FINAL EXAMINATION
Semester I: 1424-1425
Department of Mathematics
King Saud University
MATH 580: Measure Theory
Time: 3 H Full Marks: 50

Question #1.

(a) Let $\mathcal{A}$ be an algebra of subsets of a set X and let $\mu : \mathcal{A} \to [0, +\infty]$ be a set function. If $\mu(\emptyset) = 0$ then prove that μ is countably additive if and only if μ is both finitely additive and countably subadditive.

(b) If μ_1 and μ_2 are totally finite measures on a σ-algebra $\mathcal{S}$ and $\mathcal{K} := \{M \in \mathcal{S} | \mu_1(M) = \mu_2(M)\}$, then show that $\mathcal{K}$ is a monotone class.

Question #2.

(a) What do you mean by complete measure space? Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \Re^\star$ be $\mathcal{S}$-measurable function. Let μ be a measure on $(X, \mathcal{S})$. Let $g : X \to \Re^\star$ be such that $N := \{x \in X | f(x) \neq g(x)\}$ is a μ-null set. If $(X, \mathcal{S}, \mu)$ is a complete measure space, then prove that g is also $\mathcal{S}$-measurable or equivalently, prove that $g^{-1}[t, \infty] \in \mathcal{S}$ for any $t \in \Re$.

(b) For $f \in \mathbf{L}$(=the class of all measurable functions), prove that $f \in \mathcal{L}_1(X, \mathcal{S}, \mu)$(=the space of all μ-integrable functions on X) if and only if $|f| \in \mathcal{L}_1(X, \mathcal{S}, \mu)$. Also, prove that $|\int f d\mu| \leq \int |f| d\mu$.

(c) Let $(f_n)_{n \geq 1}$ be a sequence of measurable functions such that, for each n, $|f_n| \leq g$, an integrable function. Show that

$$\int \underline{\lim} f_n d\mu \leq \underline{\lim} \int f_n d\mu \leq \overline{\lim} \int f_n d\mu \leq \int \overline{\lim} f_n d\mu.$$

Question #3.

(a) Let μ, ν be measures on $(X, \mathcal{S})$. If ν is finite and $\nu << \mu$, then prove that $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $\nu(E) < \epsilon$ whenever $\mu(E) < \delta$, $E \in \mathcal{S}$; Is the condition that ν is finite necessary?

(b) State Radon-Nikodym theorem. Let $F : \Re \to \Re$ be monotonically increasing absolutely continuous function, and μ_F the Lebesgue-Stieltjes measure induced by F on $(\Re, \mathcal{B}_\Re)$($\mathcal{B}_\Re$ denotes the

σ-algebra of Borel subsets of $\Re$). Show that $\frac{d\mu_F}{d\lambda}(x) = F'(x)$, for a.e. $x(\lambda)$(here λ is the Lebesgue measure on $\Re$).

Question #4.

(a) What do you mean by signed measure? Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\int f d\mu$ exist. Define ν by

$$\nu(E) = \int_E f d\mu,$$

for $E \in \mathcal{S}$.

Find a Hahn decomposition with respect to ν and the Jordan decomposition of ν. Do you find any difference between Hahn decomposition and the Jordan decomposition?

(b) State Fubini's theorem for $f \in \mathcal{L}_1(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$, μ and ν being σ-finite measures.

Let $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B} = \mathcal{B}_{[0,1]}$, the σ-algbera of Borel subsets of $[0, 1]$ and let $\mu = \nu$ be the Lebesgue measure on $[0, 1]$. If

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } x = y. \end{cases}$$

then compute $\int_0^1 \{\int_0^1 f(x, y) d\nu(y)\} d\mu(x)$ and $\int_0^1 \{\int_0^1 f(x, y) d\mu(x)\} d\nu(y)$. Is it true that $f \in \mathcal{L}_1(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$? Justify your claim.