King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /Summer/2017
Full Mark:25. Time 1H30mn
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Question 1[4,4]. a) Find and sketch the largest local region of the $xy-$plane for which the initial value problem

$$\begin{cases} \ln(2x-4)\frac{dy}{dx} = \sqrt{y^2-1} \\ y(3) = 3. \end{cases}$$

has a unique solution.

b) Find the solution of the differential equation:

$$\frac{dy}{dx} - \frac{x+y-\sqrt{xy}}{x} = 0, \quad x > 0,\ y > 0.$$

Question 2[4,4]. a) Solve the initial value problem

$$2xydx + (4 - x^2 - y^2)dy = 0, \quad y(0) = 1,\ y > 0.$$

b) Solve the differential equation

$$\frac{dy}{dx} = (2x+y-3) + (2x+y-1)\ln(2x+y-1), \quad 2x+y-1 > 0.$$

Question 3[4]. Find the general solution of the differential equation

$$2xydy - (x+y^2)dx = 0, \quad x > 0.$$

Question 4[5]. A bar of metal was heated to a temperature 150^0C, then it was left in a medium temperature 25^0C. The temperature of the bar was reduced to half its initial value after 5 minutes. How long will it take so that the temperature of the metal bar will reach 30^0C.

1

Model answer

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Q1 a) $\dfrac{dy}{dx} = \dfrac{\sqrt{y^2-1}}{\ln(2x-4)} = f(x,y)$

$f(x,y)$ is continuous on the region

$D_1 = \{(x,y) \in \mathbb{R}^2 : x>2,\ x \neq \frac{5}{2},\ |y|\geq 1\}$ 1

$\dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{y^2-1}\ \ln(2x-4)}$

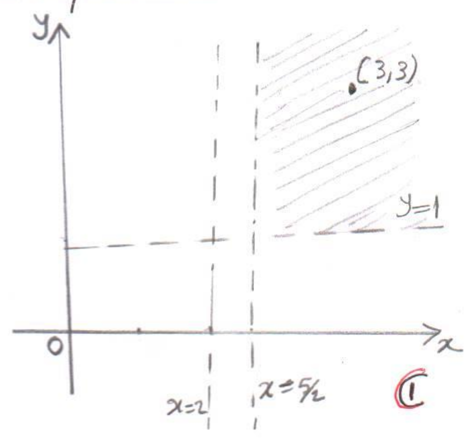
$\dfrac{\partial f}{\partial y}$ is continuous on the region

$D_2 = \{(x,y) \in \mathbb{R}^2 : x>2,\ x \neq \frac{5}{2},\ |y|>1\}$ 1

$\therefore$ Both f and $\dfrac{\partial f}{\partial y}$ are continuous on the region $D_1 \cap D_2 = D_2$

$\because (3,3) \in D = \{(x,y) \in \mathbb{R}^2 : x > \frac{5}{2},\ y>1\}$ 1

then the largest local region is D where the IVP admits a unique solution



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b) $\dfrac{dy}{dx} - \dfrac{x+y-\sqrt{xy}}{x} = 0,\ x>0,\ y>0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x+y-\sqrt{xy}}{x}$

$\Rightarrow y' = 1 + \dfrac{y}{x} - \sqrt{\dfrac{y}{x}} = F\left(\dfrac{y}{x}\right)$

(Homogeneous Eq.)

let $\boxed{u = \dfrac{y}{x}} \Rightarrow y' = xu' + u$ 1

$\Rightarrow xu' + \cancel{u} = 1 + \cancel{u} - \sqrt{u}$

$\Rightarrow \dfrac{du}{1-\sqrt{u}} = \dfrac{dx}{x}$ 1

$\Rightarrow \displaystyle\int \frac{du}{1-\sqrt{u}} = \ln x + c$

let $t = \sqrt{u} \Rightarrow$

$\displaystyle\int \frac{du}{1-\sqrt{u}} = \int \frac{2t\,dt}{1-t}$

$= \displaystyle\int \left[-2 + \frac{2}{1-t}\right] dt$ 2

$= -2t - 2\ln(1-t)$

$= -2\sqrt{u} - 2\ln(1-\sqrt{u})$

$\Rightarrow -2\sqrt{u} - 2\ln(1-\sqrt{u}) = \ln x + c$

$2\sqrt{y/x} + 2\ln(1-\sqrt{y/x}) + \ln x + C = 0$

#

2

4

Q2

a) $\underbrace{2xy}_{M}dx + (\underbrace{4-x^2-y^2}_{N})dy = 0$, $y(0)=1$, $y>0$

$\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = -2x$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ the DE is not exact

$\Rightarrow \frac{N_x - M_y}{M} = \frac{-2}{y}$

$\Rightarrow \mu(y) = e^{-\int \frac{2}{y}dy} = \frac{1}{y^2}$ ①

Multiply the DE by $\mu(y) = \frac{1}{y^2}$

$\Rightarrow \frac{2x}{y}dx + (\frac{4}{y^2} - \frac{x^2}{y^2} - 1)dy = 0$
exact DE

$\therefore \exists F(x,y)$ s.t $\frac{\partial F}{\partial x} = \frac{2x}{y}$ ①

$, \frac{\partial F}{\partial y} = \frac{4}{y^2} - \frac{x^2}{y^2} - 1$ ②

①$\Rightarrow F(x,y) = \frac{x^2}{y} + C(y)$ ③

$\frac{\partial F}{\partial y} = -\frac{x^2}{y^2} + C'(y)$ ④

②, ④ $\Rightarrow C'(y) = \frac{4}{y^2} - 1$

$\Rightarrow C(y) = -\frac{4}{y} - y + C$

$\therefore F(x,y) = \frac{x^2}{y} - \frac{4}{y} - y + C = 0$ is the soln. of DE

$\therefore \boxed{\frac{x^2}{y} - \frac{4}{y} - y + C = 0}$ ②

For $y(0)=1 \Rightarrow 0 - 4 - 1 + C = 0$

$\boxed{C = 5}$

$\Rightarrow \frac{x^2}{y} - \frac{4}{y} - y + 5 = 0$ ①

$\Rightarrow x^2 - y^2 - 4 + 5y = 0$

$\Rightarrow x^2 - (y - 5/2)^2 = 4 - \frac{25}{4} = \frac{-9}{4}$

is the soln of DE.

4

b)

$\frac{dy}{dx} = (2x+y-3) + (2x+y-1)\ln(2x+y-1)$

$, 2x+y-1 > 0$

let $u = 2x+y-1$ ①

$u' = 2 + y'$

DE $\Rightarrow u' - \cancel{2} = u - \cancel{2} + u \ln u$

$\boxed{\int \frac{du}{u(1+\ln u)} = \int dx}$ ①

$\ln(1+\ln u) = x + C$

$\Rightarrow 1 + \ln u = e^{x+C}$

$= e^x \cdot e^C$

$1 + \ln u = C_1 e^x$ ②

$\Rightarrow 1 + \ln(2x+y-1) = C_1 e^x$

is the soln. of DE where $C_1 = e^C$ #

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Q3 4

$2xy\,dy - (x+y^2)\,dx = 0, \quad x>0$

$\frac{dy}{dx} = \frac{x+y^2}{2xy}$

$\Rightarrow y' = \frac{1}{2x}y + \frac{1}{2y}$

$\Rightarrow y' - \frac{1}{2x}y = \frac{1}{2}y^{-1}$ 1

Bernoulli Eqn

$n=-1$, let $u = y^{1+1} = y^2$

$\Rightarrow y = \sqrt{u}$

$y' = \frac{1}{2\sqrt{u}}u'$

the DE $\Rightarrow \frac{1}{2\sqrt{u}}u' - \frac{1}{2x}\sqrt{u} = \frac{1}{2\sqrt{u}}$ $\times 2\sqrt{u}$

$\Rightarrow \boxed{u' - \frac{1}{x}u = 1}$ (1) 1

$\mu(x) = e^{\int P(x)dx}$ Integ. Factor

$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{\ln 1/x} = 1/x$

the solution of DE(1) is given by

$\mu u = \int \frac{1}{x}(1)\,dx = \ln x + c$

$\frac{u}{x} = \ln x + c$ 2

$\frac{y^2}{x} = \ln x + c$

$\Rightarrow y^2 = x\ln x + cx$ is the soln of the given DE. #

Q4 5

$\frac{dT}{dt} = k(T - T_s)$ Newton's law for Cooling

$T_s = 25,\ T(0) = 150,\ T(5) = 75$

$\Rightarrow \frac{dT}{dt} = k(T-25)$

$\int \frac{dT}{T-25} = k\int dt$

$\ln(T-25) = kt + \ln c$

$\ln\left(\frac{T-25}{c}\right) = kt$

$\Rightarrow \frac{T-25}{c} = e^{kt}$

$\Rightarrow T = ce^{kt} + 25$ 1

For $T(0) = 150 \Rightarrow c = 150 - 25 = 125$

$T = 125e^{kt} + 25$

For $T(5) = 75 \Rightarrow$

$75 = 125e^{5kt} + 25$ 1

$\Rightarrow 125e^{5k} = 50$

$\Rightarrow e^{5k} = \frac{50}{125}$

$\Rightarrow 5k = \ln(2/5)$

$\Rightarrow k = \frac{1}{5}\ln(2/5)$ 1

$\therefore T(t) = 125\,e^{\frac{1}{5}\ln(2/5)\cdot t} + 25$

$\Rightarrow 30 = 125e^{\frac{1}{5}\ln(2/5)\cdot t} + 25$ 2

$\Rightarrow t \simeq 17.6$ min.