King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1439/1440
M 380 – Stochastic Processes
Time: 3 hours – Female Section

Answer the following questions:

Q1: [4+4]

(a) Suppose X and Y are jointly distributed random variables having the density function $f_{XY}(x,y) = \frac{1}{y} e^{-(x/y)-y}$ for x, y >0. Find the conditional probability of X and determine the expected value for X, given that Y=y.

b) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \dots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$ where $p_{i_0} = \operatorname{pr}\{X_0 = i_0\}$

Q2: [2+4]

a) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \left\| \begin{array}{cc} 0.98 & 0.02 \\ 0.14 & 0.86 \end{array} \right\| \end{array}$$

In the long run, what is the probability that an item produced by this system is good?

b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.4$, $\Pr\{\xi_n = 1\} = 0.3$, $\Pr\{\xi_n = 2\} = 0.3$ and suppose s=0 and S=3. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

Q3: [8]

An airline reservation system has two computers, only one of which is in operation at any given time. A computer may break down on any given day with probability p. There is a single repair facility that takes 2 days to restore a computer to normal. The facilities are such that only one computer at a time can be dealt with. Form a Markov chain by taking as states the pairs (x,y), where x is the number of machines in operating condition at the end of a day and y is 1 if a day's labor has been expended on a machine not yet repaired and 0 otherwise. Also, find the system availability.

Q4: [5+4]

(a) From purchase to purchase, a particular customer switches brands among products A, B, and C according to a Markov chain whose transition probability matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{Vmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{Vmatrix} \end{array}$$

In the long run, what fraction of time does this customer purchase brand A?

(b) Let X(t) be a Yule process that is observed at a random time U, where U is uniformly distributed over [0,1). Show that $pr\{X(U)=k\}=p^k/(\beta k)$ for $k=1,2,\ldots$, with $p=1-e^{-\beta}$.

Q5: [5+4]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt}=-\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt}=\lambda p_{n-1}(t)-\lambda p_n(t),\ n=1,2,3,\ldots \qquad (2)$$

where all birth parameters are the same constant λ with initial condition X(0)=0,

Show that $p_n(t)=\dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n=0,1,2,\ldots$

(b) Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that N=X+Y=n.

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Model Answer of Final Exam S1 1439/1440

M 380 - Stochastic Process

8

4+4

Q1

a) $f_{XY}(x,y) = \frac{1}{y} e^{-(x/y)-y}$

$$f_{XY}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty f(x,y)\,dx$$

$$f_Y(y) = \frac{1}{y}\int_0^\infty e^{-(x/y)-y}\,dx$$

$$= \frac{e^{-y}}{y}\int_0^\infty e^{-x/y}\,dx$$

1.5

$$f_Y(y) = \frac{e^{-y}}{y}\left[\frac{e^{-x/y}}{-1/y}\right]_0^\infty = e^{-y}, \quad y>0$$

when $e^{-x} \to 0$ as $x \to \infty$

$$\therefore f_{XY}(x|y) = \frac{\frac{1}{y} e^{-(x/y)-y}}{e^{-y}}$$

$$f_{XY}(x|y) = \frac{1}{y} e^{-(x/y)} \quad \text{for } x,y>0$$

1.5

$\therefore X \sim \exp(1/y)$

1

$E[X|Y=y] = y$

b) $pr\{X_0=i_0, X_1=i_1, X_2=i_2, \dots, X_n=i_n\}$

$= pr\{X_0=i_0, X_1=i_1, X_2=i_2, \dots, X_{n-1}=i_{n-1}\}$

$\cdot\, pr\{X_n=i_n | X_0=i_0, X_1=i_1, X_2=i_2, \dots, X_{n-1}=i_{n-1}\}$

1 ①

$pr\{X_n=i_n | X_0=i_0, X_1=i_1, \dots, X_{n-1}=i_{n-1}\}$

$= pr\{X_n=i_n | X_{n-1}=i_{n-1}\}$ 1

②

$= P_{i_{n-1} i_n}$

Defn of Markov process

Subs. ② in ①

$pr\{X_0=i_0, X_1=i_1, X_2=i_2, \dots, X_n=i_n\}$

$= pr\{X_0=i_0, X_1=i_1, X_2=i_2, \dots, X_{n-1}=i_{n-1}\}$

$\cdot\, P_{i_{n-1} i_n}$ 1

By repeating this argument $n-1$ times

$\Rightarrow pr\{X_0=i_0, X_1=i_1, X_2=i_2, \dots, X_n=i_n\}$

$= P_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ 1

where $P_{i_0} = pr\{X_0=i_0\}$ for initial distn.

2

6 (2+4)

Q2

a) In the long run, the probability that an item produced by this system is good is given by $b/(a+b) = \frac{0.14}{(0.02+0.14)}$
$= 87.5\ \%$

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Not that:

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{cc} 0 & 1 \\ \left[\begin{array}{c} \frac{b}{a+b} \\ \frac{b}{a+b} \end{array}\right. & \left.\begin{array}{c} \frac{a}{a+b} \\ \frac{a}{a+b} \end{array}\right] \end{array}$$

b)

$$\begin{array}{c|ccccc} & -1 & 0 & 1 & 2 & 3 \\ & -1 & 0 & 0.3 & 0.3 & 0.4 \\ -1 & 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & & & & \\ 1 & 0.3 & 0.3 & 0.4 & 0 & 0 \\ 2 & 0 & 0.3 & 0.3 & 0.4 & 0 \\ 3 & 0 & 0 & 0.3 & 0.3 & 0.4 \end{array}$$

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$i \le s \Rightarrow \mathfrak{I}_{n+1} = S - d$ replenishment

, $s < i \le S \Rightarrow \mathfrak{I}_{n+1} = i - d$ non replenishment

e.g. $P_{-1\,-1} = pr(\mathfrak{I}_{n+1} = 4) = 0$

, $P_{-1\,2} = pr(\mathfrak{I}_{n+1} = 1) = 0.3$

, $P_{2\,2} = pr(\mathfrak{I}_{n+1} = 0) = 0.4$

, $P_{3\,0} = pr(\mathfrak{I}_{n+1} = 3) = 0$

. . . . .

Q3

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	(2,0)	(1,0)	(1,1)	(0,1)
(2,0)	q	p	0	0
(1,0)	0	0	q	p
(1,1)	q	p	0	0
(0,1)	0	1	0	0
	$\downarrow$ π_0	$\downarrow$ π_1	$\downarrow$ π_2	$\downarrow$ π_3

4

The limitting dist^n
$= (\pi_0, \pi_1, \pi_2, \pi_3)$

$\Rightarrow q\pi_0 + q\pi_2 = \pi_0$ ①

$\therefore \boxed{\pi_2 = \frac{p}{q}\pi_0}$ 1

, $p\pi_0 + p\pi_2 + \pi_3 = \pi_1$

, $q\pi_1 = \pi_2$

$\therefore \pi_1 = \frac{1}{q}\pi_2 = \frac{1}{q}\left(\frac{p}{q}\right)\pi_0$

$\therefore \boxed{\pi_1 = \frac{p}{q^2}\pi_0}$ ② 1

, $p\pi_1 = \pi_3$

$\therefore \pi_3 = p\left(\frac{p}{q^2}\pi_0\right) = \boxed{\frac{p^2}{q^2}\pi_0}$ ③

& $\therefore \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

$\therefore \pi_0 + \frac{p}{q^2}\pi_0 + \frac{p}{q}\pi_0 + \frac{p^2}{q^2}\pi_0 = 1$

$\therefore \left(\frac{q^2 + p + pq + p^2}{q^2}\right)\pi_0 = 1$

. . . . $\boxed{\pi_0 = \frac{q^2}{1+p^2}}$ ④

$\therefore \pi_3 = \frac{p^2}{1+p^2} \Rightarrow$ [illegible] $\therefore R_1 = 1 - \pi_3 = \frac{1}{1+p^2}$ 1

=

Q4

a)

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{array}\right] \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \pi_0 & \pi_1 & \pi_2 \end{array} \end{array}$$

The limiting distn is

$\pi = (\pi_0, \pi_1, \pi_2)$

$\Rightarrow 4\pi_0 - \pi_1 - \pi_2 = 0$ 1

$2\pi_0 - 3\pi_1 + \pi_2 = 0$ 1

$\pi_0 + \pi_1 + \pi_2 = 1$ 1

By using Cramer's rule

$$\Delta = \begin{vmatrix} 4 & -1 & -1 \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -20$$ 1

$$, \Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4$$

$$\pi_0 = \frac{\Delta_0}{\Delta} = \frac{-4}{-20} = \frac{1}{5}$$ 1

, The fraction of time that the customer purchases brand A is $\pi_A = \pi_0 = \frac{1}{5} = 20\%$

b) For Yule process 2

$$P_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \quad n \geq 1$$

$$pr\{X(U)=k\} = \int_0^1 e^{-\beta u}(1 - e^{-\beta u})^{k-1} \cdot du$$ 1

$$= \frac{1}{\beta}\int_0^1 (1 - e^{-\beta u})^{k-1} \cdot \beta e^{-\beta u} du$$

$$= \frac{1}{\beta}\left[\frac{(1 - e^{-\beta u})^k}{k}\right]_0^1$$

$$= \frac{1}{\beta k}\left[(1 - e^{-\beta})^k - 0\right]$$ 1

$$\therefore pr\{X(U)=k\} = \frac{p^k}{\beta k}, \quad k = 1, 2, \ldots$$

where $p = 1 - e^{-\beta}$

#

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5+4

4

Q5 9

5+4

a) let $X(t)$ represents the size of population with initial Condition $X(0)=0$

$$\Rightarrow P_n(0)=\begin{cases}1 & , n=0\\ 0 & , \text{otherwise}\end{cases}$$

$$① \Rightarrow \frac{dP_0(t)}{dt}=-\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)}=-\lambda dt \Rightarrow \int_0^t \frac{dP_0(u)}{P_0(u)}=-\lambda\int_0^t du$$

$$\therefore \left[\ln P_0(u)\right]_0^t=-\lambda\left[u\right]_0^t$$

$$\therefore \ln P_0(t)-\cancel{\ln P_0(0)}^{\,0}=-\lambda t$$

$$\Rightarrow \ln P_0(t)=-\lambda t \quad \therefore \boxed{P_0(t)=e^{-\lambda t}} \; ③$$

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$$② \Rightarrow \frac{dP_n(t)}{dt}=\lambda P_{n-1}(t)-\lambda P_n(t)$$

$$\frac{dP_n(t)}{dt}+\lambda P_n(t)=\lambda P_{n-1}(t) \quad , n=1,2,3,\dots$$

Multiply both sides by $e^{\lambda t}$

$$e^{\lambda t}\left[\frac{dP_n(t)}{dt}+\lambda P_n(t)\right]=\lambda P_{n-1}(t)\,e^{\lambda t}$$

$$\therefore \frac{d}{dt}\left[e^{\lambda t}P_n(t)\right]=\lambda P_{n-1}(t)\,e^{\lambda t}$$

$$\Rightarrow \int_0^t d\left[e^{\lambda t}P_n(t)\right]=\lambda\int_0^t P_{n-1}(x)\,e^{\lambda x}\,dx$$

$$e^{\lambda t}P_n(t)-\cancel{P_n(0)}_{\,0}=\lambda\int_0^t P_{n-1}(x)\,e^{\lambda x}\,dx \quad , n=1,2,\dots$$

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$$P_n(t) = \lambda e^{-\lambda t} \int_0^t P_{n-1}(x) e^{\lambda x} dx \quad , n = 1, 2, \dots \quad (4)$$

which is a recurrence relation

at $n=1$ $\quad P_1(t) = \lambda e^{-\lambda t} \int_0^t P_0(x) e^{\lambda x} dx$

$(3) \Rightarrow P_0(x) = e^{-\lambda x}$

$$\therefore P_1(t) = \lambda e^{-\lambda t} \int_0^t dx$$

$$\boxed{P_1(t) = \lambda t e^{-\lambda t}} \quad (5)$$

, at $n=2$

$(4) \Rightarrow P_2(t) = \lambda e^{-\lambda t} \int_0^t P_1(x) e^{\lambda x} dx$

$(5) \Rightarrow P_1(x) = \lambda x e^{-\lambda x}$

$$\therefore P_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x \, dx$$

$$\therefore P_2(t) = \lambda^2 e^{-\lambda t} \left[\frac{x^2}{2}\right]_0^t = \boxed{\frac{(\lambda t)^2}{2!} e^{-\lambda t}} \quad (6)$$

$(3), (5)$ and $(6) \Rightarrow P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n = 0, 1, 2, \dots$ (Poisson process)

b) $pr\{X=k \mid N=n\}$

$= pr\{X=k \mid X+Y=n\}$

$= \dfrac{pr\{X=k\} \cap pr\{X+Y=n\}}{pr\{N=n\}}$

$= \dfrac{pr\{X=k\} \cap pr\{Y=n-k\}}{pr\{N=n\}} \Rightarrow$

4

$= \dfrac{\cancel{\dfrac{e^{-\alpha}\alpha^k}{k!}} \cdot \cancel{\dfrac{e^{-\beta}\beta^{n-k}}{(n-k)!}}}{\cancel{\dfrac{e^{-(\alpha+\beta)}(\alpha+\beta)^n}{n!}}}$

$= \alpha^k \beta^{n-k} \left(\dfrac{1}{\alpha+\beta}\right)^n \binom{n}{k}$

$= \binom{n}{k} \left(\dfrac{\alpha}{\alpha+\beta}\right)^k \left(\dfrac{\beta}{\alpha+\beta}\right)^{n-k}$

$\therefore pr\{X=k \mid N=n\} = \binom{n}{k} p^k (1-p)^{n-k}$, $p = \alpha/(\alpha+\beta)$ Binomial distr.