# KING SAUD UNIVERSITY
# DEPARTMENT OF MATHEMATICS
# FINAL EXAMINATION., SEM II: 1425-26
# MATH 384: Real Analysis II TIME: 3 H FULL MARKS: 50

**Question #1**

(a) If $f : [a, b] \to \Re$ is a continuous function on $[a, b]$ then prove that it is integrable on $[a, b]$.

(b) Let $f : [0, 2] \to \Re$ be defined by $f(x) = 1$ if $x \neq 1$, and $f(1) = 0$. Show that $f$ is integrable on $[0, 2]$ and calculate its integral. Is $f$ a continuous function? Explain.

(c) Do you think that the composition of integrable functions is integrable? Discuss.

**Question #2**

(a) Let $f : [a, b] \to \Re$ be integrable on $[a, b]$, and let $|f(x)| \leq M$ for all $x \in [a, b]$. Use the inequality

$$((f(x))^2 - (f(y))^2 \leq 2M|f(x) - f(y)|$$

for $x, y \in [a, b]$ to show that $f^2$ is integrable on $[a, b]$.

(b) Let $f : [a, b] \to \Re$ be integrable on $[a, b]$ and let $F(x) = \int_a^x f$ for $x \in [a, b]$. If $f$ is continuous at a point $c \in [a, b]$ then show that $F$ is differentiable at $c$ and $F'(c) = f(c)$.

**Question #3**

(a) Show that the sequence $(\frac{x^n}{1+x^n})$ does not converge on $[0, 2]$ by showing that the limit function is not continuous on $[0, 2]$.

(b) Let $g_n(x) = nx(1 - x^2)$ for $x \in [0, 1]$, $n \in \mathbf{Z}^+$. Discuss the convergence of $(g_n)$ and $(\int_0^1 g_n dx)$.

(c) Do you think that $\sum_{n=1}^{\infty}(\frac{1}{n^2})\cos nx$ converges uniformly on $\Re$ to a countinuous function? Discuss.

**Question #4**

(a) Find the length of the set $\cup_{k=1}^{\infty}\{x : \frac{1}{2^k} \leq x < \frac{1}{2^{k-1}}\}$.

(b) Present the definition of Lebesgue outer measure. Show that (Lebesgue) outer measure of an interval $I$ is its length, that is, $m^*(I) = l(I)$.

(c) What is $\sigma$-algebra? If $\mathcal{D}$ is any class of subsets of $X$ then show that there exists a smallest $\sigma$-algebra $\mathcal{A}(\mathcal{D})$ on $X$ that contains $\mathcal{D}$.

**Question #5**

(a) What do you mean by a measurable function? Prove that a constant function is measurable.

(b) If $f$ is a non-negative measurable function then show that $f = 0$ a.e.(almost everywhere) if and only if $\int f dx = 0$.

(c) Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotone increasing for each $x$. Let $f = \lim f_n$. Prove that $\int f dx = \lim \int f_n dx$.