King Saud University: Mathematics Department M-254
Semester I (1st Midterm Exam) 1439-1440 H
Maximum Marks=25 Time Allowed: 90 Mins.

Questions : (5 + 5 + 5 + 5 + 5)

Q1: Find the first approximation of the point of intersection of the nonlinear equations $x^2 + y^2 = 1$ and $\frac{1}{3}x^2 + \frac{1}{2}y^2 = 1$ using Newton's method, starting with the initial approximation $(x_0, y_0)^T = (1, 1)^T$.

Q2: The equation $1 - 2\cos(x) + \cos^2(x) = 0$ has the root $\alpha = 0$. Develop the Modified Newton's formula for computing this root, then use it to find the second approximation by using the initial approximation $x_0 = 0.5$.

Q3: Show that the x-value of the intersection point (x, y) of the graphs $y = x^3 + 2x - 1$ and $y = \sin x$ is lying in the interval $[0.5, 1]$. Then use Secant method to find its second approximation, when $x_0 = 0.5$ and $x_1 = 0.55$. Also, find the intersection point.

Q4: Convert the equation $x^2 - 3x + 2 = 0$ to the fixed-point problem

$$x = \frac{1}{1+c}\left(cx + \frac{x^2+2}{3}\right),$$

with c a constant. Find a value of c to ensure rapid convergence of the following scheme

$$x_{n+1} = \frac{1}{1+c}\left(cx_n + \frac{x_n^2+2}{3}\right), \qquad n \geq 0,$$

at $\alpha = 1$. Compute the second approximation, starting with $x_0 = 0.5$.

Q5: To find approximation of root of quadratic equation $ax^2 + bx + c = 0$ we use the following iterative scheme

$$x_{n+1} = \frac{bx_n^2 + 2cx_n}{c - ax_n^2}, \qquad n \geq 0.$$

Show that this iterative scheme is atleast quadratic if $c \neq 0$ and $ax^2 \neq c$ at a root α. Use this iterative scheme to find the second approximation of the positive root of the equation $2x^2 - 5x = 3$, starting with $x_0 = 2.5$.

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Q1: Find the first approximation of the point of intersection of the nonlinear equations $x^2 + y^2 = 1$ and $\frac{1}{3}x^2 + \frac{1}{2}y^2 = 1$ using Newton's method, starting with the initial approximation $(x_0, y_0)^T = (1, 1)^T$.

Solution. We are given the nonlinear system

$$\begin{array}{rcl} x^2 \quad +y^2 & = & 1 \\ \frac{1}{3}x^2 \quad +\frac{1}{2}y^2 & = & 1 \end{array}$$

and it gives the functions and the first partial derivatives as follows:

$$\begin{array}{rcl} f_1(x,y) & = & x^2 + y^2 - 1, \qquad f_{1x} = 2x, \quad f_{1y} = 2y, \\ f_1(x,y) & = & \frac{1}{3}x^2 + \frac{1}{2}y^2 - 1, \quad f_{2x} = \frac{2}{3}x, \quad f_{2y} = y. \end{array}$$

At the given initial approximation $x_0 = 1$ and $y_0 = 1$, we get

$$\begin{array}{rclrclrcl} f_1(1,1) & = & 1, & \frac{\partial f_1}{\partial x} = f_{1x} & = & 2, & \frac{\partial f_1}{\partial y} = f_{1y} & = & 2, \\ f_2(1,1) & = & -\frac{1}{6}, & \frac{\partial f_1}{\partial x} = f_{2x} & = & \frac{2}{3}, & \frac{\partial f_2}{\partial y} = f_{2y} & = & 1. \end{array}$$

The Jacobian matrix J and its inverse J^{-1} at the given initial approximation can be calculated as follows:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ \frac{2}{3} & 1 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{2/3} \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & 2 \end{pmatrix}.$$

Substituting all these values in the formula, we get the first approximation as follows:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2/3} \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} -1 \\ 2.5 \end{pmatrix}.$$

Q2: The equation $1 - 2\cos(x) + \cos^2(x) = 0$ has the root $\alpha = 0$. Develop the Modified Newton's formula for computing this root, then use it to find the second approximation by using the initial approximation $x_0 = 0.5$.

Solution. Since $\alpha = 0$ is a root of $f(x)$, so

$$\begin{aligned}
f(x) &= 1 - 2\cos(x) + \cos^2(x) = (1-\cos(x))^2, & f(0) &= 0,\\
f'(x) &= 2\sin(x)(1-\cos(x)), & f'(0) &= 0,\\
f''(x) &= 2\sin^2(x) - 2\cos^2(x) + 2\cos(x), & f''(0) &= 0,\\
f'''(x) &= 4\sin(2x) - 2\sin(x), & f'''(0) &= 0,\\
f^{(4)}(x) &= 8\cos(2x) - 2\cos(x), & f^{(4)}(0) &= 6 \neq 0,
\end{aligned}$$

the function has zero of multiplicity 4. Using modified Newton's iterative formula, we get

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - 4\frac{(1-\cos(x_n))^2}{2\sin(x_n)(1-\cos(x_n))}.$$

Thus

$$x_{n+1} = x_n - \frac{2(1-\cos(x_n))}{\sin(x_n)}, \qquad n \geq 0.$$

Now evaluating this at the give approximation $x_0 = 0.5$, gives

$$x_1 = x_0 - \frac{2(1-\cos(x_0))}{\sin(x_0)} = -0.0107,$$

and

$$x_2 = x_1 - \frac{2(1-\cos(x_1))}{\sin(x_1)} = 1.0209 \times 10^{-7},$$

are the required approximations.

Q3: Show that the x-value of the intersection point (x, y) of the graphs $y = x^3 + 2x - 1$ and $y = \sin x$ is lying in the interval $[0.5, 1]$. Then use Secant method to find its second approximation, when $x_0 = 0.5$ and
$x_1 = 0.55$. Also, find the intersection point.

Solution. Since there is an intersection, so $x^3+2x-1 = \sin x$ or $= x^3+2x-\sin x-1 = 0$. Thus we have the nonlinear function of the form

$$f(x) = x^3 + 2x - \sin x - 1.$$

Note that

$$f(0.5) = -0.3544 \quad \text{and} \quad f(1.0) = 1.1585.$$

Since $f(x)$ is continuous on $[0.5, 1.0]$ and $f(0.5)f(1.0) < 0$, so that a x-value or root of $f(x) = 0$ lies in the interval $[0.5, 1.0]$.
Applying Secant iterative formula to find the approximation of this equation, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(x_n^3 + 2x_n - \sin x_n - 1)}{(x_n^3 + 2x_n - \sin x_n - 1) - (x_{n-1}^3 + 2x_{n-1} - \sin x_{n-1} - 1)}, \quad n \geq 1.$$

Finding the second approximation using the initial approximations $x_0 = 0.5$ and $x_1 = 0.55$, we get

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^3 + 2x_1 - \sin x_1 - 1)}{(x_1^3 + 2x_1 - \sin x_1 - 1) - (x_0^3 + 2x_0 - \sin x_0 - 1)} = 0.6606,$$

and $f(x_2) = f(0.6606) = 0.0473$, so

$$x_3 = x_2 - \frac{(x_2 - x_1)(x_2^3 + 2x_2 - \sin x_2 - 1)}{(x_2^3 + 2x_2 - \sin x_2 - 1) - (x_1^3 + 2x_1 - \sin x_1 - 1)} = 0.6603.$$

Thus x-value of the intersection point is $x = 0.6603$ and intersection point is $(0.6603, 0.61)$.

Q4: Convert the equation $x^2 - 3x + 2 = 0$ to the fixed-point problem

$$x = \frac{1}{1+c}\left(cx + \frac{x^2+2}{3}\right),$$

with c a constant. Find a value of c to ensure rapid convergence of the following scheme

$$x_{n+1} = \frac{1}{1+c}\left(cx_n + \frac{x_n^2+2}{3}\right), \qquad n \geq 0,$$

at $\alpha = 1$. Compute the second approximation, starting with $x_0 = 0.5$.

Solution. Given $x^2 - 3x + 2 = 0$ and it can be written as for any c

$$x(c - c + 1) = \frac{x^2+2}{3} \quad \text{or} \quad x(c+1) - xc = \frac{x^2+2}{3},$$

or

$$x(c+1) = cx + \frac{x^2+2}{3}.$$

From this we have

$$x = \frac{1}{1+c}\left(cx + \frac{x^2+2}{3}\right) = g(x),$$

and it gives the iterative scheme

$$x_{n+1} = \frac{1}{1+c}\left(cx_n + \frac{x^2+2}{3}\right) = g(x_n), \qquad n \geq 0.$$

For guaranteed the convergence will be rapid if

$$g'(1) = 0, \quad \text{gives} \quad c = -\frac{2}{3}.$$

Thus, $c = g'(1) = -\frac{2}{3}$. Now to find two iterates we have

$$\begin{aligned} x_1 &= \frac{1}{1+c}\left(cx_0 + \frac{x_0^2+2}{3}\right) = 1.25 \\ x_2 &= \frac{1}{1+c}\left(cx_1 + \frac{x_1^2+2}{3}\right) = 1.0625, \end{aligned}$$

the required approximations at the value of $c = -\frac{2}{3}$.

Q5: To find approximation of root of quadratic equation $ax^2 + bx + c = 0$ we use the following iterative scheme

$$x_{n+1} = \frac{bx_n^2 + 2cx_n}{c - ax_n^2}, \qquad n \geq 0.$$

Show that this iterative scheme is atleast quadratic if $c \neq 0$ and $ax^2 \neq c$ at a root α. Use this iterative scheme to find the second approximation of the positive root of the equation $2x^2 - 5x = 3$, starting with $x_0 = 2.5$.

Solution. Since

$$f(x) = x - g(x) = 0,$$

therefore, we have

$$cx - ax^3 = bx^2 + 2cx,$$

or

$$ax^3 + bx^2 + cx = 0.$$

Thus

$$x(ax^2 + bx + c) = 0.$$

For $x \neq 0$, we have the quadratic equation, $ax^2 + bx + c = 0$.
Since

$$g(x) = \frac{bx^2 + 2cx}{c - ax^2},$$

and

$$g'(x) = \frac{2c(ax^2 + bx + c)}{(c - ax^2)^2}.$$

At root α, $ax^2 + bx + c$ is identically zero and $g'(\alpha) = 0$ if $c - a\alpha^2 \neq 0$.
Finding the first two approximations of the positive root of $2x^2 - 5x = 3$ using the initial approximation $x_0 = 2.5$ and $a = 2, b = -5, c = -3$, we use the above iterative scheme by taking $n = 0, 1$ as follows

$$x_1 = \frac{bx_0^2 + 2cx_0}{c - ax_0^2} = 2.9839,$$

and

$$x_2 = \frac{bx_1^2 + 2cx_1}{c - ax_1^2} = 2.99999,$$

are the possible two approximations.