

Chapter 7

Eigenvalues and Eigenvectors



7.1 Eigenvalues and Eigenvectors

7.2 Diagonalization

7.1 Eigenvalues and Eigenvectors

- **Eigenvalue problem:**

If A is an $n \times n$ matrix, do there exist nonzero vectors \mathbf{x} in R^n such that $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ?

- **Eigenvalue and eigenvector:**

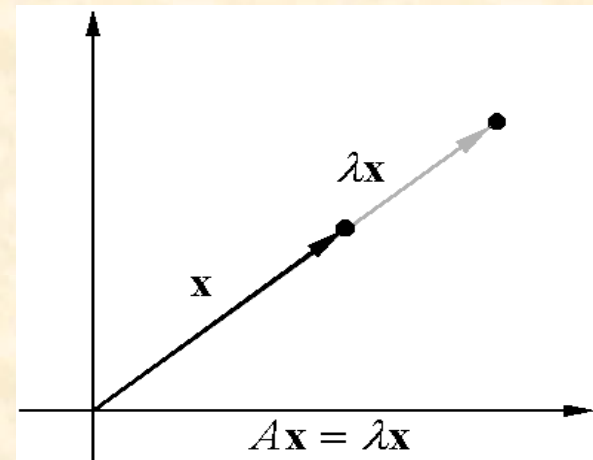
A : an $n \times n$ matrix

λ : a scalar

\mathbf{x} : a nonzero vector in R^n

$$\begin{array}{c} \text{Eigenvalue} \\ \downarrow \\ A\mathbf{x} = \lambda\mathbf{x} \\ \uparrow \quad \uparrow \\ \text{Eigenvector} \end{array}$$

- **Geometrical Interpretation**



- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2x_1$$

Eigenvalue
↓
Eigenvalue
↑
Eigenvalue
↑
Eigenvector

$$Ax_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)x_2$$

Eigenvalue
↓
Eigenvalue
↑
Eigenvalue
↑
Eigenvector

- **Thm 7.1: (The eigenspace of A corresponding to λ)**

If A is an $n \times n$ matrix with an eigenvalue λ , then the set of all eigenvectors of λ together with the zero vector is a subspace of \mathbb{R}^n . This subspace is called **the eigenspace of λ** .

Pf:

x_1 and x_2 are eigenvectors corresponding to λ

(i.e. $Ax_1 = \lambda x_1$, $Ax_2 = \lambda x_2$)

$$(1) A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$$

(i.e. $x_1 + x_2$ is an eigenvector corresponding to λ)

$$(2) A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$$

(i.e. cx_1 is an eigenvector corresponding to λ)

- Ex 3: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

If $\mathbf{v} = (x, y)$

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

For a vector on the x -axis

Eigenvalue $\lambda_1 = -1$

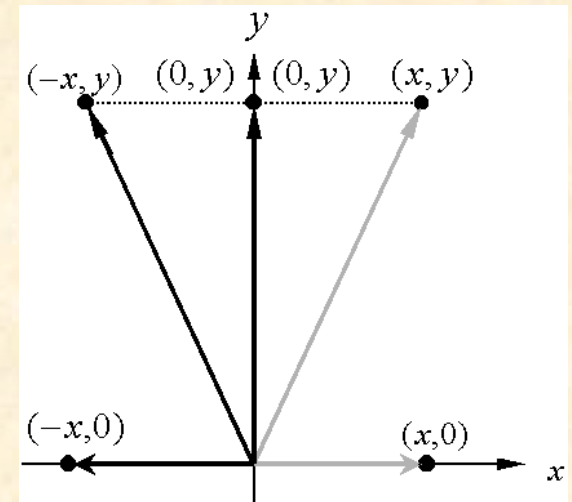
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

For a vector on the y-axis

Eigenvalue $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \neq 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis.



The eigenspace corresponding to $\lambda_1 = -1$ is the x-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

- **Thm 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n \times n}$)**

Let A is an $n \times n$ matrix.

(1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.

(2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)x = 0$.

- **Note:**

$$Ax = \lambda x \implies (\lambda I - A)x = 0 \quad (\text{homogeneous system})$$

If $(\lambda I - A)x = 0$ has nonzero solutions iff $\det(\lambda I - A) = 0$.

- **Characteristic polynomial of $A \in M_{n \times n}$:**

$$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- **Characteristic equation of A :**

$$\det(\lambda I - A) = 0$$

- Ex 4: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\ &= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0 \end{aligned}$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalues : $\lambda_1 = -1, \lambda_2 = -2$

$$(1) \lambda_1 = -1 \Rightarrow (\lambda_1 \mathbf{I} - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2) \lambda_2 = -2 \Rightarrow (\lambda_2 \mathbf{I} - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Check : $Ax = \lambda_i x$

- **Ex 5: (Finding eigenvalues and eigenvectors)**

Find the eigenvalues and corresponding eigenvectors for the matrix A . What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda I - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\} : \text{the eigenspace of } A \text{ corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- **Notes:**

- (1) If an eigenvalue λ_1 occurs as a multiple root (*k times*) for the characteristic polynomial, then λ_1 has multiplicity *k*.
- (2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

-
- **Ex 6** : Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Sol: Characteristic equation:

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 1)^2 (\lambda - 2) (\lambda - 3) = 0 \end{aligned}$$

Eigenvalues : $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

$$(1)\lambda_1 = 1 \Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the eigenspace of } A \text{ corresponding to } \lambda = 1$$

$$(2)\lambda_2 = 2 \Rightarrow (\lambda_2 I - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$$(3)\lambda_3 = 3 \Rightarrow (\lambda_3 I - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- **Thm 7.3: (Eigenvalues of triangular matrices)**

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- **Ex 7: (Finding eigenvalues for diagonal and triangular matrices)**

$$(a) A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b) A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a) |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b) \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- **Eigenvalues and eigenvectors of linear transformations:**

A number λ is called an eigenvalue of a linear transformation $T : V \rightarrow V$ if there is a nonzero vector \mathbf{x} such that $T(\mathbf{x}) = \lambda\mathbf{x}$.

The vector \mathbf{x} is called an eigenvector of T corresponding to λ , and the set of all eigenvectors of λ (with the zero vector) is called the eigenspace of λ .

■ Ex 8: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Sol:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda + 2)^2 (\lambda - 4)$$

eigenvalues : $\lambda_1 = 4, \lambda_2 = -2$

The eigenspaces for these two eigenvalues are as follows.

$$B_1 = \{(1, 1, 0)\}$$

Basis for $\lambda_1 = 4$

$$B_2 = \{(1, -1, 0), (0, 0, 1)\}$$

Basis for $\lambda_2 = -2$

■ **Notes:**

(1) Let $T:R^3 \rightarrow R^3$ be the linear transformation whose standard matrix is A in Ex. 8, and let B' be the basis of R^3 made up of three linear independent eigenvectors found in Ex. 8. Then A' , the matrix of T relative to the basis B' , is diagonal.

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$


Eigenvectors of A

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

(2) The main diagonal entries of the matrix A' are the eigenvalues of A .

Keywords in Section 7.1:

- eigenvalue problem: مسألة القيمة الذاتية
- eigenvalue: قيمة ذاتية
- eigenvector: متجه ذاتي
- characteristic polynomial: متعددة الحدود المميزة
- characteristic equation: المعادلة المميزة
- eigenspace: فضاء ذاتي
- multiplicity: تعددية

7.2 Diagonalization

- **Diagonalization problem:**

For a square matrix A , does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- **Diagonalizable matrix:**

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**.

(P diagonalizes A)

- **Notes:**

(1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

(2) The eigenvalue problem is related closely to the diagonalization problem.

- **Thm 7.4: (Similar matrices have the same eigenvalues)**

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf:

A and B are similar $\Rightarrow B = P^{-1}AP$

$$\begin{aligned} |\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda IP - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\ &= |\lambda I - A| \end{aligned}$$

A and B have the same characteristic polynomial.

Thus A and B have the same eigenvalues.

■ Ex 1: (A diagonalizable matrix)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = -2$, $\lambda_3 = -2$

(1) $\lambda = 4 \Rightarrow$ Eigenvector: $p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (See p.403 Ex.5)

(2) $\lambda = -2 \Rightarrow$ Eigenvector: $p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (See p.403 Ex.5)

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

■ **Notes:**

$$(1) P = [p_2 \quad p_1 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2) P = [p_2 \quad p_3 \quad p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- **Thm 7.5: (Condition for diagonalization)**

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Pf:

(\Rightarrow) A is diagonalizable

there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal

Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$ and $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$

$$\begin{aligned} PD &= [p_1 \mid p_2 \mid \cdots \mid p_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= [\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n] \end{aligned}$$

$$AP = A[p_1 \mid p_2 \mid \cdots \mid p_n] = [Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n]$$

$$\therefore AP = PD$$

$$\therefore Ap_i = \lambda_i p_i, \quad i = 1, 2, \dots, n$$

(i.e. the column vector p_i of P are eigenvectors of A)

$\therefore P$ is invertible $\Rightarrow p_1, p_2, \dots, p_n$ are linearly independent.

$\therefore A$ has n linearly independent eigenvectors.

(\Leftarrow) A has n linearly independent eigenvectors p_1, p_2, \dots, p_n

with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

i.e. $Ap_i = \lambda_i p_i, \quad i = 1, 2, \dots, n$

Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$

$$\begin{aligned}
AP &= A[p_1 \mid p_2 \mid \cdots \mid p_n] \\
&= [Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n] \\
&= [\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n] \\
&= [p_1 \mid p_2 \mid \cdots \mid p_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD
\end{aligned}$$

$\because p_1, p_1, \dots, p_n$ are linearly independent $\Rightarrow P$ is invertible

$$\therefore P^{-1}AP = D$$

$\Rightarrow A$ is diagonalizable

Note: If n linearly independent vectors do not exist, then an $n \times n$ matrix A is not diagonalizable.

- Ex 4: (A matrix that is not diagonalizable)

Show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Eigenvalue : $\lambda_1 = 1$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Eigenvector : } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A does not have two ($n=2$) linearly independent eigenvectors, so A is not diagonalizable.

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors p_1, p_2, \dots, p_n
for A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

Step 2: Let $P = [p_1 \mid p_2 \mid \dots \mid p_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \text{ where } Ap_i = \lambda_i p_i, \quad i = 1, 2, \dots, n$$

Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D .

■ Ex 5: (Diagonalizing a matrix)

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

Eigenvalues : $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$

$$\lambda_1 = 2$$

$$\Rightarrow \lambda_1 \mathbf{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2$$

$$\Rightarrow \lambda_2 \mathbf{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3$$

$$\Rightarrow \lambda_3 \mathbf{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

-
- **Notes:** k is a positive integer

$$(1) D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2) D = P^{-1}AP$$

$$\Rightarrow D^k = (P^{-1}AP)^k$$

$$= (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP$$

$$= P^{-1}AA\cdots AP$$

$$= P^{-1}A^kP$$

$$\therefore A^k = PD^kP^{-1}$$

- **Thm 7.6: (Sufficient conditions for diagonalization)**

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

-
- Ex 7: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol: Because A is a triangular matrix,
its eigenvalues are the main diagonal entries.

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable. (Thm.7.6)

- **Ex 8: (Finding a diagonalizing matrix for a linear transformation)**

Let $T:R^3 \rightarrow R^3$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

Find a basis B for R^3 such that the matrix for T relative to B is diagonal.

Sol: The standard matrix for T is given by

$$A = [T(e_1) \quad T(e_2) \quad T(e_3)] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From Ex. 5, there are three distinct eigenvalues

$$\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

so A is diagonalizable. (Thm. 7.6)

Thus, the three linearly independent eigenvectors found in Ex. 5

$$p_1 = (-1, 0, 1), p_2 = (1, -1, 4), p_3 = (-1, 1, 1)$$

can be used to form the basis B . That is

$$B = \{p_1, p_2, p_3\} = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}$$

The matrix for T relative to this basis is

$$\begin{aligned} D &= \begin{bmatrix} [T(p_1)]_B & [T(p_2)]_B & [T(p_3)]_B \\ [Ap_1]_B & [Ap_2]_B & [Ap_3]_B \\ [\lambda_1 p_1]_B & [\lambda_2 p_2]_B & [\lambda_3 p_3]_B \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

Keywords in Section 7.2:

- diagonalization problem: مسألة التقطير (الجدولة)
- diagonalization: تقطير (جدولة)
- diagonalizable matrix: مصفوفة قابلة للتقطير (للجدولة)