King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2 1442
M 380 - Stochastic Processes
Time: 3 hours - Marks: 20

Answer the following questions:

Q1: [4+4]

(a) Given the joint probability mass function of two random variables X and Y

as in the following table:

X \ Y	-1	0	1
-1	1/9	0	2/9
0	2/9	1/9	0
1	0	2/9	1/9

i) Find $\rho(X,Y)$

ii) Determine whether X and Y are two independent random variables or not, Justify your answer.

(b) Suppose that X and Y are two independent random variables, each having the same exponential distribution with parameter α. What is the conditional probability density function for X, given that $Z = X + Y = z$? with clarifying, the name of the distribution.

(Hint, $f_{X,Z}(x,z) = \alpha^2 e^{-\alpha z}$ for $0 \le x \le z$)

Q2: [4+4]

(a) Let X and Y be independent Poisson distributed random variables having means μ and ν, respectively. Determine the probability distribution of their sum $Z = X + Y = n$.

(b) Suppose that $\xi_1, \xi_2, \ldots, \xi_N$ are independent and identically distributed with $\Pr\{\xi_k = \pm 1\} = \frac{1}{2}$. Let N be independent of $\xi_1, \xi_2, \ldots$ and follow the geometric probability mass function

$P_N(k) = \alpha(1-\alpha)^k$ for $k = 0, 1, \ldots,$ where $0 < \alpha < 1$. Determine the mean and variance of the random sum

$Z = \xi_1 + \xi_2 + \ldots + \xi_N$.

Q3: [4+4]

(a) Customers arrive at a facility and wait there until K customers have accumulated. Upon the arrival of the K th customer, all are instantaneously served, and the process repeats. Let $\xi_0, \xi_1, \ldots$ denote the arrivals in successive periods, assumed to be independent random variables whose distribution is given by

$\Pr(\xi_k = 0) = \alpha, \ \Pr(\xi_k = 1) = 1 - \alpha,$ where $0 < \alpha < 1$. Let X_n denote the number of customers in the system at time n. Then, $\{X_n\}$ is a Markov chain on the states $0, 1, \ldots, K-1$. With $K = 3$, give the transition probability matrix for $\{X_n\}$.

(b) Let $\{X(t);\ t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the numerical values to two decimal places for the following probabilities:

(i) $\Pr\{X(1) \leq 2\}$

(ii) $\Pr\{X(1) = 1 \text{ and } X(2) = 3\}$

(iii) $\Pr\{X(1) \geq 2 \mid X(1) \geq 1\}$

Q4: [4+4]

(a) Suppose that a production process changes state according to a Markov process whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.3 & 0.5 & 0 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{array} \right\| \end{array}$$

It's known that $\pi_1 = \frac{119}{379} = 0.3140$ and $\pi_2 = \frac{81}{379} = 0.2137$. Determine the limiting probabilities π_0 and π_3.

(b) Let X_n denote the condition of a machine at the end of period n for n=1,2,.... Let X_0 be the condition of the machine at the start. Consider the condition of a machine at any time can be observed and classified as being in one of the following three states:

State 1: Good operating order, State 2: Deteriorated operating order and State 3: In repair.

Assume that $\{X_n\}$ is a Markov chain with transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{array} \right\| \end{array}$$

and starts in state $X_0 = 1$.

(i) Find $\Pr\{X_4 = 1\}$.

(ii) Calculate the limiting distribution.

(iii) What is the long run rate of repairs per unit time?

Q5: [4+4]

(a) A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1,\ \lambda_1 = 2,\ \lambda_2 = 3$ and $\lambda_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

(b) A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0,\ \mu_1 = 2,\ \mu_2 = 3$ and $\mu_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

Model Answer

Q1: [4+4]

(a)

Y \ X	-1	0	1	$P_X(x)$
-1	1/9	0	2/9	1/3
0	2/9	1/9	0	1/3
1	0	2/9	1/9	1/3
$P_Y(y)$	1/3	1/3	1/3	Sum=1

$E(X)=0,\ E(X^2)=\frac{2}{3},\ Var(X)=\frac{2}{3}$

$E(Y)=0,\ E(Y^2)=\frac{2}{3},\ Var(Y)=\frac{2}{3}$

$E(XY)=0$

$Cov(X,Y)=E(XY)-E(X)E(Y)=0$

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=\frac{0}{2/3}=0$$

$\Rightarrow X$ and Y are not correlated

$\because$ For example, $P(X=0,Y=1)=0,$ but $P(X=0)P(Y=1)=\frac{1}{9}$

$\Rightarrow P(X=0,Y=1)\neq P(X=0)P(Y=1)$

$\therefore$ X and Y are not independent r.vs

(b)

$\because$ $X\sim \exp(\alpha)$ and $Y\sim \exp(\alpha)$

and $Z=X+Y$

$\therefore$ $Z\sim gamma(2,\alpha)$

$$\Rightarrow f(z)=\frac{\alpha^2}{\Gamma(2)}ze^{-\alpha z}=\alpha^2 ze^{-\alpha z}$$

$$f(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)}$$
$$= \frac{\alpha^2 e^{-\alpha z}}{\alpha^2 z e^{-\alpha z}} = \frac{1}{z}, \quad 0 \le x \le z$$

$\therefore\ X|Z = z$ is uniformly distributed over the interval $[0, z]$

Q2: [4+4]

(a)

$$\because\ pr(Z=n) = \sum_{k=0}^{n} pr\{X=k\} pr\{Y=n-k\}$$
$$= \sum_{k=0}^{n} \frac{\mu^k e^{-\mu} v^{(n-k)} e^{-v}}{k!(n-k)!}$$
$$= \frac{e^{-(\mu+v)}}{n!} \sum_{k=0}^{n} \frac{n! \mu^k v^{(n-k)}}{k!(n-k)!}$$
$$\therefore\ pr(Z=n) = e^{-(\mu+v)} \frac{(\mu+v)^n}{n!} \quad \text{(by using binomial formula)}$$
$\therefore$ Z is a poisson distributed with parameter $\mu + v$.

(b)

$$Z = \xi_1 + \xi_2 + \ldots + \xi_N.$$

$$E(\xi_k) = \mu = 0,\ \text{Var}(\xi_k) = \sigma^2 = 1$$
$$E(N) = v = \frac{1-\alpha}{\alpha},\quad \text{Var}(N) = \tau^2 = \frac{1-\alpha}{\alpha^2}$$
$$\because E(Z) = \mu v$$
$$\therefore E(Z) = 0$$
$$\because\ \text{Var(Z)} = v\sigma^2 + \mu^2 \tau^2$$
$$\therefore\ \text{Var(Z)} = \frac{1-\alpha}{\alpha}(1) + 0\left(\frac{1-\alpha}{\alpha^2}\right)$$
$$= \frac{1-\alpha}{\alpha}$$

Q3: [4+4]

(a)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \alpha & 1-\alpha & 0 \\ 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

(b)

For Poisson Process

$$\Pr\{X(s+t) - X(s) = k\}$$
$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \; k = 0, 1, 2, \ldots$$

(i)

$$\Pr\{X(1) \le 2\} = \Pr\{X(1) = 0\} + \Pr\{X(1) = 1\} + \Pr\{X(1) = 2\}$$
$$= \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} = 5e^{-2} \simeq 0.68$$

(ii)

$$\Pr\{X(1) = 1 \text{ and } X(2) = 3\}$$
$$= \Pr\{X(1) = 1\} \Pr\{X(2) = 3\}$$
$$= \Pr\{X(1) - X(0) = 1\} \Pr\{X(2) - X(1) = 2\}$$
$$= \frac{2^1 e^{-2}}{1!} . \frac{2^2 e^{-2}}{2!} = 4e^{-4} \simeq 0.07$$

(iii)

$$\Pr\{X(1)\ge 2 | X(1)\ge 1\}$$

$$= \frac{\Pr\{X(1)\ge 2 \text{ and } X(1)\ge 1\}}{\Pr\{X(1)\ge 1\}}$$

$$= \frac{\Pr\{X(1)\ge 2\}}{1-\Pr\{X(1)<1\}}$$

$$= \frac{1-\Pr\{X(1)<2\}}{1-\Pr\{X(1)<1\}}$$

$$= \frac{1-[\Pr\{X(1)=0\}+\Pr\{X(1)=1\}]}{1-\Pr\{X(1)=0\}}$$

$$= \frac{1-\frac{e^{-2}}{0!}-\frac{2e^{-2}}{1!}}{1-\frac{e^{-2}}{0!}} = \frac{1-3e^{-2}}{1-e^{-2}} = 0.68696$$

Q4: [4+4]

(a)

$\pi_0 = \frac{117}{379} = 0.3087$ and $\pi_3 = \frac{62}{379} = 0.1636$

(b)

(i)

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{Vmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{Vmatrix} \end{array}$$

$$\because \ \Pr\{X_4=1\} = \Pr\{X_4=1 | X_0=1\}\Pr\{X_0=1\}$$

$$= P_{11}^4, \ \Pr\{X_0=1\}=1$$

$$\mathrm{P}^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{Vmatrix} 0.81 & 0.18 & 0.01 \\ 0.1 & 0.81 & 0.09 \\ 0.9 & 0.1 & 0 \end{Vmatrix} \end{array}$$

$$\text{and} \quad P^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{ccc} 0.6831 & 0.2926 & 0.0243 \\ 0.2430 & 0.6831 & 0.0739 \\ 0.7390 & 0.2430 & 0.0180 \end{array}\right\| \end{array}$$

$\therefore \ \Pr\{X_4 = 1\} = 0.6831$

(ii) To get the limiting distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\pi_G, \pi_D, \pi_R)$

Solving the following equations

$$\pi_1 = 0.9\pi_1 + \pi_3 \qquad (1)$$
$$\pi_2 = 0.1\pi_1 + 0.9\pi_2 \qquad (2)$$
$$\pi_3 = 0.1\pi_2 \qquad (3)$$
$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

$(1) \Rightarrow \pi_3 = 0.1\pi_1$

$(2) \Rightarrow \pi_2 = \pi_1$

also, (3) $\pi_3 = 0.1\pi_2$

$(4) \Rightarrow \pi_1 + \pi_1 + 0.1\pi_1 = 1$

$\therefore \ \pi_1 = \frac{10}{21}$

$\Rightarrow \therefore \ \pi_2 = \frac{10}{21}$ and π_3

$\therefore \ \pi = (\frac{10}{21}, \frac{10}{21}, \frac{1}{21})$

(iii) $\pi_R = \pi_3 = \frac{1}{21} = 0.0476$

Q5: [4+4]

(a)

For pure birth process, the transition probabilities are given by

$$p_0(t) = e^{-\lambda_0 t}, \quad (1)$$

$$p_1(t) = \lambda_0 \left[\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right], \quad (2)$$

and $p_n(t) = pr\{X(t) = n | X(0) = 0\}$

$$= \lambda_0 \lambda_1 \ldots \lambda_{n-1} \left[B_{0,n} e^{-\lambda_0 t} + \ldots + B_{k,n} e^{-\lambda_k t} + \ldots + B_{n,n} e^{-\lambda_n t} \right], \ n > 1, \quad (3)$$

where

$$B_{k,n} = \prod_{i=0}^{n}\left(\frac{1}{\lambda_i - \lambda_k}\right)\ i \neq k,\ 0 < k < n,$$

$$B_{0,n} = \prod_{i=1}^{n}\left(\frac{1}{\lambda_i - \lambda_0}\right)$$

and

$$B_{n,n} = \prod_{i=0}^{n-1}\left(\frac{1}{\lambda_i - \lambda_n}\right)$$

at $n = 0$ (1) $\Rightarrow p_0(t) = e^{-\lambda_0 t}, \quad \lambda_0 = 1$

$\therefore\ p_0(t) = e^{-t}$

at $n = 1$ (2) $\Rightarrow p_1(t) = \left[e^{-t} - e^{-2t}\right]$

at $n = 2$ (3) $\Rightarrow p_2(t) = \lambda_0\lambda_1\left[B_{0,2}e^{-\lambda_0 t} + B_{1,2}e^{-\lambda_1 t} + B_{2,2}e^{-\lambda_2 t}\right],$

where, $B_{0,2} = \dfrac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}$

$= \frac{1}{2},$

$$B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}$$

$= -1$

and

$$B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$= \frac{1}{2}$

$\therefore\ p_2(t) = 2\left[\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right]$

at $n = 2$ (3) $\Rightarrow p_3(t) = \lambda_0\lambda_1\lambda_2\left[B_{0,3}e^{-\lambda_0 t} + B_{1,3}e^{-\lambda_1 t} + +B_{2,3}e^{-\lambda_2 t} + B_{3,3}e^{-\lambda_3 t}\right]$

where $B_{0,3} = \dfrac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)(\lambda_3 - \lambda_0)} = \frac{1}{8}$,

$B_{1,3} = \dfrac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} = -\frac{1}{3}$,

$B_{2,3} = \dfrac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} = \frac{1}{4}$,

and

$B_{3,3} = \dfrac{1}{(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} = -\frac{1}{24}$.

$$\therefore\ p_3(t) = 6\left[\tfrac{1}{8}e^{-t} - \tfrac{1}{3}e^{-2t} + \tfrac{1}{4}e^{-3t} - \tfrac{1}{24}e^{-5t}\right]$$

(b)

For pure death process, the transition probabilities are given by

$$p_N(t) = e^{-\mu_N t} \qquad (1)$$

and for $n < N$

$$p_n(t) = pr\left\{X(t) = n \middle| X(0) = N\right\}$$
$$= \mu_{n+1}\mu_{n+2}\ \ldots\ \mu_N\left[A_{n,n}e^{-\mu_n t} + \ \ldots + A_{k,n}e^{-\mu_k t} + \ \ldots + A_{N,n}e^{-\mu_N t}\right] \quad (2)$$

Where $A_{k,n} = \prod_{i=N}^{n} \dfrac{1}{(\mu_i - \mu_k)}$, $\ i \neq k,\ n \leq k \leq N,\ i = N, N\text{-}1,\ \ldots,\ n$ (3)

For N=3 $\quad (1) \Rightarrow \mathrm{p}_3(\mathrm{t}) = \mathrm{e}^{-\mu_3 t}$

$\therefore\ \mathrm{p}_3(\mathrm{t}) = \mathrm{e}^{-5t}$ (I)

For n=2 $\quad (2) \Rightarrow \mathrm{p}_2(\mathrm{t}) = \mu_3\left[A_{2,2}e^{-\mu_2 t} + A_{3,2}e^{-\mu_3 t}\right]$

$$(3) \Rightarrow A_{2,2} = \prod_{i=3}^{2} \frac{1}{(\mu_i - \mu_2)}\ ,\ i \neq 2$$
$$= \frac{1}{\mu_3 - \mu_2} = \frac{1}{2}$$

$$, A_{3,2}=\prod_{i=3}^{2}\frac{1}{(\mu_i-\mu_3)},\ i\neq 3$$

$$=\frac{1}{\mu_2-\mu_3}=-\frac{1}{2}$$

$$\therefore\ p_2(t)=5\left[\frac{1}{2}e^{-3t}-\frac{1}{2}e^{-5t}\right] \qquad \text{(II)}$$

For n=1 $\quad (2)\Rightarrow p_1(t)=\mu_2\mu_3\left[A_{1,1}e^{-\mu_1 t}+A_{2,1}e^{-\mu_2 t}+A_{3,1}e^{-\mu_3 t}\right]$

$$(3)\Rightarrow\ A_{1,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_1)},\ i\neq 1$$

$$=\frac{1}{(\mu_3-\mu_1)(\mu_2-\mu_1)}=\frac{1}{3}$$

$$A_{2,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_2)},\ i\neq 2$$

$$=\frac{1}{(\mu_3-\mu_2)(\mu_1-\mu_2)}=-\frac{1}{2}$$

$$, A_{3,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_3)},\ i\neq 3$$

$$=\frac{1}{(\mu_2-\mu_3)(\mu_1-\mu_3)}=\frac{1}{6}$$

$$\therefore\ p_1(t)=15\left[\frac{1}{3}e^{-2t}-\frac{1}{2}e^{-3t}+\frac{1}{6}e^{-5t}\right] \qquad \text{(III)}$$

Using (I), (II) and (III) we can get $p_0(t)$ as follows

$$\therefore\ p_0(t)=1-\left[p_1(t)+p_2(t)+p_3(t)\right]$$

$$=1-\left[5e^{-2t}-\frac{15}{2}e^{-3t}+\frac{5}{2}e^{-3t}+\frac{5}{2}e^{-5t}-\frac{5}{2}e^{-5t}+e^{-5t}\right]$$

$$=1-5e^{-2t}+5e^{-3t}-e^{-5t} \qquad \text{(IV)}$$
