King Saud University
College of Sciences
Department of Mathematics

Second Mid Term, S2 1441
M 380 – Stochastic Processes
Time: 90 minutes - Male Section

Answer the following questions:

Q1: [4+5]

a) For the Markov process $\{X_t\}$, $t=0,1,2,\ldots,n$ with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.1$, $\Pr\{\xi_n = 1\} = 0.5$, $\Pr\{\xi_n = 2\} = 0.4$ and suppose $s=0$ and $S=2$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

Q2: [3+6]

a) Let X_n denote the weather of the nth day with $X_n = 1$ meaning "rainy" and $X_n = 2$ meaning "dry". Suppose that $\{X_n\}$, $n = 0,1,2,\ldots$ evolves as a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left\| \begin{array}{cc} 0.6 & 0.4 \\ 0.3 & 0.7 \end{array} \right\| \end{array}$$

Given that, the probability of dry weather on 1st June equals $\frac{5}{8}$. What's the probability that the weather will be rainy on 3rd June.

b) Determine whether the transition matrix

$$\mathbf{P}= \begin{matrix} & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0.1 & 0.6 & 0.3 \\ 2 & 0 & 0 & 1 \end{matrix}$$

represents an absorbing Markov chain or not, sketch Markov chain diagram and then find each of the following:

i) Starting in state 1, determine the probability that the Markov chain ends in state 0.

ii) Determine the mean time to absorption.

Q3: [3+4]

a) Let $X=\begin{cases} 0 & \text{if } N=0 \\ \xi_1+\xi_2+\ \dots\ +\xi_N & \text{if } N>0 \end{cases}$ be a random sum and assume that $E(\xi_k)=\mu$, $E(N)=\upsilon$

Prove that $E(X)=\mu\upsilon$

b) The number of accidents occurring in a factory in a week is a Poisson random variable with mean 2. The number of individuals injured in different accidents is independently distributed, each with mean 3 and variance 4. Determine the mean and variance of the number of individuals injured in a weak.

The Model Answer

Q1: [4+5]

a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\} . \Pr\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\} . P_{i_{n-1} i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

b)

$$\begin{array}{c|cccc|} & -1 & 0 & 1 & 2 \\ -1 & 0 & 0.4 & 0.5 & 0.1 \\ 0 & 0 & 0.4 & 0.5 & 0.1 \\ 1 & 0.4 & 0.5 & 0.1 & 0 \\ 2 & 0 & 0.4 & 0.5 & 0.1 \end{array}$$

Where

$$P_{ij} = \begin{cases} pr = (\xi_n = 2 - j), \ i \le 0 & \text{replenishment} \\ pr = (\xi_n = i - j), \ 0 < i \le 2 & \text{without replenishment} \end{cases}$$

Q2: [3+6]

a) X_n , $n = 0,1,2,\ldots$ denotes the weather of the nth day with $X_n = 1$ meaning "rainy" and $X_n = 2$ meaning "dry"

The probability of dry weather on 1st June equals $\frac{5}{8}$

$\therefore P^0 = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$ is the initial Prob. distribution

$$\mathbf{P}^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.48 & 0.52 \\ 0.39 & 0.61 \end{bmatrix}$$

$$\therefore \ \Pr(X_2 = 1) = P_1^2$$

$$= \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} 0.48 \\ 0.39 \end{bmatrix}$$

$$= 0.4238$$

b)

i)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

$$u = pr\left\{X_T = 0 \middle| X_0 = 1\right\}$$
$$u_1 = p_{10} + p_{11}u_1$$
$$u_1 = 0.1 + 0.6u_1$$

$\therefore \ u_1 = u_{10} = \frac{1}{4}$ is the prob. that Markov chains ends in state 0

ii)

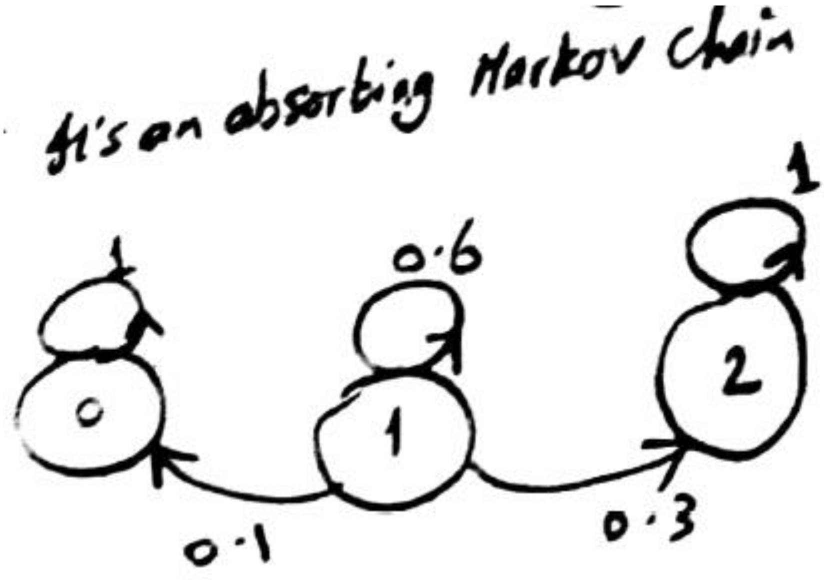
The mean time to absorption can be found as follows

$$v = E\left\{T \middle| X_0 = 1\right\}$$
$$v_1 = 1 + p_{11}v_1$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1$$
$$v_1 = \frac{5}{2}$$

$$\therefore v_1 = v_{10} = \frac{5}{2}$$



Q3: [3+4]

a)

The random sum is

$$X=\begin{cases} 0 & \text{if } N=0 \\ \xi_1+\xi_2+\ \dots\ +\xi_N & \text{if } N>0 \end{cases}$$

$\because$ E(X)= $\sum_{n=0}^{\infty} E[X \mid N=n] P_N(n)$

$$=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_{\mathrm{N}} \mid N=n] P_N(n)$$

$$=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_{\mathrm{n}} \mid N=n] P_N(n)$$ ....Prop.of cond. expectation

$$=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_{\mathrm{n}}] P_N(n),$$ where N is independent of $\xi_1,\ \xi_2,\ ...$

and $\because$ E(ξ_k)=μ, k=1,2,...,n

$\therefore$ E(X)=$\sum_{n=1}^{\infty} n\mu P_N(n)$

$$=\mu\sum_{n=1}^{\infty} nP_N(n)$$

=μE(N), E(N)=υ

$\therefore$ E(X)=μv

b)

N ~ Poisson (2)

N is the # of accidents in aweek

ξ_k is the # of individuals injured for kth accident

$E(\xi_k)=3,\ \ \mathrm{var}(\xi_k)=4$

$E(N)=2,\ \mathrm{var}(N)=2$

$\therefore\ E(X)=\mu v=3(2)=6$

$\mathrm{var}(X)=v\sigma^2+\mu^2\tau^2$

$\therefore$ var(X)=2(4)+9(2)=26