



Answer the following questions:

Q1: [4+4]

a) For the Markov process $\{X_t\}$, $t=0,1,2,\dots,n$ with states $i_0, i_1, i_2, \dots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$

b) A Markov chain X_0, X_1, X_2, \dots has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{vmatrix} \end{matrix}$$

and initial distribution $p_0=0.5$ and $p_1=0.5$ Determine the probabilities

$$\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$$

Q2: [6+3]

a) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.3$, $\Pr\{\xi_n = 1\} = 0.2$, $\Pr\{\xi_n = 2\} = 0.5$ and suppose $s=0$ and $S=3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n .

b) Let $\{X_n\}$ be a Markov chain for daily weather with two states 0, 1 (0 for dry day and 1 for rainy day) has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix} \end{matrix}$$

What's the probability for the weather to be dry today and rainy on the coming two days?

Q3: [8]

A Markov chain X_0, X_1, X_2, \dots has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.5 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{vmatrix} \end{matrix}$$

Every period that the process spends in state 0 incurs a cost \$4. Every period that the process spends in state 1 incurs a cost of \$7. Every period that the process spends in state 2 incurs a cost of \$5. What is the long run mean cost per period associated with this Markov chain?

Q1: [4+4]

a)

$$\begin{aligned} &\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot \Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot P_{i_{n-1}i_n} \quad \text{Definition of Markov} \end{aligned}$$

By repeating this argument $n-1$ times

$$\begin{aligned} &\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n} \quad \text{where } p_{i_0} = \Pr\{X_0 = i_0\} \text{ is obtained from the initial distribution of the process.} \end{aligned}$$

b)

$$\begin{aligned} \text{i) } \Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} &= p_1 P_{11} P_{10}, \quad p_1 = \Pr\{X_0 = 1\} \\ &= 0.5(0.2)(0.4) \\ &= 0.04 \end{aligned}$$

$$\begin{aligned} \text{ii) } \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\} &= p_1 P_{11} P_{10}, \quad p_1 = \Pr\{X_1 = 1\} \\ \Pr\{X_1 = 1\} &= \Pr(X_1 = 1 | X_0 = 0) \Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1) \Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2) \Pr(X_0 = 2) \\ &= P_{01} p_0 + P_{11} p_1 + P_{21} p_2 \\ &= 0.3(0.5) + 0.2(0.5) = 0.25 \end{aligned}$$

Q2: [6+3]

a)

$$\begin{array}{c} -1 \quad 0 \quad 1 \quad 2 \quad 3 \\ -1 \left\| \begin{array}{ccccc} 0 & 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.5 & 0.2 & 0.3 \\ 1 & 0.5 & 0.2 & 0.3 & 0 \\ 2 & 0 & 0.5 & 0.2 & 0.3 \\ 3 & 0 & 0 & 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

$$P_{ij} = \Pr(\xi_{n+1} = S - j), \quad i \leq s \quad \text{for replenishment}$$

$$P_{-1,-1} = \Pr(\xi_{n+1} = 4) = 0, \quad P_{01} = \Pr(\xi_{n+1} = 2) = 0.5$$

$$P_{ij} = \Pr(\xi_{n+1} = i - j), \quad s < i \leq S \quad \text{for non-replenishment}$$

$$P_{1,-1} = \Pr(\xi_{n+1} = 2) = 0.5, \quad P_{11} = \Pr(\xi_{n+1} = 0) = 0.3, \quad P_{21} = \Pr(\xi_{n+1} = 1) = 0.2$$

b)

$$\Pr(X_2=1|X_0=0) = P_{01}^2$$

$$P^2 = \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix} \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix}$$
$$= \begin{vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{vmatrix}$$

$$\therefore \Pr(X_2=1|X_0=0) = 0.39$$

Q3: [8]

$$\pi_j = \sum_{k=0}^2 \pi_k P_{kj}$$

at $j=0$

$$\Rightarrow \pi_0 = 0.5\pi_0 + 0.5\pi_1 + 0.3\pi_2$$
$$\therefore 5\pi_0 - 5\pi_1 - 3\pi_2 = 0 \quad (1)$$

at $j=1$

$$\Rightarrow \pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2$$
$$\therefore 2\pi_0 - 9\pi_1 + 2\pi_2 = 0 \quad (2)$$

$$\text{and } \therefore \pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

\therefore By solving equations (1), (2) and (3)

We get $\pi_0 = 0.4205$, $\pi_1 = 0.1818$, $\pi_2 = 0.3977$

The long run mean cost per unit period is

$$C = \sum_{j=0}^2 \pi_j c_j$$
$$= \pi_0 c_0 + \pi_1 c_1 + \pi_2 c_2$$
$$= \$ 4.9431$$