Discrete Mathematics (MATH 151)

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1. Graphs
   - Graphs and Graph Models
     - Graph
     - Graph Models
   - Graph Terminology and Special Types of Graphs
   - Representing Graphs and Graph Isomorphism
   - Connectivity

Graph

Definition 2.1

A graph $G = (V, E)$ consists of V , a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Remark

The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an infinite graph, and in comparison, a graph with a finite vertex set and a finite edge set is called a finite graph.

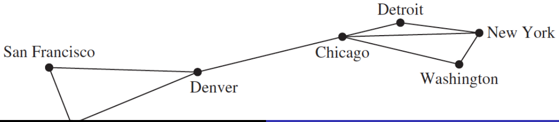
Graph

Definition 2.2

A simple graph, also called a strict graph, is an unweighted, undirected graph containing no graph loops or multiple edges

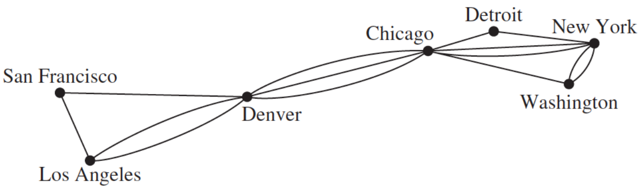
Example 2.1

Suppose that a network is made up of data centers and communication links between computers.We can represent the location of each data center by a point and each communications link by a line segment



Definition 2.3

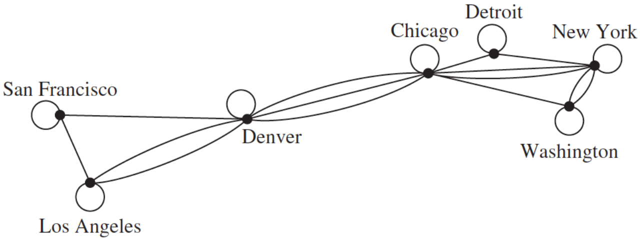
a multigraph is a graph which is permitted to have multiple edges (also called parallel edges), that is, edges that have the same end vertices. Thus two vertices may be connected by more than one edge.



A Computer Network with Multiple Links between Data Centers.

Definition 2.4

A *pseudograph* *is a non-simple graph in which both graph loops and multiple edges are permitted*



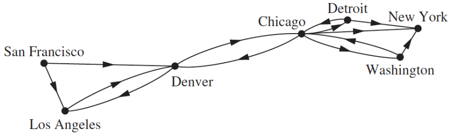
A Computer Network with Diagnostic Links.

Definition 2.5

A *directed graph* *(or digraph)* (V, E) *consists of a nonempty set of vertices* V *and a set of directed edges (or arcs)* E*. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at* u *and end at* v*.*

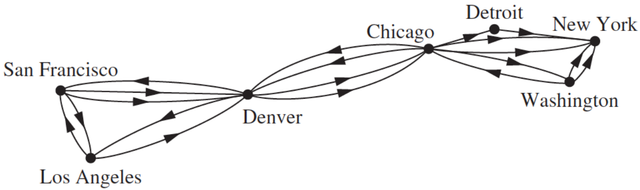
Definition 2.6

A *simple directed graph* *is a directed graph having no multiple edges or graph loops*



Definition 2.7

A directed multigraphs is a directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex are used to model such networks.



A Computer Network with Multiple One-Way Links.

Definition 2.8

Agraph with both directed and undirected edges is called a mixed graph.

Example 2.2

A mixed graph might be used to model a computer network containing links that operate in both directions and other links that operate only in one direction.

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Remark 2.1

Although the terminology used to describe graphs may vary, three key questions can help us understand the structure of a graph:

1. *Are the edges of the graph undirected or directed (or both)?*
2. *If the graph is undirected, are multiple edges present that connect the same pair of vertices?*
   If the graph is directed, are multiple directed edges present?
3. *Are loops present?*

SOCIAL NETWORKS

social networks

In these graph models, individuals or organizations are represented by vertices; relationships between individuals or organizations are represented by edges.

Example 2.3

A cquaintanceship and Friendship Graphs: We can use a simple graph to represent whether two people know each other, that is, whether they are acquainted, or whether they are friends.
No multiple edges and usually no loops are used.

Example 2.4

A cquaintanceship and Friendship Graphs: We can use a simple graph to represent whether two people know each other, that is, whether they are acquainted, or whether they are friends.
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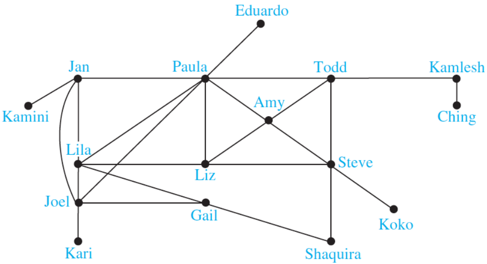
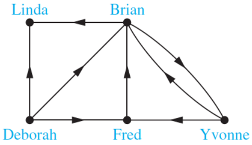


An Acquaintanceship Graph.

Example 2.5

Influence Graphs: *Each person of the group is represented by a vertex. There is a directed edge from vertex a to vertex b when the person represented by vertex a can influence the person represented by vertex b.*

This graph does not contain loops and it does not contain multiple directed edges.



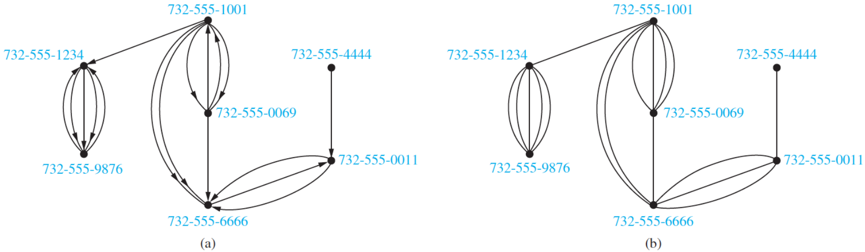
An Influence Graph.

COMMUNICATION NETWORKS

We can model different communications networks using vertices to represent devices and edges to represent the particular type of communications links of interest.

Example 2.6

Call Graphs a directed multigraph can be used to model calls where each telephone number is represented by a vertex and each telephone call is represented by a directed edge.
When we care only whether there has been a call connecting two telephone numbers, we use an undirected graph with an edge connecting telephone numbers when there has been a call between these numbers.


732-555-1001
732-555-1234
732-555-4444
732-555-0069
732-555-9876
732-555-0011
732-555-6666
(a)
732-555-1001
732-555-1234
732-555-4444
732-555-0069
732-555-9876
732-555-0011
732-555-6666
(b)

A Call Graph.

Basic Terminology

Definition 2.9

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Definition 2.10

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v. If A is a subset of V , we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \cup_{v \in A} N(v)$.

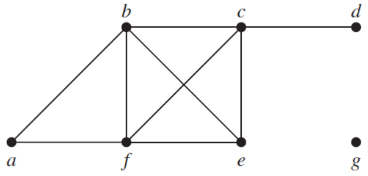
Basic Terminology

Definition 2.11

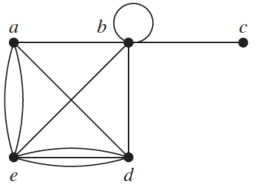
The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Example 2.7

What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure



G



H

Solution

In G:
deg(a) = 2, deg(b) = deg(c) = deg(f) = 4,
deg(d) = 1, deg(e) = 3, and deg(g) = 0.
The neighborhoods of these vertices are
$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$,
$N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \phi$.
In H:
deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, and deg(d) = 5.
The neighborhoods of these vertices are
$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$,
$N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

Theorem 2.1

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

Example 2.8

How many edges are there in a graph with 10 *vertices each of degree six?*

Solution:

Because the sum of the degrees of the vertices is $6 \times 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

Theorem 2.2

An undirected graph has an even number of vertices of odd degree.

Definition 2.12

When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u.
The vertex u is called the initial vertex of (u, v),
and v is called the terminal or end vertex of (u, v).
The initial vertex and terminal vertex of a loop are the same.

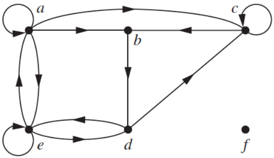
Basic Terminology

Definition 2.13

In a graph with directed edges the in-degree of a vertex v, denoted by $deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex.
(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Example 2.9

Find the in-degree and out-degree of each vertex in the graph G with directed edges.



G

Solution

The in-degree in G are: $deg^{-}(a) = 2, deg^{-}(b) = 2,$
$deg^{-}(c) = 3, deg^{-}(d) = 2, deg^{-}(e) = 3,$ and $deg^{-}(f) = 0.$
The out-degree are :$deg^{+}(a) = 4, deg^{+}(b) = 1,$

Theorem 2.3

Let $G = (V, E)$ be a graph with directed edges. Then

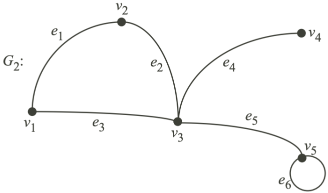
$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

The graph $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if

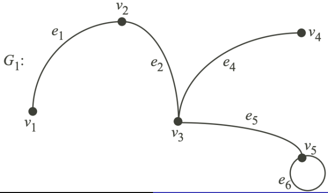
1. $V_1 \in V_2$ and
2. Every edge of G_1 is also an edge of G_2.

Example

We have the graph



and G_1 subgraph

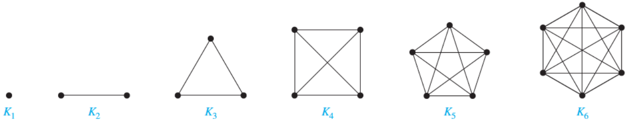


Some Special Simple Graphs

Definition 2.14

A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.

The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$,

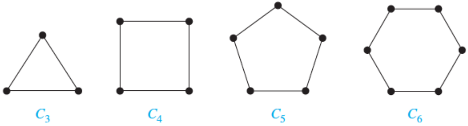


Some Special Simple Graphs

Definition 2.15

A cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

The cycles C_3, C_4, C_5, and C_6

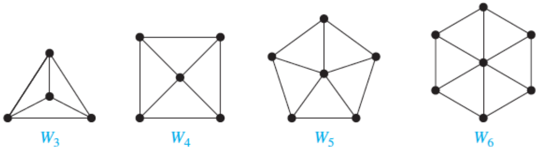


Some Special Simple Graphs

Definition 2.16

Wheels We obtain a wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

The wheels W_3, W_4, W_5, and W_6

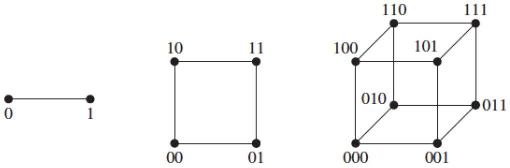


Some Special Simple Graphs

Definition 2.17

n-Cubes An n-dimensional hypercube, or n-cube, denoted by textcolorredQ_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

The Q_1, Q_2, and Q_3



Q_1 Q_2 Q_3

Bipartite Graphs

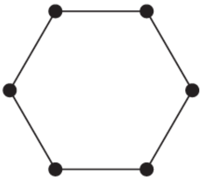
Definition 2.18

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

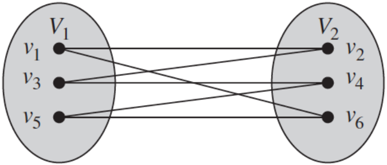
Bipartite Graphs

Example 2.10

C_6 is bipartite, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



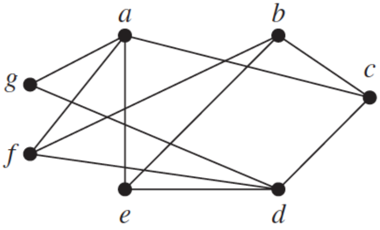
C_6



Bipartite Graphs

Example 2.11

the graph G is bipartite?



G

Bipartite Graphs

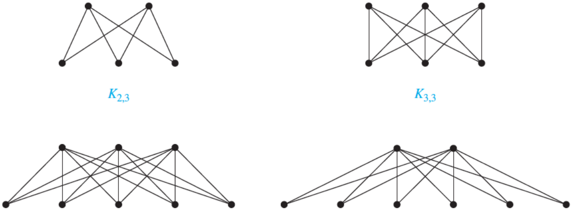
Theorem 2.4

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Bipartite Graphs

Definition 2.19

A *complete bipartite graph* $K_{m,n}$ *is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices* *if and only if* *one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs* $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, *and* $K_{2,6}$



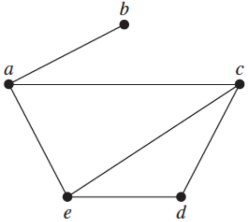
Definition 2.20

A matching of graph G is a subgraph of G such that every edge shares no vertex with any other edge. That is, each vertex in matching M has degree one.

Adjacency lists

Example 2.12

Use adjacency lists to describe the simple graph given in Figure.

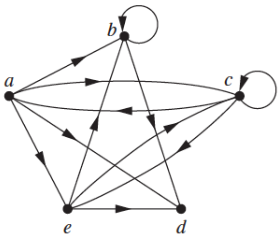


Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Adjacency lists

Example 2.13

Represent the directed graph shown in Figure, by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.



Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

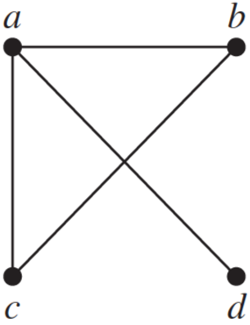
Adjacency Matrices

Suppose that $G = (V, E)$ is a simple graph where$|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The adjacency matrix A (orA_G) of G, with respect to this listing of the vertices, is the$n \times n$ zero - one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{if } (v_i, v_j) \text{ otherwise.} \end{cases}$$

Example 2.14

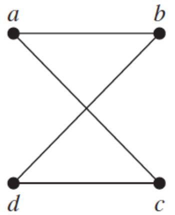
Use an adjacency matrix to represent the graph shown in Figure



$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example 2.15

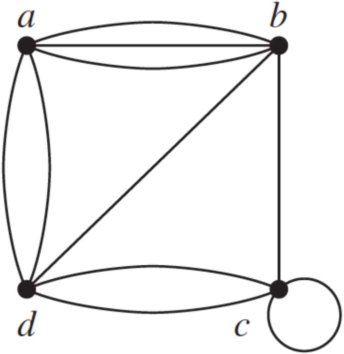
Use an adjacency matrix to represent the graph shown in Figure



$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Example 2.16

Use an adjacency matrix to represent the graph shown in Figure



$$A_G = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

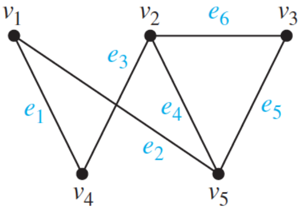
Definition 2.21

Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix *with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where*

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.17

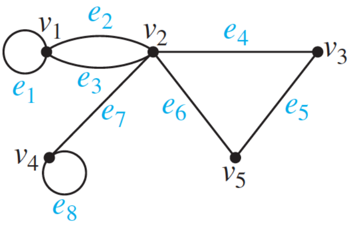
Represent the graph shown in Figure with an incidence matrix.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Example 2.18

Represent the pseudograph shown in Figure using an incidence matrix.



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

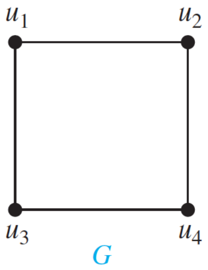
Isomorphism of Graphs

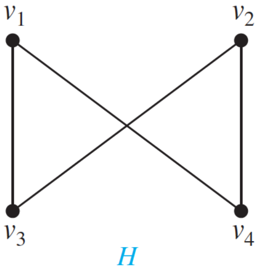
Definition 2.22

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one to one and on to function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism. Two simple graphs that are not isomorphic are called nonisomorphic.

Example 2.19

Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure , are isomorphic.
If we take $f : V \to W$ such that
$f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3$ *and* $f(u_4) = v_2$



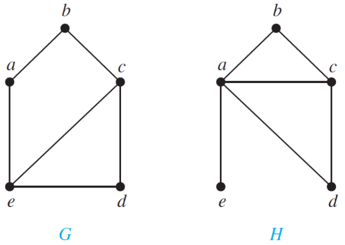


For instance,

1. isomorphic simple graphs must have the same number of vertices, because there is a one-to-one correspondence between the sets of vertices of the graphs.
2. Isomorphic simple graphs also must have the same number of edges, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges.
3. the degrees of the vertices in isomorphic simple graphs must be the same. That is, a vertex v of degree d in G must correspond to a vertex $f(v)$ of degree d in H, because a vertex w in G is adjacent to v if and only if $f(v)$ and $f(w)$ are adjacent in H.

Example 2.20

Show that the graphs displayed in Figure are not isomorphic.

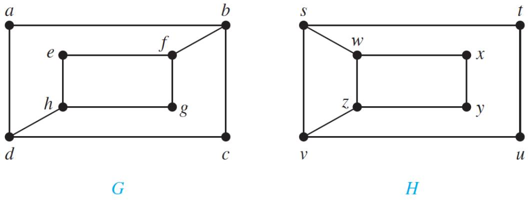


Solution

Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

Example 2.21

Determine whether the graphs shown in Figure are isomorphic.



Solution

G and H are not isomorphic. To see this, note that because $deg(a) = 2$ in G, a must correspond to either t, u, x, or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G. It follows that G and H are not isomorphic.

Connectivity

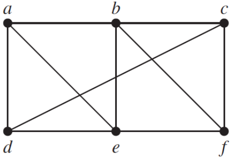
Definition 2.23

Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, ..., x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, ..., n$, the endpoints x_{i-1} and x_i . When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, ..., x_n$ (because listing these vertices uniquely determines the path). The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to pass through the vertices $x_1, x_2, ..., x_{n-1}$ or traverse the edges $e_1, e_2, ..., e_n$. A path or circuit is simple if it does not contain the same edge more than once.

Connectivity

Example:

1. a, d, c, f, e is a simple path of length 4, because $\{a, d\}, \{d, c\}, \{c, f\}$, and $\{f, e\}$ are all edges.
2. b, c, f, e, b is a circuit of length 4 because $\{b, c\}, \{c, f\}, \{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b.
3. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge a, b twice.
4. d, e, c, a is not a path, because $\{e, c\}$ is not an edge.



Connectivity

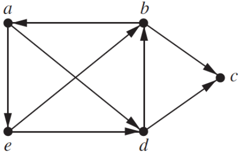
Definition 2.24

Let n be a nonnegative integer and G a directed graph.
A path of length n from u to v in G is a sequence of edges $e_1, e_2, ..., e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with en associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, ..., x_n$. A path of length greater than zero that begins and ends at the same vertex is called a circuit or cycle. A path or circuit is called simple if it does not contain the same edge more than once.

Connectivity

Example:

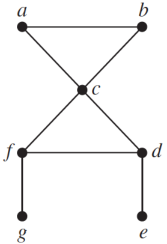
1. a, e, d, b, c is a simple path of length 4, because $\{a, e\}, \{e, d\}, \{d, b\}$, and $\{b, c\}$ are all edges.
2. a, e, d, b, a is a circuit of length 4 because $\{a, e\}, \{e, d\}, \{d, b\}$, and $\{b, a\}$ are edges, and this path begins and ends at b.
3. The path b, a, d, b, a, e, which is of length 5, is not simple because it contains the edge a, b twice.
4. e, d, c, a is not a path, because $\{c, a\}$ is not an edge.



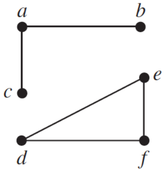
Connectedness in Undirected Graphs

Definition 2.25

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.
An undirected graph that is not connected is called disconnected.
We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.



G_1



G_2

CONNECTED COMPONENTS

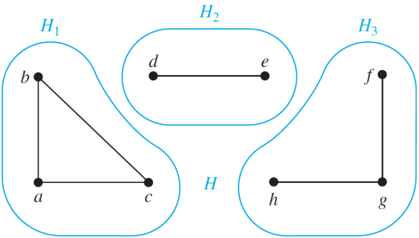
Theorem 2.5

There is a simple path between every pair of distinct vertices of a connected undirected graph.

CONNECTED COMPONENTS

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

CONNECTED COMPONENTS



The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3, shown in Figure. These three subgraphs are the connected components of H.

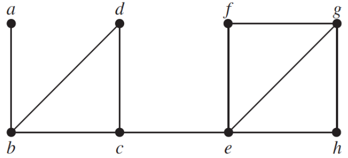
How Connected is a Graph?

A cut vertex is a vertex that when removed (with its boundary edges) from a graph creates more components than previously in the graph.

A cut edge is an edge that when removed (the vertices stay in place) from a graph creates more components than previously in the graph.

Example:

Find the cut vertices and cut edges in the graph G_1 shown in Figure.

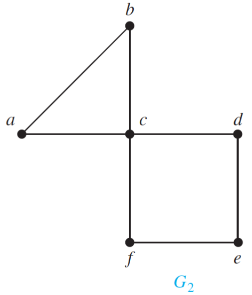


G_1

The cut vertices of G_1 are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph.
The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G_1.

Example:

Find the cut vertices and cut edges in the graph G_2 shown in Figure.



The cut vertex of G_2 is c. There are no cut edges.

Connectedness in Directed Graphs

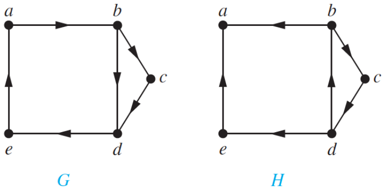
Definition 2.26

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Definition 2.27

A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Connectedness in Directed Graphs (Example)



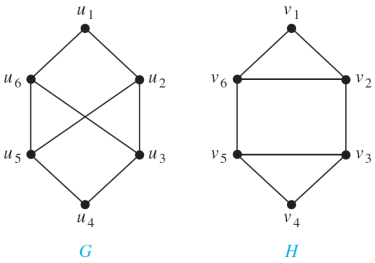
1. G is strongly connected because there is a path between any two vertices in this directed graph.
2. G is also weakly connected.
3. H is not strongly connected. There is no directed path from a to b in this graph.
4. H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H.

Paths and Isomorphism

There are several ways that paths and circuits can help determine whether two graphs are isomorphic.

Example:

Determine whether the graphs G and H shown in Figure are isomorphic.

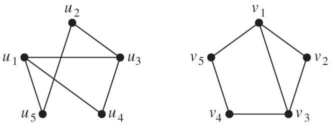


1. Both G and H have six vertices and eight edges.
2. Each has four vertices of degree three, and two vertices of degree two.
3. The three invariants-number of vertices, number of edges, and degrees of vertices-all agree for the two graphs.
4. H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1, whereas G has no simple circuit of length three, as can be determined by inspection (all simple circuits in G have length at least four).

Because the existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

Example:

Determine whether the graphs G and H shown in Figure are isomorphic.



G *H*

1. Both G and H have five vertices and six edges,
2. Both have two vertices of degree three and three vertices of degree two,
3. Both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five.

Because all these isomorphic invariants agree, *G* and *H* may be isomorphic.

To find a possible isomorphism,
we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths u_1, u_4, u_3, u_2, u_5 in G and v_3, v_2, v_1, v_5, v_4 in H

1. both go through every vertex in the graph; start at a vertex of degree three; go through vertices of degrees two, three, and two, respectively; and end at a vertex of degree two.
2. By following these paths through the graphs, we define the mapping f
3. the mapping f with $f(u_1) = v_3, f(u_4) = v_2, f(u_3) = v_1, f(u_2) = v5$, and $f(u_5) = v_4$.

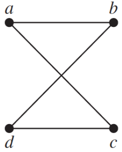
G and H are isomorphic,

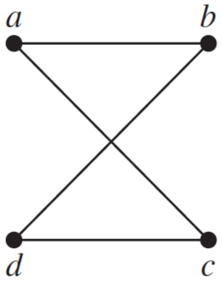
Theorem 2.6

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, ..., v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j , where r is a positive integer, equals the $(i, j)^{th}$ entry of A^r.

Example:

How many paths of length four are there from a to d in the simple graph G in Figure?





$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \qquad \mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

the number of paths of length 4 from a to d is 8 the $(1,4)^{th}$ entry of A^4.