Discrete Mathematics (MATH 151)

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Mathematical Induction

Introduction

- P(n) is a propositional function.
- P(n) is true for all positive integers n
- **mathematical induction** can be used to prove statements that assert that P(n) is true.

PRINCIPLE OF MATHEMATICAL INDUCTION

To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete two steps:

1. basis step, where we show that P(1) is true
2. inductive step, where we show that for all positive integers k, if P(k) is true, then P(k + 1) is true.
   $(P(k) \to P(k+1)$ is ture for all positive integers k)

Mathematical Induction

Remark 2.1

this proof technique can be stated as
$[P(1) \wedge (\forall k P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$

Remark

In a proof by mathematical induction it is NOT assumed that P(k) is true for all positive integers! It is only shown that if it is assumed that P(k) is true, then P(k + 1) is also true.

Mathematical Induction (Example)

Example 2.1

Show that if n is a positive integer, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Mathematical Induction (Example)

Solution

- BASIS STEP: we show that $P(1)$ is true, $1 = \frac{1(1+1)}{2}$
- INDUCTIVE STEP: If we assume that P(k) holds for any arbitrary positive integer k. $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$, Under this assumption, it must be shown that P(k + 1) is true, namely, that $1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$ is also true. When we add$k + 1$ to both sides of the equation in $P(k)$, we obtain
  $1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2}$
  This last equation shows that P(k + 1) is true under the assumption that P(k) is true.
  This completes the inductive step.

Mathematical Induction (Example)

Example 2.2

Show that if n is a positive integer, then

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Mathematical Induction (Example)

Solution

- BASIS STEP: P(1) states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1. The basis step is complete.
- INDUCTIVE STEP: If we assume that P(k) holds for any arbitrary positive integer k.
  $p(k): 1+3+5+\cdots+(2k-1)=k^2$ we have to show that
  $p(k+1): 1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2$ True
  $1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2$

Mathematical Induction (Example)

Example 2.3

Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Mathematical Induction (Example)

Solution

- BASIS STEP: P(0) is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.
- INDUCTIVE STEP: If we assume that P(k) holds for any arbitrary positive integer k.
  $P(k) : 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$ we have to show that
  $P(k+1) : 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$ is True
  $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} =$
  $2 \times 2^{k+1} - 1 = 2^{k+2} - 1$
  We have completed the inductive step.

Mathematical Induction (Example)

Example 2.4

Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \quad \text{where} \quad r \neq 1$$

for all nonnegative integers n

Mathematical Induction (Example)

Solution

- BASIS STEP: P(0) is true because,
  $\frac{ar^{0+1}-a}{r-1} = \frac{ar-a}{r-1} = \frac{a(r-1)}{r-1} = a$
- INDUCTIVE STEP: If we assume that P(k) holds for any arbitrary positive integer k. That is, P(k) is the statement that $P(k) : a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1}-a}{r-1}$ we have to show that $P(k+1) : a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2}-a}{r-1}$
  $a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
  $= \frac{ar^{k+1}-a}{r-1} + \frac{ar^{k+1}(r-1)}{r-1} = \frac{ar^{k+1}-a+ar^{k+2}-ar^{k+1}}{r-1} = \frac{ar^{k+2}-a}{r-1}$
  So if the inductive hypothesis P(k) is true, then P(k + 1) must also be true. This completes the inductive argument.

Mathematical Induction (Example)

Example 2.5

Use mathematical induction to show that if n is a positive integer, then

$$1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Mathematical Induction (Example)

Solution

- BASIS STEP: $P(1)$ is true.
- INDUCTIVE STEP: If we assume that P(k) holds for any arbitrary positive integer k. That is, P(k) is the statement that
  $P(k): \quad 1+4+9+\cdots+k^2 \quad = \frac{k(k+1)(2k+1)}{6}$
  and we have to show that:
  $P(k+1): 1+4+9+\cdots+k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$
  $1+4+9+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$
  $\frac{k(k+1)(2k+1)}{6}+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

This last equation shows that P(k + 1) is true under the assumption that P(k) is true. This completes the inductive step.

Mathematical Induction (Example)

Example 2.6

Use mathematical induction to show that if n is a positive integer, then

$$1.2^1 + 2.2^2 + 3.2^3 + \cdots + n.2^n = 2 + (n-1).2^{n+1}$$

Mathematical Induction (Example)

Solution

- BASIS STEP: $P(1)$ is true.
- INDUCTIVE STEP: If we assume that P(k) holds for any arbitrary positive integer k. That is,
  P(k): $1.2^1 + 2.2^2 + 3.2^3 + \cdots + k.2^k = 2 + (k-1).2^{k+1}$ and we have to show that P(k+1) is true
  $1.2^1 + 2.2^2 + 3.2^3 + \cdots + k.2^k + (k+1).2^{k+1} =$
  $2 + (k-1).2^{k+1} + (k+1).2^{k+1} =$
  $2 + [(k-1) + (k+1)].2^{k+1} = 2 + 2.k.2^{k+1} = 2 + k.2^{k+2}$

This last equation shows that P(k + 1) is true under the assumption that P(k) is true. This completes the inductive step.

Strong Induction

STRONG INDUCTION

To prove that P(n) is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:
BASIS STEP: We verify that the proposition $P(1)$ is true.
INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Strong Induction(Example)

Example 2.7

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution

BASIS STEP: P(2) is true, because 2 can be written as the product of one prime, itself.
INDUCTIVE STEP: The inductive hypothesis is the assumption that P(j) is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k.

Mathematical Induction (Example)

Solution

To complete the inductive step, it must be shown that P(k + 1) is true under this assumption, that is, that k + 1 is the product of primes. There are two cases to consider, namely, when k + 1 is prime and when k + 1 is composite. If k + 1 is prime, we immediately see that P(k + 1) is true. Otherwise, k + 1 is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if k + 1 is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

Review and Examples

Use Mathematical Induction to show that if n is a positive integer, then

1. $1.2^1 + 2.2^2 + 3.2^3 + \cdots + n2^n = 2 + (n-1)2^{n+1}$
2. $3 + 3.4 + 3.4^2 + \cdots + 3.4^{n-1} = 4^n - 1 \qquad n \geq 1$
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

Strong Induction(Example)

Example 2.8

Assume $\{a_n\}_{n=1}^{\infty}$ *is a sequence defined as:*
$a_1 = 3, a_2 = 6, \quad a_n = a_{n-1} + a_{n-2} : \forall n \geq 3$
prove that $3|a_n$ *for all positive integer* n.

Solution

Let $P(n)$ be the proposition,
$P(n): \ 3|a_n, \forall n \geq 1 \Rightarrow a_n = 3C : \quad C \in \mathbb{N}$

1. Basis step $P(1) : 3|a_1 = 3 \Rightarrow P(1)$ is true.
   $P(2) : 3|a_2 = 6 \Rightarrow P(2)$ is true.
2. Inductive step: We assume that:
   $P(1), P(2), \ldots, P(k-2), P(k-1), P(k)$ are all true for $k \geq 2$
   Under this assumption, it must be shown that $P(k+1)$ is also true .

Strong Induction(Example)

a_{k-1}, a_k both are true,
$P(k) : 3|a_k \Rightarrow a_k = 3C_1 : C_1 \in \mathbb{N}$
$P(k-1) : 3|a_{k-1} \Rightarrow a_{k-1} = 3C_2 : C_2 \in \mathbb{N}$
$a_{k+1} = a_k + a_{k-1} = 3C_1 + 3C_2 = 3(C_1 + C_2) = 3C;$
$C = (C_1 + C_2) \in \mathbb{N}$
$a_{k+1} = 3c \Rightarrow 3|a_{k+1} \Rightarrow P(k+1)$ is true.
Then $P(n)$ is true $\forall n \geq 1$