Chapter 5

Dot, Inner and Cross Products

5.1 Length of a vector

5.2 Dot Product

5.3 Inner Product

5.4 Cross Product

5.1 Length and Dot Product in R^n

- Length:

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2} + \cdots + v_n^{\,2}}$$

- Notes: The length of a vector is also called its **norm**.

- Notes: Properties of length

(1) $\|\mathbf{v}\| \geq 0$

(2) $\|\mathbf{v}\| = 1 \Rightarrow \mathbf{v}$ is called a **unit vector**.

(3) $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = 0$

(4) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

- Ex 1:

(a) In R^5, the length of $\mathbf{v}=(0,-2,1,4,-2)$ is given by

$$\|\mathbf{v}\|=\sqrt{0^2+(-2)^2+1^2+4^2+(-2)^2}=\sqrt{25}=5$$

(b) In R^3 the length of $\mathbf{v}=(\frac{2}{\sqrt{17}},\frac{-2}{\sqrt{17}},\frac{3}{\sqrt{17}})$ is given by

$$\|\mathbf{v}\|=\sqrt{\left(\frac{2}{\sqrt{17}}\right)^2+\left(\frac{-2}{\sqrt{17}}\right)^2+\left(\frac{3}{\sqrt{17}}\right)^2}=\sqrt{\frac{17}{17}}=1$$

（$\mathbf{v}$ is a unit vector）

- A standard unit vector in R^n:

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0),(0,1,\cdots,0),(0,0,\cdots,1)\}$$

- Ex:

the standard unit vector in R^2: $\{i, j\} = \{(1,0),(0,1)\}$

the standard unit vector in R^3: $\{i, j, k\} = \{(1,0,0),(0,1,0),(0,0,1)\}$

- Notes: (Two nonzero vectors are parallel)

$$\mathbf{u} = c\mathbf{v}$$

(1) $c > 0 \Rightarrow$ **u** and **v** have the same direction

(2) $c < 0 \Rightarrow$ **u** and **v** have the opposite direction

- Thm 5.1: (Length of a scalar multiple)

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Pf:

$$\mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\Rightarrow c\mathbf{v} = (cv_1, cv_2, \cdots, cv_n)$$

$$\|c\mathbf{v}\| = \|(cv_1, cv_2, \cdots, cv_n)\|$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$

$$= |c|\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

$$= |c|\,\|\mathbf{v}\|$$

- Thm 5.2: (Unit vector in the direction of **v**)

If **v** is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

has length 1 and has the same direction as **v**. This vector **u** is called the **unit vector in the direction of v**.

Pf:

$$\mathbf{v} \text{ is nonzero} \Rightarrow \|\mathbf{v}\| \neq 0 \Rightarrow \frac{1}{\|\mathbf{v}\|} > 0$$

$$\Rightarrow \mathbf{u} = \frac{\mathbf{1}}{\|\mathbf{v}\|}\mathbf{v} \quad (\mathbf{u} \text{ has the same direction as } \mathbf{v})$$

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1 \quad \text{(u has length 1)}$$

- Notes:

(1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of **v**.

(2) The process of finding the unit vector in the direction of **v** is called **normalizing** the vector **v.**

- Ex 2: (Finding a unit vector)

Find the unit vector in the direction of $\mathbf{v}=(3,-1,2)$,

and verify that this vector has length 1.

Sol:

$$\mathbf{v}=(3,-1,2) \quad \Rightarrow \|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3,-1,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3,-1,2)=\left(\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}}\right)$$

$$\because \quad \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

$\therefore \quad \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

- Distance between two vectors:

The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- Notes: (Properties of distance)

（1） $d(\mathbf{u}, \mathbf{v}) \geq 0$

（2） $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

（3） $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Ex 3: (Finding the distance between two vectors)

The distance between **u**=(0, 2, 2) and **v**=(2, 0, 1) is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(0-2, 2-0, 2-1)\|$$
$$= \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

Keywords in Section 5.1:

- length: طول
- norm: معيار
- unit vector: متجه الوحدة
- standard unit vector : متجه الوحدة الأساسي
- normalizing: معايرة
- distance: المسافة
- angle: زاوية
- triangle inequality: متباينة المثلث
- Pythagorean theorem: نظرية فيثاغورس

5.2 Dot Product

- Dot product in R^n:

  The **dot product** of $\mathbf{u}=(u_1\,,u_2\,,\cdots,u_n)$ and $\mathbf{v}=(v_1\,,v_2\,,\cdots,v_n)$

  is the scalar quantity

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

- Ex 4: (Finding the dot product of two vectors)

  The dot product of **u**=(1, 2, 0, -3) and **v**=(3, -2, 4, 2) is

$$\mathbf{u}\cdot\mathbf{v}=(1)(3)+(2)(-2)+(0)(4)+(-3)(2)=-7$$

- Thm 5.3: (Properties of the dot product)

If **u**, **v**, and **w** are vectors in R^n and c is a scalar,

then the following properties are true.

(1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(2) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(3) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$

(4) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

(5) $\mathbf{v}\cdot\mathbf{v}\geq 0$, and $\mathbf{v}\cdot\mathbf{v}=0$ if and only if $\boldsymbol{v}=0$

- Euclidean *n*-space:

  R^n was defined to be the *set* of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called **Euclidean *n*-space**.

- Ex 5: (Finding dot products)

$$\mathbf{u}=(2\,,-2)\,,\ \mathbf{v}=(5\,,8),\mathbf{w}=(-4\,,3)$$

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (c) $\mathbf{u}\cdot(2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})$

Sol:

(a) $\mathbf{u}\cdot\mathbf{v}=(2)(5)+(-2)(8)=-6$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=-6\mathbf{w}=-6(-4\,,3)=(24\,,-18)$

(c) $\mathbf{u}\cdot(2\mathbf{v})=2(\mathbf{u}\cdot\mathbf{v})=2(-6)=-12$

(d) $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=(-4)(-4)+(3)(3)=25$

(e) $\mathbf{v}-2\mathbf{w}=(5-(-8)\,,8-6)=(13\,,2)$

$$\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})=(2)(13)+(-2)(2)=26-4=22$$

- Ex 6: (Using the properties of the dot product)

Given $\mathbf{u}\cdot\mathbf{u}=39 \quad \mathbf{u}\cdot\mathbf{v}=-3 \qquad \mathbf{v}\cdot\mathbf{v}=79$

Find $(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v})$

Sol:

$$\begin{aligned}(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+\mathbf{v})+2\mathbf{v}\cdot(3\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot(3\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+(2\mathbf{v})\cdot(3\mathbf{u})+(2\mathbf{v})\cdot\mathbf{v}\\ &= 3(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+6(\mathbf{v}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v})\\ &= 3(\mathbf{u}\cdot\mathbf{u})+7(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{v}\cdot\mathbf{v})\\ &= 3(39)+7(-3)+2(79)=254\end{aligned}$$

- Thm 5.4: (The Cauchy - Schwarz inequality)

  If **u** and **v** are vectors in R^n, then

  $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ ($|\mathbf{u}\cdot\mathbf{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$)

- Ex 7: (An example of the Cauchy - Schwarz inequality)

  Verify the Cauchy - Schwarz inequality for **u**=(1, -1, 3) and **v**=(2, 0, -1)

Sol: $\mathbf{u}\cdot\mathbf{v} = -1,\quad \mathbf{u}\cdot\mathbf{u} = 11,\quad \mathbf{v}\cdot\mathbf{v} = 5$

$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

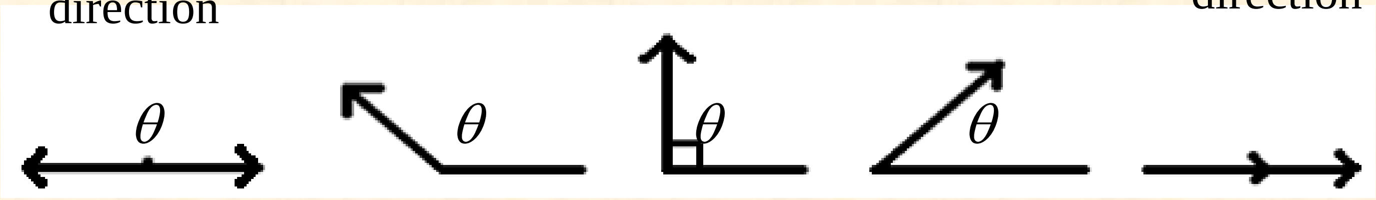
$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

- The angle between two vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

Opposite direction	$\mathbf{u}\cdot\mathbf{v} < \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} = \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} > \mathbf{0}$	Same direction



$\theta = \pi$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$	$0 < \theta < \frac{\pi}{2}$	$\theta = 0$
$\cos = -1$	$\cos < 0$	$\cos = 0$	$\cos > 0$	$\cos = 1$

- Note:

The angle between the zero vector and another vector is not defined.

- Ex 8: (Finding the angle between two vectors)

$$\mathbf{u} = (-4, 0, 2, -2) \quad \mathbf{v} = (2, 0, -1, 1)$$

Sol:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{2^2 + (0)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\mathbf{u}\cdot\mathbf{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = -\frac{12}{\sqrt{144}} = -1$$

$\Rightarrow \theta = \pi \qquad \therefore$ $\mathbf{u}$ and $\mathbf{v}$ have opposite directions. $(\mathbf{u} = -2\mathbf{v})$

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal if

  $$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector.

- Ex 10: (Finding orthogonal vectors)

Determine all vectors in R^n that are orthogonal to **u**=(4, 2).

Sol:

$$\mathbf{u} = (4\,,\,2) \quad \text{Let} \quad \mathbf{v} = (v_1\,,\,v_2)$$

$$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v} = (4\,,\,2)\cdot(v_1\,,\,v_2)$$

$$= 4v_1 + 2v_2$$

$$= 0$$

$$\begin{bmatrix} 4 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \quad v_1 = \frac{-t}{2} \quad , \quad v_2 = t$$

$$\therefore \quad \mathbf{v} = \left(\frac{-t}{2}, t\right), \qquad t \in R$$

- Thm 5.5: (The triangle inequality)

If **u** and **v** are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

Pf:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \quad \le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2$$

$$\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Note:

Equality occurs in the triangle inequality if and only if the vectors **u** and **v** have the same direction.

- Thm 5.6: (The Pythagorean theorem)

  If **u** and **v** are vectors in R^n, then **u** and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

- Dot product and matrix multiplication:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ in R^n is represented as an $n\times 1$ column matrix)

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = [u_1 \quad u_2 \quad \cdots \quad u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1v_1 + u_2v_2 + \cdots + u_nv_n]$$

Keywords in Section 5.2:

- dot product: الضرب النقطي
- Euclidean *n*-space: فضاء نوني اقليدي
- Cauchy-Schwarz inequality: متباينة كوشي-شوارز

- triangle inequality: متباينة المثلث
- Pythagorean theorem: نظرية فيثاغورس

5.3 Inner Product

- Inner product:

Let **u**, **v**, and **w** be vectors in a vector space *V*, and let *c* be any scalar. An inner product on *V* is a function that associates a real number <**u**, **v**> with each pair of vectors **u** and **v** and satisfies the following axioms.

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(2) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(3) $c \langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n)$$

$$< \mathbf{u}\,,\, \mathbf{v} > = \text{general inner product for vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space.

Vector space: $(V,\ +,\ \bullet)$

Inner product space: $(V,\ +,\ \bullet,\ <,>)$

- Ex 1: (The Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\mathbf{u} = (u_1, u_2, \cdots, u_n) \quad , \quad \mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

- Ex 2: (A different inner product for R^n)

Show that the function defines an inner product on R^2, where $\mathbf{u}=(u_1\,,u_2)$ and $\mathbf{v}=(v_1\,,v_2)$.

$$\langle \mathbf{u}\,,\mathbf{v}\rangle = u_1v_1 + 2u_2v_2$$

Sol:

$$(a)\quad \langle \mathbf{u}\,,\mathbf{v}\rangle = u_1v_1 + 2u_2v_2 = v_1u_1 + 2v_2u_2 = \langle \mathbf{v}\,,\mathbf{u}\rangle$$

$$(b)\quad \mathbf{w}=(w_1\,,w_2)$$

$$\begin{aligned}\Rightarrow \langle \mathbf{u}\,,\mathbf{v}+\mathbf{w}\rangle &= u_1(v_1+w_1)+2u_2(v_2+w_2)\\ &= u_1v_1+u_1w_1+2u_2v_2+2u_2w_2\\ &= (u_1v_1+2u_2v_2)+(u_1w_1+2u_2w_2)\\ &= \langle \mathbf{u}\,,\mathbf{v}\rangle+\langle \mathbf{u}\,,\mathbf{w}\rangle\end{aligned}$$

$$(c)\quad c\langle \mathbf{u}, \mathbf{v}\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v}\rangle$$

$$(d)\quad \langle \mathbf{v}, \mathbf{v}\rangle = v_1{}^2 + 2v_2{}^2 \geq 0$$

$$\langle \mathbf{v}, \mathbf{v}\rangle = 0 \Rightarrow v_1{}^2 + 2v_2{}^2 = 0 \qquad \Rightarrow \quad v_1 = v_2 = 0 \quad (\mathbf{v} = 0)$$

- Note: (An inner product on R^n)

$$\langle \mathbf{u}, \mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n\,, \qquad c_i > 0$$

- Ex 3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3.

$$\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$

Then $<\mathbf{v}, \mathbf{v}> = (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied.

Thus this function is not an inner product on R^3.

- Thm 5.7: (Properties of inner products)

  Let **u, v**, and **w** be vectors in an inner product space *V*, and let *c* be any real number.

  (1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

  (2) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

  (3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

- Norm (length) of **u**:

  $$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Note:

  $$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

- Distance between **u** and **v**:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are **orthogonal** if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

- Notes:

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector.

(2) $\|\mathbf{v}\| \neq 1$, $\mathbf{v} \neq 0$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of **v**)

not a unit vector

- Ex 6: (Finding inner product)

$$\langle p\ ,\ q\rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$$ is an inner product

Let $p(x) = 1 - 2x^2,\ q(x) = 4 - 2x + x^2$ be polynomials in $P_2(x)$

$(a)\ \langle p, q\rangle = ?\quad (b)\ \|q\| = ?\quad (c)\ d(p, q) = ?$

Sol:

$$(a)\ \langle p, q\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$$

$$(b)\ \|q\| = \sqrt{\langle q, q\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$(c)\ \because p - q = -3 + 2x - 3x^2$$

$$\therefore d(p, q) = \|p - q\| = \sqrt{\langle p - q, p - q\rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

- Properties of norm:

  (1) $\|\mathbf{u}\| \geq 0$

  (2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

  (3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance:

  (1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

  (2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Thm 5.8：

Let **u** and **v** be vectors in an inner product space *V*.

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\| \qquad \text{Theorem 5.4}$$

(2) Triangle inequality:

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \qquad \text{Theorem 5.5}$$

(3) Pythagorean theorem :

u and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \text{Theorem 5.6}$$

- Orthogonal projections in inner product spaces:

Let **u** and **v** be two vectors in an inner product space *V*, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$$

- Note:

If **v** is a init vector, then $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 1$.

The formula for the orthogonal projection of **u** onto **v** takes the following simpler form.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v}$$

- Ex 10: (Finding an orthogonal projection in R^3)

  Use the Euclidean inner product in R^3 to find the orthogonal projection of u=(6, 2, 4) onto v=(1, 2, 0).

Sol:

$$\because \langle \mathbf{u}, \mathbf{v} \rangle = (6)(1) + (2)(2) + (4)(0) = 10$$

$$\langle \mathbf{v}, \mathbf{v} \rangle = 1^2 + 2^2 + 0^2 = 5$$

$$\therefore \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \tfrac{10}{5}(1, 2, 0) = (2, 4, 0)$$

- Note:

$\boldsymbol{u} - \text{proj}_v \boldsymbol{u} = (6, 2, 4) - (2, 4, 0) = (4, -2, 4)$ is orthogonal to $v = (1,2,0)$.

- Thm 5.9: (Orthogonal projection and distance)

Let **u** and **v** be two vectors in an inner product space *V*, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\mathbf{u}, \text{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}, c\mathbf{v}), \qquad c \neq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Keywords in Section 5.2:

- inner product: ضرب داخلي
- inner product space: فضاء الضرب الداخلي
- norm: معيار
- distance: مسافة
- angle: زاوية
- orthogonal: متعامد
- unit vector: متجه وحدة
- normalizing: معايرة
- Cauchy – Schwarz inequality: متباينة كوشي - شوارز
- triangle inequality: متباينة المثلث
- Pythagorean theorem: نظرية فيثاغورس
- orthogonal projection: اسقاط عمودي

5.4 Cross Product

- Cross product in R^3:

The **cross product** of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$

is the vector quantity

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \left(u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1\right)$$

- Ex 11: (Finding the cross product of two vectors)

The cross product of **u**=(1, 2, 0) and **v**=(3, -2, 4) is

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \left(8, -4, -8\right)$$

Keywords in Section 5.4:

- Cross product: ضرب خارجي
- inner product space: فضاء الضرب الداخلي
- norm: معيار
- distance: مسافة
- angle: زاوية
- orthogonal: متعامد