α^* is the optimal strategy. QED

It follows from Theorem 2.1 that if we find the solution to the Hamilton-Jacobi-Bellman (HJB) equation as described in items 1–5, we have the (unique) minimum probability of ruin and corresponding optimal strategies π^* and q^*.

3. INVESTING SURPLUS IN RISKY AND RISKLESS ASSETS

In this section we allow the insurer to invest optimally in a risky asset whose price process S follows geometric Brownian motion as in equation (2.3) as well as in a riskless asset with constant continuous rate of return $r \geq 0$. We do not allow the insurer to purchase reinsurance, so $q_t \equiv 0$ in equation (2.4). In Section 3.1 we do not restrict the amount invested in the risky asset. However, in Section 3.2 we restrict it to lie between 0 and the current surplus; that is, we do not allow the insurer to sell the risky asset short or to borrow money in order to invest in the risky asset. Finally, in Section 3.3 we allow the insurer to borrow money in order to invest in the risky asset but at a rate R higher than the riskless rate r. In each case we change the HJB equation in Theorem 2.1 appropriately.

3.1 Not Restricting the Amount Invested in the Risky Asset

From Theorem 2.1 we know that the minimum probability of ruin ψ is the unique solution of the following HJB equation:

$$\begin{cases} (ru + \theta a)\psi' + \dfrac{1}{2} b^2\psi'' + \min\limits_{\pi} \left[(\mu - r)\pi\psi' + \left(\dfrac{1}{2}\sigma^2\pi^2 + \sigma\pi b\rho_S\right) \psi''\right] = 0, \\ \psi(0) = 1, \quad \lim\limits_{u\to\infty} \psi(u) = 0. \end{cases} \tag{3.1}$$

According to Pestien and Sudderth (1985) and applied in Browne (1995), the optimal amount invested in the risky asset π^* is the amount that maximizes the drift divided by the square of the volatility of the surplus process. Specifically, we find π to maximize

$$f(\pi) = \frac{ru + (\mu - r)\pi + \theta a}{\sigma^2\pi^2 + 2\sigma\pi b\rho_S + b^2}. \tag{3.2}$$

Thus,

$$\pi^*(u) = \frac{1}{\mu - r}\left[-(ru + \theta a) + \sqrt{(ru + X)^2 + Y^2}\right], \tag{3.3}$$

in which $X = \theta a - b\rho_S\ (\mu - r)/\sigma$, and $Y = b\sqrt{1 - \rho_S^2}\ (\mu - r)/\sigma$.

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We hypothesize that ψ is convex on $\mathbf{R}^+$; thus, π^* is also given via the first-order condition from (3.1). Because the unique solution of the HJB equation equals the minimum probability of ruin from Theorem 2.1, it is perfectly acceptable to hypothesize additional properties for ψ to find that solution. It follows from the assumption of convexity that

$$\pi^*(u) = -\frac{\mu - r}{\sigma^2}\frac{\psi'(u)}{\psi''(u)} - \frac{b\rho_S}{\sigma}. \tag{3.4}$$

Equate equations (3.3) and (3.4) and solve the resulting equation for ψ. Because of the cumbersome nature of the solution, we present a few intermediate steps. First,

$$\psi'(u) = \psi'(0)\, \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_0^u \frac{dv}{-(rv + X) + \sqrt{(rv + X)^2 + Y^2}}\right]. \tag{3.5}$$

It follows from equation (3.5) that

$$\begin{aligned}\psi(u) &= 1 + \int_0^u \psi'(v)dv \\ &= 1 + \psi'(0) \int_0^u \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_0^v \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}}\right] dv.\end{aligned} \tag{3.6}$$

We can use the boundary condition $\lim_{u\to\infty} \psi(u) = 0$ to determine $\psi'(0)$. Indeed,

$$0 = 1 + \psi'(0) \int_0^\infty \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_0^v \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}}\right] dv. \tag{3.7}$$

Thus, we have the following theorem.

Theorem 3.1

The minimum probability of ruin for the model in this section is given by

$$\psi(u) = 1 - \frac{\displaystyle\int_0^u \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_0^v \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}}\right] dv}{\displaystyle\int_0^\infty \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_0^v \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}}\right] dv}. \tag{3.8}$$

The corresponding optimal investment strategy in the risky asset π^* is as in equation (3.3).

PROOF

It is straightforward to show that ψ as given in equation (3.8) is decreasing and convex and solves the HJB equation in (3.1). Thus, Theorem 2.1 implies that ψ and π^* in equations (3.8) and (3.3), respectively, are the minimum probability of ruin and the corresponding optimal investment strategy. QED

One can obtain special cases from our result by considering various values for the parameters. In particular, we have the following well-known result; for example, see Klugman, Panjer, and Willmot (1998, Corollary 6.10).

Corollary 3.2

When $\mu = r = 0$, then the probability of ruin is given by $\psi(u) = \exp\,(-2\theta\alpha/b^2\, u)$.

There is a simple intuitive explanation for the fact that ψ is an exponential function in this special case. Suppose the insurer begins with an initial surplus of $u + v$ and divides this amount into two pots:

one with u and the other with v. To be ruined, the insurer must first lose the pot with u, and after this occurs, the insurer must subsequently lose the one with v. We therefore have the identity

$$\psi(u + v) = \psi(u)\psi(v).$$

As is well known, with even the mildest type of regularity condition (e.g., measurability), the only positive solutions to this equation are exponential functions (Aczél 1966, Section 2.1, Theorem 1).

The above argument relies on the fact that the risk process has continuous paths, so the surplus at time of ruin will be 0. This is not true when the risk process involves jumps, as in the familiar compound Poisson process. In that case the ruining claim will cause a deficit. After losing the initial u, the insurer will have something less than v in the remaining pot. Recall that the formula for ruin probability in that case (Bowers et al. 1997, Theorem 13.4.1) involves the surplus at time of ruin.

3.2 No Borrowing and No Short Selling

In this section we restrict π^* so that $0 \le \pi^*(u) \le u$ for $u \ge 0$, and we obtain the corresponding minimum probability of ruin ψ. To this end we hypothesize a form for the optimal amount invested in the risky asset. We consider π^* in the unrestricted case, as given in equation (3.3). It is easy to show that π^* in equation (3.3) is decreasing with respect to u. Thus, if $\pi^*(0) < 0$ in equation (3.3), then the optimal investment strategy in the restricted case is $\pi^* \equiv 0$, and the probability of ruin is the solution of the ordinary differential equation in (3.1) with $\pi^* \equiv 0$. The resulting probability of ruin ψ is

$$\psi(u) = \frac{\Phi\left(-\sqrt{\frac{2}{r}}\,\frac{ru + \theta a}{b}\right)}{\Phi\left(\sqrt{\frac{2}{r}}\,\frac{\theta a}{b}\right)}, \tag{3.9}$$

in which Φ is the cumulative distribution function of the standard normal random variable. See Example 2 in Section 3.2 of Gerber (1975) for an early reference of this result.

For the remainder of this section, suppose that $\pi^*(0) \ge 0$ in equation (3.3), or equivalently,

$$b\,\frac{\mu - r}{\sigma} \ge 2\theta a \rho_S. \tag{3.10}$$

Because the unconstrained π^* in equation (3.3) decreases with respect to u, suppose that the optimal investment strategy in the constrained case is of the form

$$\pi^*(u) = \begin{cases} u, & \text{if } 0 \le u \le u_l; \\ \dfrac{1}{\mu - r}\left[-(ru + \theta a) + \sqrt{(ru + X)^2 + Y^2}\right], & \text{if } u_l < u < u_0; \\ 0, & \text{if } u_0 \le u \end{cases} \tag{3.11}$$

for some $0 \le u_l < u_0 \le \infty$. Note that it is not automatic that the optimal restricted investment strategy will be of the form in equation (3.11); that is, we cannot simply state that the optimal restricted π^* is obtained by truncating π^* from equation (3.3). However, if we can show that π^* in equation (3.11) and the resulting ψ satisfy the conditions of Theorem 2.1, then π^* in equation (3.11) is indeed the optimal investment strategy.

We use a subscript l to denote that u_l is the lending level above which the individual invests a positive amount of wealth in the riskless asset. We use a subscript 0 to denote that u_0 is the level above which the individual invests *no* wealth in the risky asset. If we assume that π^* is continuous, then

$$u_l = \frac{1}{\mu + r}\left[-(\theta a + r\rho_S b/\sigma) + \sqrt{(\theta a + r\rho_S b/\sigma)^2 - (\mu + r)\{2\theta a\rho_S b/\sigma - (\mu - r)(b/\sigma)^2\}}\right], \tag{3.12}$$

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and

$$u_0 = \begin{cases} \infty, & \text{if } \rho_S \leq 0; \\ \dfrac{b\,\dfrac{\mu - r}{\sigma} - 2\theta a \rho_S}{2r\rho_S}, & \text{if } \rho_S > 0. \end{cases} \tag{3.13}$$

Note that if $\rho_S \leq 0$, then the unconstrained π in equation (3.3) is always positive. The intuition behind this is that if $\rho_S < 0$, then the insurance losses and the price of the risky asset are *positively* correlated; see equation (2.1). Thus, to hedge the insurance losses, the insurer wants to invest a positive amount in the risky asset.

Consider the three cases in equation (3.11).

Case A

$0 \leq u \leq u_l$.

By substituting $\pi^*(u) = u$ into the HJB equation in (3.1) and by solving the resulting differential equation, we obtain

$$\psi(u) = 1 + \psi'(0) \int_0^u \frac{g(v)}{g(0)}\, dv, \tag{3.14}$$

in which

$$g(u) = h(u)e^{j(u)}, \tag{3.15}$$

and h and j are, respectively, given by

$$h(u) = (u^2 + 2u\rho_S b/\sigma + (b/\sigma)^2)^{-\mu/\sigma^2} \tag{3.16}$$

and

$$j(u) = \frac{2\mu}{\sigma} \frac{1}{b\sqrt{1 - \rho_S^2}} \left(\frac{b\rho_S}{\sigma} - \frac{\theta a}{\mu} \right) \arctan \frac{\sigma u + b\rho_S}{b\sqrt{1 - p_S^2}}. \tag{3.17}$$

Case B

$u_l < u < u_0$.

By substituting $\pi^*(u) = 1/(\mu - \mathrm{r}) \left[-(ru + \theta a) + \sqrt{(ru + X)^2 + Y^2} \right]$ into the HJB equation in (3.1) and by solving the resulting differential equation as in Section 3.1, we obtain

$$\psi(u) = \psi(u_l) + \psi'(u_l) \int_{u_l}^u \exp\left[-\left(\frac{\mu - r}{\sigma} \right)^2 \int_{u_l}^v \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}} \right] dv. \tag{3.18}$$

Case C

$u_0 \leq u$.

By substituting $\pi^*(u) \equiv 0$ into the HJB equation in (3.1), we obtain

$$\psi(u) = \psi(u_0) \frac{\Phi\left(-\sqrt{\dfrac{2}{r}} \dfrac{ru + \theta a}{b} \right)}{\Phi\left(-\sqrt{\dfrac{2}{r}} \dfrac{ru_0 + \theta a}{b} \right)}, \tag{3.19}$$

a scaling of ψ in equation (3.9).

To determine $\psi'(0)$, $\psi(u_l)$, $\psi'(u_l)$, and $\psi(u_0)$ we require that ψ and ψ' be continuous at $u = u_l$ and $u = u_0$. This requirement leads to the following four equations that one can solve for the four unknowns:

$$1 + \psi'(0) \int_0^{u_l} \frac{g(v)}{g(0)}\, dv = \psi(u_l), \tag{3.20}$$

$$\psi'(0) \frac{g(u_l)}{g(0)} = \psi'(u_l), \tag{3.21}$$

$$\psi(u_l) + \psi'(u_l) \int_{u_l}^{u_0} \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_{u_l}^{v} \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}} \right] dv = \psi(u_0), \tag{3.22}$$

and

$$\begin{aligned} &\psi'(u_l) \exp\left[-\left(\frac{\mu - r}{\sigma}\right)^2 \int_{u_l}^{u_0} \frac{dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}} \right] \\ &= -\frac{\sqrt{2r}}{b} \psi(u_0) \frac{\phi\left(-\sqrt{\frac{2}{r}} \frac{ru_0 + \theta a}{b}\right)}{\Phi\left(-\sqrt{\frac{2}{r}} \frac{ru_0 + \theta a}{b}\right)}, \end{aligned} \tag{3.23}$$

in which ϕ is the probability density function of the standard normal random variable.

Note that if $u_0 = \infty$, then we can immediately say that $\psi(u_0) = 0$, and we can use equations (3.20)–(3.22) to solve for the remaining three unknowns: $\psi'(0)$, $\psi(u_l)$, and $\psi'(u_l)$.

Thus, because a constrained version of Theorem 2.1 is valid, we have the following theorem.

Theorem 3.3

If no borrowing and no short selling are allowed, and if inequality (3.10) *holds, then the minimum probability of ruin is given by*

$$\psi(u) = \begin{cases} 1 + \psi'(0) \int_0^u \frac{g(v)}{g(0)}\, dv, & 0 \le u \le u_l; \\ \psi(u_l) + \psi'(u_l) \int_{u_l}^{u} \exp\left[- \int_{u_l}^{v} \frac{(\mu - r)^2/\sigma^2\, dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}} \right] dv, & u_l < u < u_0; \\ \psi(u_0) \frac{\Phi\left(-\sqrt{\frac{2}{r}} \frac{ru + \theta a}{b}\right)}{\Phi\left(-\sqrt{\frac{2}{r}} \frac{ru_0 + \theta a}{b}\right)}, & u_0 \le u, \end{cases} \tag{3.24}$$

in which $\psi'(0)$, $\psi(u_l)$, $\psi'(u_l)$, *and* $\psi(u_0)$ *solve equations* (3.20)–(3.23) *if* $\rho_S > 0$. *If* $\rho_S \le 0$, *then* $u_0 = \infty$, *and* $\psi'(0)$, $\psi(u_l)$, *and* $\psi'(u_l)$ *solve equations* (3.20)–(3.22). *The corresponding optimal investment strategy in the risky asset is given in equation* (3.11) *with* u_l *and* u_0 *as in equations* (3.12) *and* (3.13), *respectively.*

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3.3 Borrowing at a Higher Rate and No Short Selling

In this section we continue to prohibit short selling of the risky asset, but we allow the insurer to borrow at a rate $R \in (r, \mu)$. One can readily adapt Theorem 2.1 to this case and show that the minimum probability of ruin is the unique solution of the following HJB equation:

$$\begin{cases} 0 = \min\Big[(ru + \theta a)\psi' + \frac{1}{2} b^2\psi'' + \min_{0 \le \pi \le u} \left((\mu - r)\pi\psi' + \left(\frac{1}{2}\sigma^2\pi^2 + \sigma\pi b\rho_S\right)\psi'' \right), \\ \qquad (Ru + \theta a)\psi' + \frac{1}{2} b^2\psi'' + \min_{\pi \ge u} \left((\mu - R)\pi\psi' + \left(\frac{1}{2}\sigma^2\pi^2 + \sigma\pi b\rho_S\right)\psi'' \right)\Big]; \\ \psi(0) = 1, \quad \lim_{u\to\infty} \psi(u) = 0. \end{cases} \tag{3.25}$$

By way of explanation of equation (3.25), the first term in the minimization arises from the insurer's investing less than current surplus in the risky asset, $0 \le \pi \le u$, with the remainder $u - \pi$ invested in the riskless asset at rate r. The second term in the minimization arises from the insurer's investing more than current surplus in the risky asset, $\pi \ge u$, and borrowing the excess $\pi - u$ at rate R. Whichever of these two strategies yields the minimum is the optimal strategy.

Again, if $\pi^*(0) < 0$ in equation (3.3), then the optimal $\pi^* \equiv 0$ because we do not allow short selling, and the minimum probability of ruin ψ is given by equation (3.9). Therefore, assume that inequality (3.10) holds, and by analogy with equation (3.11), hypothesize that the optimal investment strategy is given by

$$\pi^*(u) = \begin{cases} \frac{1}{\mu - R}\left[-(Ru + \theta a) + \sqrt{(Ru + X_R)^2 + Y_R^2}\right], & \text{if } 0 \le u \le u_b, \\ u, & \text{if } u_b < u \le u_l, \\ \frac{1}{\mu - r}\left[-(ru + \theta a) + \sqrt{(ru + X)^2 + Y^2}\right], & \text{if } u_l < u < u_0, \\ 0, & \text{if } u_0 \le u, \end{cases} \tag{3.26}$$

in which $0 \le u_b < u_l < u_0$, $X_R = \theta a - b\rho_S \frac{\mu - R}{\sigma}$, and $Y_R = b\sqrt{1 - \rho_S^2}\,\frac{\mu - R}{\sigma}$. We use a subscript b to denote that u_b is the borrowing level below which the individual borrows a positive amount at the rate R. It follows that u_b is as in equation (3.12) with r replaced by R, and u_l and u_0 are given in equations (3.12) and (3.13), respectively.

Note that if we compute u_b to be a nonpositive number, then simply set it equal to 0, we are back in the no-borrowing and no-short-selling case of Section 3.2. Thus, without loss of generality, assume that $u_b > 0$, or equivalently,

$$\left[-(\theta a + R\rho_S b/\sigma) + \sqrt{(\theta a + R\rho_S b/\sigma)^2 - (\mu + R)\{2\theta a\rho_S b/\sigma - (\mu - R)(b/\sigma)^2\}}\right] > 0. \tag{3.27}$$

By considering the four cases as suggested in equation (3.26), we proceed as in Section 3.2 and obtain the following theorem.

Theorem 3.4

If borrowing is allowed at rate $R \in (r, \mu)$ but no short selling is allowed, and if inequalities (3.10) *and* (3.27) *hold, then the minimum probability of ruin is given by*

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$$\psi(u) = \begin{cases} 1 + \psi'(0) \int_0^u \exp\left[-\int_0^v \frac{(\mu - R)^2/\sigma^2 dw}{-(Rw + X_R) + \sqrt{(Rw + X_R)^2 + Y_R^2}} \right] dv, & 0 \le u \le u_b; \\ \psi(u_b) + \psi'(u_b) \int_{u_b}^u \frac{g(v)}{g(u_b)} dv, & u_b < u \le u_l; \\ \psi(u_l) + \psi'(u_l) \int_{u_l}^u \exp\left[-\int_{u_l}^v \frac{(\mu - r)^2/\sigma^2 dw}{-(rw + X) + \sqrt{(rw + X)^2 + Y^2}} \right] dv, & u_l < u < u_0; \\ \psi(u_0) \frac{\Phi\left(-\sqrt{\frac{2}{r}} \frac{ru + \theta a}{b} \right)}{\Phi\left(-\sqrt{\frac{2}{r}} \frac{ru_0 + \theta a}{b} \right)}, & u_0 \le u, \end{cases} \tag{3.28}$$

in which $\psi'(0)$, $\psi(u_b)$, $\psi'(u_b)$, $\psi(u_l)$, $\psi'(u_l)$, *and* $\psi(u_0)$ *are determined by continuity of* ψ *and* ψ' *at* $u = u_b$, u_l, and u_0 *if* $\rho_S > 0$. *If* $\rho_S \le 0$, *then* $u_0 = \infty$, *and we determine the remaining unknowns accordingly. The corresponding optimal investment strategy in the risky asset is given in equation* (3.26).

In the last theorem of this section, we examine the limiting behavior of this case as R approaches μ.

Theorem 3.5

If $\rho_S \ge 0$, *then as* R *approaches* μ, *the minimum probability of ruin and optimal investment strategy approach the corresponding functions in the no-borrowing and no-short-selling case specified in Theorem 3.2. If* $\rho_S < 0$, *then as* R *approaches* μ, *the no-borrowing case is not the limit of the borrowing case.*

PROOF

The theorem is an easy result of the following observations: If $\rho_S \ge 0$, then $\lim_{R\to\mu-} u_b \le 0$. On the other hand, if $\rho_S < 0$, then $\lim_{R\to\mu-} u_b > 0$. QED

4. PURCHASING QUOTA-SHARE REINSURANCE AND INVESTING IN A RISKLESS ASSET

In this section we allow the insurer to buy quota-share reinsurance optimally, as in Schmidli (2001), but assume that the surplus is invested in a riskless asset that earns interest at the constant continuous rate $r \ge 0$. According to Theorem 2.1, the minimum probability of ruin ψ solves the HJB equation

$$\begin{cases} ru\psi' + \min_{0 \le q \le 1} \left[(\theta - \eta q) a \psi' + \frac{1}{2}(1 - q)^2 b^2 \psi'' \right] = 0, \\ \psi(0) = 1, \quad \lim_{u\to\infty} \psi(u = 0). \end{cases} \tag{4.1}$$

As in Schmidli (2001), there are two main cases to consider, namely, $\theta < \eta \le 2\theta$ and $\eta > 2\theta$. To find the optimal proportion to reinsure in equation (4.1), we use the method of Browne (1995, Section 7.2) and Pestien and Sudderth (1985) and find q to maximize the drift divided by the square of volatility of the surplus process

$$f(q) = \frac{ru + (\theta - \eta q)a}{(1 - q)^2 b^2}. \tag{4.2}$$

The resulting optimal proportion to reinsure is given by

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$$q^*(u) = \begin{cases} 0, & 0 \le ru \le a(\eta/2 - \theta), \\ 2\dfrac{\theta}{\eta} - 1 + \dfrac{2ru}{\eta a}, & a(\eta/2 - \theta) < ru \le a(\eta - \theta), \\ 1, & ru > a(\eta - \theta). \end{cases} \tag{4.3}$$

Note that if $\eta \le 2\theta$, then the first two cases in equation (4.3) collapse into one case, namely, $0 \le ru \le a(\eta - \theta)$, because if $\eta \le 2\theta$ and $ru \le a(\eta/2 - \theta)$, then $u \le 0$. This collapse makes the case $\eta \le 2\theta$ easier.

We use equation (4.3) in what follows to determine the minimum probability of ruin ψ. We consider the solution of equation (4.1) case by case.

Case A

$\eta > 2\theta$.

Case A.1

$0 \le ru \le a(\eta/2 - \theta)$.

Substitute $q^*(u) \equiv 0$ for q in equation (4.1) to obtain

$$(ru + \theta a)\psi' + \frac{1}{2} b^2\psi'' = 0, \tag{4.4}$$

with $\psi(0) = 1$. It follows that

$$\psi(u) = 1 + k\sqrt{\frac{\pi}{r}} \cdot b \left[\Phi\left(\sqrt{\frac{2}{r}} \cdot \frac{ru + \theta a}{b}\right) - \Phi\left(\sqrt{\frac{2}{r}} \cdot \frac{\theta a}{b}\right)\right], \tag{4.5}$$

in which

$$k = \psi'(0) \exp\left(\frac{\theta^2 a^2}{rb^2}\right). \tag{4.6}$$

Case A.2

$a(\eta/2 - \theta) < ru \le a(\eta - \theta)$.

Substitute $q^*(u) = 2\dfrac{\theta}{\eta} - 1 + \dfrac{2ru}{\eta a}$ for q in equation (4.1) to obtain

$$\psi' + \frac{2b^2}{\eta^2 a^2}((\eta - \theta)a - ru)\psi'' = 0. \tag{4.7}$$

It follows that

$$\psi(u) = \psi\left(\frac{a}{r}\left(\frac{\eta}{2} - \theta\right)\right) + \psi'\left(\frac{a}{r}\left(\frac{\eta}{2} - \theta\right)\right)\frac{\eta a b^2}{\eta^2 a^2 + 2rb^2}\left\{1 - \left[2\,\frac{(\eta - \theta)a - ru}{\eta\alpha}\right]^{\frac{\eta^2 a^2 + 2rb^2}{2rb^2}}\right\}. \tag{4.8}$$

Case A.3

$ru > a(\eta - \theta)$.

Because $q^*(u) \equiv 1$ and because there is enough surplus to fund the cost of the reinsurance risklessly, there is no chance for ruin; therefore, $\psi(u) \equiv 0$ in this case.

By imposing continuity of ψ and ψ' at the boundaries of the three regions in Cases A.1–3, we can solve for k, $\psi(a(\eta/2 - \theta)/r)$, and $\psi'(a(\eta/2 - \theta)/r)$. Indeed, we obtain

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$$-k^{-1} = \sqrt{\frac{\pi}{r}} \cdot b \left[\Phi\left(\sqrt{\frac{2}{r}} \cdot \frac{\eta a}{2b}\right) - \Phi\left(\sqrt{\frac{2}{r}} \cdot \frac{\theta a}{b}\right) \right] + \exp\left(-\frac{\eta^2 a^2}{4rb^2}\right) \frac{\eta a b^2}{\eta^2 a^2 + 2rb^2}, \tag{4.9}$$

$$\psi\left(\frac{a}{r}\left(\frac{\eta}{2} - \theta\right)\right) = 1 + k\sqrt{\frac{\pi}{r}} \cdot b\left[\Phi\left(\sqrt{\frac{2}{r}} \cdot \frac{\eta a}{2b}\right) - \Phi\left(\sqrt{\frac{2}{r}} \cdot \frac{\theta a}{b}\right)\right], \tag{4.10}$$

and

$$\psi'\left(\frac{a}{r}\left(\frac{\eta}{2} - \theta\right)\right) = k \exp\left(-\frac{\eta^2 a^2}{4rb^2}\right), \tag{4.11}$$

Once we calculate k via equation (4.9), then we can get ψ for Case A.1 by substituting for k into equation (4.5). We can also get ψ for Case A.2 by calculating $\psi(a(\eta/2 - \theta)/r)$ and $\psi'(a(\eta/2 - \theta)/r)$ via equations (4.10) and (4.11), respectively, and substituting into equation (4.8) appropriately.

Case B

$\theta < \eta \le 2\theta$.

Case B.1

$0 \le ru \le a(\eta - \theta)$.

Substitute $q^*(u) = 2\,\dfrac{\theta}{\eta} - 1 + \dfrac{2ru}{\eta a}$ for q in equation (4.1) and solve to obtain

$$\psi(u) = 1 + \psi'(0)\, \frac{2(\eta - \theta)ab^2}{\eta^2 a^2 + 2rb^2} \left\{1 - \left[1 - \frac{ru}{(\eta - \theta)a}\right]^{\frac{\eta^2 a^2 + 2rb^2}{2rb^2}}\right\}. \tag{4.12}$$

Case B.2

$ru > a(\eta - \theta)$.

Because $q^*(u) \equiv 1$, there is no chance for ruin; therefore, $\psi(u) \equiv 0$, as in Case A.3.

By imposing continuity of ψ at $u = a(\eta - \theta)/r$, we obtain

$$0 = \psi\left(\frac{a}{r}(\eta - \theta)\right) = 1 + \psi'(0)\, \frac{2(\eta - \theta)ab^2}{\eta^2 a^2 + 2rb^2}. \tag{4.13}$$

It follows that equation (4.12) becomes

$$\psi(u) = \left[1 - \frac{ru}{(\eta - \theta)a}\right]^{\frac{\eta^2 a^2 + 2rb^2}{2rb^2}}. \tag{4.14}$$

We summarize the results of this section in the following theorem.

Theorem 4.1

If $\eta > 2\theta$, then the minimum probability of ruin ψ is given by equation (4.5) *when* $0 \le ru \le a(\eta/2 - \theta)$, *by equation* (4.8) *when* $a(\eta/2 - \theta) < ru \le a(\eta - \theta)$, *and by* 0 *when* $ru > a(\eta - \theta)$. *If* $\theta < \eta \le 2\theta$, *then the minimum probability of ruin is given by equation* (4.14) *when* $0 \le ru \le a(\eta - \theta)$ and by 0 *when* $ru > a(\eta - \theta)$. *The optimal quota-share reinsurance strategy* q^* *is given by equation* (4.3).

When $r = 0$, then our results reduce to those of Schmidli (2001), which we formalize in the following corollary.

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Corollary 4.2

If $r = 0$, then the minimum probability of ruin $\psi(u) = \exp(-\kappa u)$, in which κ equals

$$\kappa = \begin{cases} \dfrac{\eta^2}{\eta - \theta} \dfrac{a}{2b^2}, & \theta < \eta \leq 2\theta, \\ 2\,\dfrac{\theta a}{b^2}, & \eta > 2\theta. \end{cases} \tag{4.15}$$

The corresponding optimal proportion to reinsure is given by

$$q^* = \begin{cases} 2\,\dfrac{\theta}{\eta} - 1, & \theta < \eta \leq 2\theta, \\ 0, & \eta > 2\theta. \end{cases} \tag{4.16}$$

5. PURCHASING QUOTA-SHARE REINSURANCE AND INVESTING IN THE RISKY MARKET

In this section we allow the insurer to buy quota-share reinsurance optimally and assume that surplus is allocated optimally between a risky asset whose price follows a geometric Brownian motion as described in Section 2. According to Theorem 2.1, the minimum probability of ruin ψ is the unique solution to the HJB equation

$$\begin{cases} ru\psi' + \min\limits_{\pi, 0 \leq q \leq 1} \left[((\mu - r)\pi + (\theta - \eta q)a)\psi' + \dfrac{1}{2}(\sigma^2\pi^2 + 2\sigma\pi b\rho_S\,(1 - q) + (1 - q)^2 b^2)\psi'' \right] \\ \quad = 0; \\ \psi(0) = 1, \lim\limits_{u\to\infty} \psi(u) = 0. \end{cases} \tag{5.1}$$

To solve this optimization problem, we maximize the drift divided by the square of the volatility of the surplus process with respect to q and π:

$$f(q, \pi) = \frac{ru + (\mu - r)\pi + (\theta - \eta q)a}{\sigma^2\pi^2 + 2\sigma b\rho_S\pi(1 - q) + b^2(1 - q)^2}. \tag{5.2}$$

Assume that we have an interior solution to this problem, that is, assume the optimal q lies in (0, 1). Use the first-order conditions to get two equations for q and π. By taking the derivative of f with respect to π and setting the derivative equal to 0, we obtain

$$-\sigma^2\pi^2(\mu - r) - 2\sigma(\sigma\pi + b\rho_S(1 - q))(ru + (\theta - \eta q)a) + b^2(1 - q)^2(\mu - r) = 0. \tag{5.3}$$

Similarly, by taking the derivative of f with respect to q and setting the derivative equal to 0, we obtain

$$\begin{aligned} &\pi^2(2\sigma b\rho_S(\mu - r) - \sigma^2\eta a) + 2b\pi[\sigma\rho_S(ru - (\eta - \theta)a) + b(1 - q)(\mu - r)] \\ &\quad + b^2(1 - q)[2ru + (\theta - \eta q)a - (\eta - \theta)a] = 0. \end{aligned} \tag{5.4}$$

Theoretically, one could eliminate π^2 from these two equations, solve for π, substitute for π in one of equations (5.3) or (5.4), and solve the resulting expression for q. However, this path is rather torturous. Instead, we hypothesize that q is related to π linearly; specifically, we assume that there exists a constant A such that

$$q(u) = 1 - A\pi(u), \quad u \geq 0. \tag{5.5}$$

By substituting this expression for q into equations (5.3) and (5.4), we find that our hypothesis was correct. Indeed, we get

$$\pi^*(u) = 2\,\frac{ru - (\eta - \theta)a}{\sigma}\,\frac{\eta\rho_S x - y}{\eta^2x^2 - 2\eta\rho_S xy + y^2} \tag{5.6}$$

and

$$q^*(u) = 1 + 2\,\frac{ru - (\eta - \theta)a}{b}\,\frac{\eta x - \rho_S y}{\eta^2x^2 - 2\eta\rho_S xy + y^2}, \tag{5.7}$$

in which

$$x = \frac{a}{b} \quad \text{and} \quad y = \frac{\mu - r}{\sigma}. \tag{5.8}$$

Recall that these expressions for π^* and q^* assume that we have an interior solution for q^*. If the right-hand side of equation (5.7) is less than 0 or greater than 1, then we do not have an interior solution for q^*, and we have to truncate this expression by 0 or 1. By carefully working through the algebra, we discover that we have three cases to handle, each of which subsumes two or three subcases. For convenience, we make the following definition:

$$B = \frac{\eta x - \rho_S y}{\eta^2x^2 - 2\eta\rho_S xy + y^2}. \tag{5.9}$$

Case A

$\eta x - \rho_S y > 0$ and $(\eta - \theta)a - \dfrac{b}{2B} > 0$.

Case A.1

$1 + 2B\,\dfrac{ru - (\eta - \theta)a}{b} \le 0$.

Then $q^*(u) \equiv 0$, and π^* solves equation (5.3) with $q = 0$. Thus,

$$\pi^*(u) = \frac{1}{\mu - r}\left[-(ru + \theta a) + \sqrt{(ru + X)^2 + Y^2}\right], \tag{5.10}$$

in which X and Y are given as in Section 3. From the work in that section, we have that

$$\psi(u) = 1 + \psi'(0)\int_0^u \exp\left\{-\frac{1}{b^2(1 - \rho_S^2)}\int_0^v \left[(rw + X) + \sqrt{(rw + X)^2 + Y^2}\right]dw\right\} dv. \tag{5.11}$$

Case A.2

$0 < 1 + 2B\,\dfrac{ru - (\eta - \theta)a}{b} < 1$.

Then π^* and q^* are given by equations (5.6) and (5.7), respectively. We can solve for ψ by using the HJB equation. Indeed, from the first-order condition for π in equation (5.1), we have that

$$\pi^*(u) = -\frac{\mu - r}{\sigma^2}\frac{\psi'}{\psi''} + 2\,\frac{\rho_S B}{\sigma}\,(ru - (\eta - \theta)a)). \tag{5.12}$$

If we equate this with the expression for π^* in equation (5.6) and solve for ψ''/ψ', we obtain

$$\frac{\psi''(u)}{\psi'(u)} = \frac{1}{2B}\,\frac{\eta x - \rho_S y}{1 - \rho_S^2}\,\frac{1}{ru - (\eta - \theta)a}. \tag{5.13}$$

It follows that

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$$\psi(u) = \psi\left[\frac{2B(\eta-\theta)a-b}{2Br}\right] + \psi'\left[\frac{2B(\eta-\theta)a-b}{2Br}\right]\frac{b}{\frac{\eta x-\rho_S y}{1-\rho_S^2}+2Br}\left\{1-\left[2B\,\frac{(\eta-\theta)a-ru}{b}\right]^{\frac{\eta x-\rho_S y}{2Br(1-\rho_S^2)}+1}\right\}. \tag{5.14}$$

Alternatively, we could have substituted π^* and q^* from equations (5.6) and (5.7), respectively, into equation (5.1) and solved for ψ.

Case A.3

$1 + 2B\,\frac{ru-(\eta-\theta)a}{b} \geq 1.$

Then $q^*(u) \equiv 1$, and π^* is not unique. All we require is that π^* is such that wealth will not drop below $(\eta-\theta)a/r$, the amount required to fund the excess cost of the reinsurance. Note that $\psi(u) \equiv 0$ because all of the risk has been reinsured, and there is sufficient surplus to fund the cost of this reinsurance risklessly. In Case C below, we will encounter a situation in which all of the risk is reinsured, but there is not enough surplus to fund the cost of it without investing in the risky asset.

We can determine the values of $\psi'(0)$, $\psi[(\eta-\theta)a/r - b/(2Br)]$, and $\psi'[(\eta-\theta)a/r - b/(2Br)]$ by assuming that ψ and ψ' are continuous at $u = (\eta-\theta)a/r - b/(2Br)$ and $u = (\eta-\theta)a/r$. If we do this, then we obtain the following three equations that relate the three unknowns:

$$1 + \psi'(0)\int_0^{(\eta-\theta)a/r-b/2Br}\exp\left\{\frac{-1}{b^2(1-\rho_S^2)}\int_0^v\left[(rw+X)+\sqrt{(rw+X)^2+Y^2}\right]dw\right\}dv = \psi\left[\frac{2B(\eta-\theta)a-b}{2Br}\right], \tag{5.15}$$

$$\psi'(0)\exp\left\{\frac{-1}{b^2(1-\rho_S^2)}\int_0^{(\eta-\theta)a/r-b/2Br}\left[(rw+X)+\sqrt{(rw+X)^2+Y^2}\right]dw\right\} = \psi'\left[\frac{2B(\eta-\theta)a-b}{2Br}\right], \tag{5.16}$$

and

$$\psi\left[\frac{2B(\eta-\theta)a-b}{2Br}\right] + \frac{b}{\frac{\eta x-\rho_S y}{1-\rho_S^2}+2Br}\,\psi'\left[\frac{2B(\eta-\theta)a-b}{2Br}\right] = 0. \tag{5.17}$$

Case B

$\eta x - \rho_S y > 0$ and $(\eta-\theta)a - \frac{b}{2B} \leq 0.$

In this case $q^* > 0$, so the first two subcases of Case A collapse to $0 \leq ru < (\eta-\theta)a$.

Case B.1

$0 \leq ru < (\eta-\theta)a.$

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Then $q^*(u) = 1 + 2B \dfrac{ru - (\eta - \theta)a}{b} \in (0, 1)$. By following the same technique used to solve Case A.2, we have that ψ is given by

$$\psi(u) = 1 + \psi'(0) \frac{(\eta - \theta)a}{r} \frac{2Br}{\dfrac{\eta x - \rho_S y}{1 - \rho_S^2} + 2Br} \left\{1 - \left[1 - \frac{ru}{(\eta - \theta)a}\right]^{\frac{\eta x - \rho_S y}{2Br(1-\rho_S^2)}+1}\right\}. \tag{5.18}$$

Case B.2

$ru \geq (\eta - \theta)a.$

Then $q^*(u) \equiv 1$, and π^* is any strategy that keeps wealth above $(\eta - \theta)a/r$. It follows that $\psi(u) \equiv 0$.

By imposing continuity of ψ at $u = (\eta - \theta)a/r$, we obtain that for $0 \leq ru < (\eta - \theta)a$, ψ is given by

$$\psi(u) = \left[1 - \frac{ru}{(\eta - \theta)a}\right]^{\frac{\eta x - \rho_S y}{2Br(1 - \rho_S^2)}+1}. \tag{5.19}$$

Case C

$\eta x - \rho_S y \leq 0.$

In this case $q^*(u) \equiv 1$, intuitively reasonable because reinsurance is inexpensive relative to the return on the risky asset. Therefore, the only control remaining is the amount to invest in the risky asset.

Case C.1

$0 \leq ru < (\eta - \theta)a.$

By substituting $q = 1$ into equation (5.3), we obtain that

$$\pi^*(u) = 2 \frac{(\eta - \theta)a - ru}{\mu - r}. \tag{5.20}$$

By calculating the first-order condition for π^* from equation (5.1) with $q = 1$, we can solve for ψ:

$$\psi(u) = 1 + \psi'(0) \frac{(\eta - \theta)a}{\dfrac{1}{2}y^2 + r} \left\{1 - \left[1 - \frac{ru}{(\eta - \theta)a}\right]^{\frac{y^2}{2r}+1}\right\}. \tag{5.21}$$

Case C.2

$ru \geq (\eta - \theta)a.$

In this case the surplus is sufficient to fund the excess cost of the reinsurance as in Cases A.3 and B.2, so $\psi(u) \equiv 0$.

By imposing continuity of ψ at $u = (\eta - \theta)a/r$, we obtain that for $0 \leq ru < (\eta - \theta)a$, ψ is given by

$$\psi(u) = \left[1 - \frac{ru}{(\eta - \theta)a}\right]^{\frac{y^2}{2r}+1}. \tag{5.22}$$

6. NUMERICAL EXAMPLES

We consider two examples in this section. In the first, we demonstrate the results of this paper with a

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common set of parameters for the insurance and financial markets. In the second, we compare the optimal investment for the model in Section 3.3 with the optimal investment when claims follow a compound Poisson process.

6.1 Comparison of Optimal Strategies across Various Models

For this example, we take the following values for the parameters. The notation is consistent with that in the rest of the paper.

- Initial surplus $u = 1$.
- The parameters for the claim process C are $a = 1$, $b = 0.5$, and $\theta = 0.1$.
- The parameters for the financial market are $r = 0.04$, $R = 0.06$, $\mu = 0.08$, and $\sigma = 0.20$. We assume that the correlation between the risky asset and the claim process is $\rho_S = -0.20$.
- For the loading on the reinsurance premium, we consider two values: $\eta_1 = 0.15 \in (\theta, 2\theta)$ and $\eta_2 = 0.30 > 2\theta$.

For the various models considered in this paper, we present the probabilities of ruin $\psi(1)$, the corresponding optimal investment $\pi^*(1)$, and optimal quota-share reinsurance $q^*(1)$, when appropriate.

Investment in the Risky and Riskless Assets with No Restrictions

From Section 3.1, when $u = 1$, the probability of ruin $\psi(1) = 0.2268$ and amount invested in the risky asset equals 1.1904. Thus, the insurer borrows 0.1904 to invest in the risky asset. Note that, under this model, the insurer pays $r = 4\%$ interest on borrowed money, which might be considered unrealistic.

Investment in the Risky and Riskless Assets with No Borrowing and No Short Selling

From Section 3.2, the probability of ruin $\psi(1) = 0.2371$, necessarily at least as great as 0.2268, the probability when there is no restriction on π^*. The amount invested in the risky asset $\pi^*(1) = 1$ because $u_l = 1.1667$. Also, note that $u_0 = \infty$ in this case because $\rho_S < 0$.

Investment in the Risky and Riskless Assets with Borrowing and No Short Selling

From Section 3.3, $u_b = 0.8628$ and $u_l = 1.1667$; therefore, the insurer will neither borrow at rate $R = 0.06$ nor invest at rate $r = 0.04$ when $u = 1$. It follows that the probability of ruin $\psi(1) = 0.2342$, necessarily between the probability of ruin when the insurer can borrow at the rate of $r = 0.04$ and when the insurer cannot borrow at all.

Quota-Share Reinsurance with Investment in the Riskless Asset Only

From Section 4, when the reinsurance premium loading $\eta_1 = 0.15$, the optimal proportion reinsured $q^*(1) = 0.8667$. Because q^* is so high, the probability of ruin is quite low, namely, 0.0327.

When the reinsurance premium loading $\eta_2 = 0.30$, the optimal proportion reinsured $q^*(1) = 0$. The corresponding probability of ruin $\psi(1) = 0.2968$.

Quota-Share Reinsurance with Investment in the Risky and Riskless Assets

From Section 5, when the reinsurance premium loading $\eta_1 = 0.15$, the optimal proportion reinsured $q^*(1) = 0.9117$. Because q^* is so high, the probability of ruin is very low, namely, 0.0079. The optimal amount invested in the risky asset $\pi^*(1) = 0.1688$.

When the reinsurance premium loading $\eta_1 = 0.30$, the optimal proportion reinsured $q^*(1) = 0.0857$. The probability of ruin $\psi(1) = 0.2179$. The optimal amount invested in the risky asset $\pi^*(1) = 1.1429$.

Table 1
Probabilities of Ruin, Optimal Investment, and Optimal Quota-Share Reinsurance

	$\psi(1)$	$\pi^*(1)$	$q^*(1)$
Both assets (unrestricted)	0.2268	1.1904	n.a.
Both assets (no borrowing)	0.2371	1.0000	n.a.
Both assets (borrowing)	0.2342	1.0000	n.a.
η_1 = 0.15 + Riskless	0.0327	n.a.	0.8667
η_1 = 0.30 + Riskless	0.2968	n.a.	0.0000
η_1 = 0.15 + Both	0.0079	0.1688	0.9117
η_1 = 0.30 + Both	0.2179	1.1429	0.0857

In Table 1 we summarize the above results. Note that, in general, the values for ψ, π^*, and q^* depend on $u = 1$. Presented in tabular form, it is easier to see the inverse relation between π^* and q^* in the last two cases, which is due to the fact that $\rho_S < 0$.

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6.2 Comparison with Compound Poisson Case

In this example we compare the optimal investment strategy for our model in Section 3.3, as explicitly specified in equation (3.3), with the optimal investment strategy when claims follow a compound Poisson process. Liu and Yang (2004) considered such a model and obtained a numerical scheme for computing the optimal strategy in this case. If one supposes that claims actually do follow a compound Poisson process, then one can treat the Brownian motion in this paper as an approximation to the compound Poisson. If one relies on such an approximation, then one hopes that the optimal investment strategy given in equation (3.3) is close to the optimal strategy, as described in Liu and Yang (2004).

We consider the specific compound Poisson claim process as given in Example 3.1 of Liu and Yang (2004). They suppose that claims occur at a Poisson rate of $\lambda = 3$ and that the severity of claims is exponential with mean 1. Therefore the corresponding values for the approximating Brownian motion process are $a = 3 \cdot 1 = 3$ and $b = \sqrt{3 \cdot 2 \cdot 1^2} = \sqrt{6}$. The proportional premium loading is $\theta = 0.20$. Finally, the parameters of the financial market are $r = 0.04$, $\mu = 0.10$, and $\sigma = 0.30$. We assume that the claim process is independent of the price process of the risky asset, that is, $\rho_S = 0$.

In Table 2 we present the optimal investment for the compound Poisson process, π^P, and for the approximating Brownian motion process, π^B, as a function of surplus. We also include the percentage difference between the two investment strategies and the probability of ruin under the compound Poisson model.

Here π^B is consistently lower than π^P for $u \geq 0.14$. We hypothesize that this negative bias occurs because the volatility of claims $b \approx 2.45$ is significantly large relative to the drift of claims $a = 3$. Therefore, we are very likely to have negative claims with this model, so the insurer does not need to invest as much money in the risky asset as compared to the compound Poisson model, for which claims can only be positive. Considering the small magnitude of a/b, we are surprised that the two investment strategies are as close as they turned out to be.

On the other hand, when we compared the optimal investment strategies for the model in Example 3.2 of Liu and Yang (2004), where claim severities follow a Gamma distribution (not shown here), the negative bias was dramatic and of the order of -44%. Furthermore, in their Example 3.3, where claim severities follow a Pareto distribution, the optimal investment strategy increases with respect to surplus, and our optimal investment in equation (3.3) always decreases with respect to surplus.

7. SUMMARY AND CONCLUSIONS

In this paper we minimized the probability of ruin for an insurer who faces a claim process that follows Brownian motion with drift. We concentrated on this case because of the analytically pleasing results.

Table 2

Optimal Investment for Compound Poisson Process and Approximating Brownian Motion Process as a Function of Surplus, Percentage Difference between the Two Investment Strategies, and Probability of Ruin under Compound Poisson Model

u	$\pi^P(u)$	$\pi^B(u)$	%Δ	$\psi^P(u)$
0.1	2.70	2.89	7.4%	0.764
0.5	3.27	2.84	−13.2	0.696
1.0	3.22	2.77	−14.1	0.619
1.5	3.16	2.70	−14.6	0.550
2.0	3.10	2.63	−15.1	0.487
3.0	3.00	2.51	−16.1	0.381
4.0	2.90	2.40	−17.1	0.296
5.0	2.81	2.30	−18.1	0.228
6.0	2.72	2.21	−19.0	0.175
7.0	2.65	2.12	−19.9	0.133
8.0	2.57	2.04	−20.8	0.101
9.0	2.51	1.96	−21.7	0.076
10.0	2.44	1.89	−22.5	0.057
12.5	2.30	1.74	−24.6	0.027
15.0	2.18	1.60	−26.5	0.012

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We allowed the insurer to use two controls to minimize the probability of ruin: (1) investing in a risky asset and (2) buying quota-share reinsurance.

There are several possible extensions to the work here. One could restrict the investment in the risky asset to prohibit borrowing and short selling while buying quota-share reinsurance. One could also allow borrowing at a higher rate, prohibit short selling, and allow purchasing of quota-share reinsurance.

One could consider other claim processes, such as a compound Poisson process. For example, Hipp and Plum (2000) minimize the probability of ruin when claims follow a compound Poisson process while investing in a risky asset; their work is related to that in Section 3, except that they set $r = 0$. Liu and Yang (2004) extend the work of Hipp and Plum (2000) to the case for which the riskless asset earns a positive return, as in Section 3 of this paper. Schmidli (2001) minimizes the probability of ruin by purchasing quota-share reinsurance when the risk follows a compound Poisson process. Schmidli (2002) extends that case by investing in risky and riskless assets, as well as purchasing quota-share reinsurance as in Section 5 of this paper; however, no closed-form solutions exist for that case.

One could allow for additional controls of the surplus process. For example, Hipp and Taksar (2000) minimize the probability of ruin by controlling the rate of new business accumulation. Another control is that of the true optimal reinsurance (from the perspective of minimizing the probability of ruin), which we hypothesize is stop-loss reinsurance and not quota-share reinsurance.

Finally, one could consider alternative risk measures. For example, one might minimize the expected discounted time of ruin. Alternatively, one could minimize the expected discounted shortfall before or at some specified date. This last area of extension is the subject of our future research.

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