King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S1
1439/1440
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [5+3]

a) Given the joint probability mass functions of two random variables X and Y as in the following table:

X \ Y	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

Find Cov (X,Y) and Cov (2X,3Y)

b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.2 & 0.1 & 0.7 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{array}\right\| \end{array}$$

Find $pr\{X_2 = 1, X_3 = 1 \mid X_1 = 0\}$.

Q2: [5+3]

a) Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 1 & 0 & 0.7 & 0.2 & 0.1 \\ 2 & 0 & 0 & 0.9 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Starting in state 0, determine the mean time to absorption.

b) The probability of the thrower winning in the dice game is $p=0.4929$. Suppose player A is the thrower and begins the game with \$5, and player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

Q3: [9]

Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.6; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.4; if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.8; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.1. Transform this model into a Markov chain, and then find the transition probability matrix. Find also the long run fraction of days in which it is sunny.

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Model Answer of second Mid Term Exam

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Q1

a)

X \ Y	1	2	3	P(x)
1	1/12	1/6	0	1/4
2	0	1/9	1/5	14/45
3	1/18	1/4	2/15	79/180
P(y)	5/36	19/36	1/3	$\Sigma \cdot\cdot = 1$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= E(XY) - \mu_X \mu_Y$$

$$E(X) = \sum_X x\, p(x)$$
$$= 1(1/4) + 2(14/45) + 3(79/180)$$
$$\therefore \mu_X = \frac{197}{90} \simeq \boxed{2.1889} \quad 1$$

$$, E(Y) = \sum_Y y\, p(y)$$
$$= 1(5/36) + 2(19/36) + 3(1/3)$$
$$= \frac{79}{36} \simeq \boxed{2.1944} \quad 1$$

$$, E(XY) = \sum_X \sum_Y xy\, p(x,y)$$
$$= 1/12 + 2(1/6) + 4(1/9) + 6(1/5)$$
$$+ 3(1/18) + 6(1/4) + 9(2/15)$$
$$= \frac{887}{180} \simeq \boxed{4.9278} \quad 1$$

$$Cov(X,Y) = \frac{887}{180} - \frac{197}{90}\left(\frac{79}{36}\right)$$
$$= \frac{403}{3240}$$
$$Cov(X,Y) \simeq \boxed{0.1244} \quad 1$$
$$Cov(2X,3Y) = 6\, Cov(X,Y)$$
$$Cov(2X,3Y) \simeq \boxed{0.74} \quad 1$$

b)

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \end{array}$$

$$pr\{X_2 = 1, X_3 = 1 \mid X_1 = 0\}$$
$$= pr\{X_3 = 1 \mid X_2 = 1, \cancel{X_1 = 0}\} \cdot pr\{X_2 = 1 \mid X_1 = 0\} \quad 1$$
$$= pr\{X_3 = 1 \mid X_2 = 1\} \cdot pr\{X_2 = 1 \mid X_1 = 0\} \quad 1$$
$$= P_{11}\, P_{01} \quad 1$$
$$= 0.3(0.1) = 0.03$$

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Q2 8

a)

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0.4 & 0.3 & 0.2 & 0.1 \\
1 & 0 & 0.7 & 0.2 & 0.1 \\
2 & 0 & 0 & 0.9 & 0.1 \\
3 & 0 & 0 & 0 & 1
\end{array}
$$

Mean time to absorption is given by

$$v_i = E\{T \mid X_0 = i\}$$

$$i = 0, 1, 2$$

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$$v_0 = 1 + P_{00} v_0 + P_{01} v_1 + P_{02} v_2$$
$$v_1 = 1 + P_{10} v_0 + P_{11} v_1 + P_{12} v_2$$
$$v_2 = 1 + P_{20} v_0 + P_{21} v_1 + P_{22} v_2$$

$\Rightarrow$ $0.6 v_0 - 0.3 v_1 - 0.2 v_2 = 1$ ①

1 $0.3 v_1 - 0.2 v_2 = 1$ ②

1 $0.1 v_2 = 1 \Rightarrow \boxed{v_2 = 10}$ ③

1 Subs. ③ in ②

$\Rightarrow 0.3 v_1 - 0.2(10) = 1$

1/2 $\Rightarrow \boxed{v_1 = 10}$ ④

Subs. ④, ③ in ①

$0.6 v_0 = 6$

1/2 $\therefore \boxed{v_0 = v_{03} = 10}$

هو متوسط زمن الامتصاص من الحالة
0 إلى الحالة 3 .

b)

$u_i = Pr\{X_n$ reaches state 0 before state $N \mid X_0 = i\}$

, X_n is the fortune for player A at stage n

$$u_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}, \; p \neq q$$

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$$\Rightarrow u_i = \frac{\left[\left(\frac{0.5071}{0.4929}\right)^5 - \left(\frac{0.5071}{0.4929}\right)^{15}\right]}{\left[1 - \left(\frac{0.5071}{0.4929}\right)^{15}\right]}$$

, $\boxed{u_i = 0.71273}$

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	(S,S)	(S,C)	(C,S)	(C,C)
(S,S)	0.6	0.4	0	0
(S,C)	0	0	0.4	0.6
(C,S)	0.8	0.2	0	0
(C,C)	0	0	0.1	0.9
	$\downarrow$ π_0	$\downarrow$ π_1	$\downarrow$ π_2	$\downarrow$ π_3

4

$$\Rightarrow 6\pi_0 + 8\pi_2 = 10\pi_0 \quad (1)$$

$$4\pi_0 + 2\pi_2 = 10\pi_1 \quad (2)$$

$$4\pi_1 + \pi_3 = 10\pi_2 \quad (3)$$

$$\& \ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \quad (4)$$

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$$(1) \Rightarrow \boxed{\pi_2 = \tfrac{1}{2}\pi_0}, \quad (2) \Rightarrow \overbrace{4\pi_0 + \pi_0}^{5\pi_0} = 10\pi_1 \Rightarrow \boxed{\pi_1 = \tfrac{1}{2}\pi_0}$$

$$\therefore \boxed{\pi_1 = \pi_2 = \tfrac{1}{2}\pi_0} \quad (5)$$

$$(3) \Rightarrow \boxed{\pi_3} = (10-4)\cdot\tfrac{1}{2}\pi_0 = \boxed{3\pi_0} \quad (6)$$

Subs (5), (6) in (4) $\Rightarrow$

$$\pi_0\left(1 + \tfrac{1}{2} + \tfrac{1}{2} + 3\right) = 1$$

$$\therefore \boxed{\pi_0 = \tfrac{1}{5}} \quad (7)$$

$$\& \ \boxed{\pi_1} = \pi_2 = \tfrac{1}{2}\left(\tfrac{1}{5}\right) = \boxed{\tfrac{1}{10}} \quad (8)$$

$$, \ \boxed{\pi_3} = 3\left(\tfrac{1}{5}\right) = \boxed{\tfrac{3}{5}}$$

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In the long run $\Rightarrow$ the limiting dist: is $\pi = \left(\tfrac{1}{5}, \tfrac{1}{10}, \tfrac{1}{10}, \tfrac{3}{5}\right)$

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and the fraction of days in which it's sunny $= \pi_0 + \pi_1 = 1/5 + 1/10 = \boxed{\tfrac{3}{10}}$