# Discrete Mathematics (MATH 151)

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1. The Foundations: Logic and Proofs
   - Propositional Logic
   - Propositional Equivalences
   - Predicates and Quantifiers

# Proposition (or statement)

## Definition 2.1

*Proposition A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.*

## Example 2.1

1. *Washington, D.C., is the capital of the United States of America.*
2. *Toronto is the capital of Canada.*
3. $1 + 1 = 3$
4. $2 + 3 = 5$

*Propositions 1 and 4 are true, whereas 2 and 3 are false.*

# Proposition (or statement)

## Remark

Some sentences that are not propositions

Example:

1. What time is it?
2. Read this carefully.
3. $x + 1 = 3$
4. $x + y = z$

   Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false.

# Proposition (or statement)

- We use letters to denote propositional variables (or statement variables).
- The conventional letters used for propositional variables are p, q, r, s, . . . .
- The truth value of a proposition is true, denoted by T, if it is a true proposition,
- The truth value of a proposition is false, denoted by F, if it is a false proposition.

# Negation of proposition

## Definition 2.2

***Negation of proposition***

*Let $p$ be a proposition. The negation of $p$, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement It is not the case that $p$. The proposition $\neg p$ is read not $p$. The truth value of the negation of $p$, $\neg p$, is the opposite of the truth value of $p$.*

## Example 2.2

*The negation of the proposition "Michael's PC runs Linux" is "It is not the case that Michael's PC runs Linux."*
*Or more simply "Michael's PC does not run Linux."*

# Negation of proposition

The truth table for the negation of a proposition.

| P | $\neg P$ |
|---|---|
| T | F |
| F | T |

This table displays the truth table for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

# Conjunction of tow propositions

## Definition 2.3

***conjunction***

*Let $p$ and $q$ be propositions. The conjunction of $p$ and $q$, denoted by $p \wedge q$, is the proposition $p$ and $q$. The conjunction $p \wedge q$ is true when both $p$ and $q$ are true and is false otherwise.*

# Conjunction of tow propositions

## Example 2.3

*The conjunction of the propositions $p \wedge q$ where p is "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz." "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz."*

The truth table for the Conjunction of tow propositions

$$p \wedge q$$

.

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Disjunction of tow propositions

**Definition 2.4**

***disjunction***

*Let $p$ and $q$ be propositions. The disjunction of $p$ and $q$, denoted by $p \vee q$, is the proposition "$p$ or $q$." The disjunction $p \vee q$ is false when both $p$ and $q$ are false and is true otherwise.*

# Disjunction of tow propositions

**Example 2.4**

*The disjunction of the propositions $p \vee q$ where $p$ is "Rebecca's PC has more than 16 GB free hard disk space" and $q$ is the proposition "The processor in Rebecca's PC runs faster than 1 GHz." "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."*

The truth table for the Disjunction of tow propositions

$$p \vee q$$

.

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Examples

## Examples

Let p "Today is Friday" and q "It is raining today",

- $p \wedge q$ is "Today is Friday and it is raining today".
  This proposition is true only on rainy Fridays and is false on any other rainy day or on Fridays when it does not rain.
- $p \vee q$ is "Today is Friday or it is raining today".
  This proposition is true on any day that is a Friday or a rainy day(including rainy Fridays) and is false on any day other than Friday when it also does not rain.

# Exclusive or of tow propositions

## Definition 2.5

***Exclusive or***

*Let $p$ and $q$ be propositions. The exclusive or of $p$ and $q$, denoted by $p \oplus q$, is the proposition that is true when exactly one of $p$ and $q$ is true and is false otherwise.*

# Exclusive or of tow propositions

**Example 2.5**

*The exclusive or of the propositions p "Today is Friday" and q "It is raining today",$p \oplus q$ is "Either today is Friday or it is raining today, but not both". This proposition is true on any day that is a Friday or a rainy day(not including rainy Fridays) and is false on any day other than Friday when it does not rain or rainy Fridays.*

The truth table for the Exclusive Or of Two Propositions.

$$p \oplus q$$

.

| p | q | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# Conditional Statements of tow propositions

## Definition 2.6

***Conditional statement***

*Let p and q be propositions. The conditional statement* $p \to q$ *is the proposition "if p, then q."*
*The conditional statement* $p \to q$ *is* ***false*** *when p is true and q is false, and* ***true*** *otherwise. In the conditional statement* $p \to q$*, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).*

# Conditional Statement of tow propositions

**Example 2.6**

*"If it is Friday then it is raining today" is a proposition which is of the form $p \to q$. The above proposition is true if it is not Friday(premise is false) or if it is Friday and it is raining, and it is false when it is Friday but it is not raining.*

TheTruth Table for the Conditional Statement of Two Propositions. $p \to q$.

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Biconditional statement

## Definition 2.7

***biconditional Statement of tow propositions***

*Let p and q be propositions. The biconditional statement* $p \leftrightarrow q$ *is the proposition "p if and only if q." The biconditional statement* $p \leftrightarrow q$ *is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.*

# Biconditional Statement of tow propositions

**Example 2.7**

*"It is raining today if and only if it is Friday today."*
*is a proposition which is of the form $p \leftrightarrow q$.*
*The above proposition is true if it is not Friday and it is not raining or if it is Friday and it is raining, and it is false when it is not Friday or it is not raining.*

The truth table for the for the biconditional statement of Two Propositions. $p \leftrightarrow q$.

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Precedence of Logical Operators

| Operator | Precedence |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\longleftrightarrow$ | 5 |

- Truth Tables of Compound Propositions

**Example 2.8**

*Construct the truth table of the compound proposition*

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

# Truth Tables of Compound Propositions

| The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$ | | | | | |
|---|---|---|---|---|---|
| p | q | $\neg q$ | $p \vee \neg q$ | $p \wedge q$ | $(p \vee \neg q) \rightarrow (p \wedge q)$ |
| T | T | F | T | T | T |
| T | F | T | T | F | F |
| F | T | F | F | F | T |
| F | F | T | T | F | F |

# Logic and Bit Operations

## Bit Operations

Computer bit operations correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators $\wedge$, $\vee$ and $\oplus$.

| x | y | $x \vee y$ | $x \wedge y$ | $x \oplus y$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 |

# Logic and Bit Operations

**Definition 2.8**

*A bit string is a sequence of zero or more bits. The* ***length*** *of this string is the number of bits in the string.*

**Example 2.9**

101010011 *is a bit string of length nine.*

**Exercise 1**

*Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101.*

# Bit Operations

**Exercise 1**

Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101.

# Bit Operations

### Exercise 1

Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101.

### Solution 1

*01 1011 0110*
*11 0001 1101*

---

*11 1011 1111* *bitwise OR*
*01 0001 0100* *bitwise AND*
*10 1010 1011* *bitwise XOR*

# Propositional Equivalences

## Definition 2.9

***tautology:***

*A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.*

***contradiction:***

*A compound proposition that is always false.*

***contingency:***

*A compound proposition that is neither a tautology nor a contradiction.*

## Example 2.10

$(p \wedge q) \rightarrow (p \vee q)$ *is a tautology.*

# Propositional Equivalences

## Example 2.11

*We can construct examples of tautologies and contradictions using just one propositional variable.*
*Consider the truth tables of* $p \vee \neg p$ *and* $p \wedge \neg p$.
$p \vee \neg p$ *is a tautology, because it is always true,*
*and* $p \wedge \neg p$ *is a contradiction, because it is always false*

## Truth table

| p | $\neg p$ | $p \vee \neg p$ | $p \wedge \neg p$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

Table: Tautology and Contradiction

# Logical Equivalences

**Definition 2.10**

*The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.*
*The notation $p \equiv q$ denotes that p and q are logically equivalent.*

**Example 2.12**

*Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.*

# Logical Equivalences

## Truth table

| p | q | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

Table: Truth table for $\neg(p \vee q)$ and $\neg p \wedge \neg q$

# Logical Equivalences

## Example 2.13

*Show that* $p \to q$ *and* $\neg p \vee q$ *are logically equivalent.*

## Solution

| p | q | $p \to q$ | $\neg p$ | $\neg p \vee q$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

Table: Truth table for $p \to q$ and $\neg p \vee q$

# Logical Equivalences

## Exercise 2

*Let $p$, $q$ and $r$ three propositions, show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.*

# Logical Equivalences

## Exercise 2

*Let $p$, $q$ and $r$ three propositions, show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.*

## Remark 2.1

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ *This is the distributive law of disjunction over conjunction.*

# Solution

| p | q | r | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

Table: A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

# Logical Equivalences

**Remark 2.2**

*This table contains some important equivalences. In these equivalences, $T$ denotes the compound proposition that is always true and $F$ denotes the compound proposition that is always false.*

# Logical Equivalences

| Equivalence | Name |
|---|---|
| $p \wedge T \equiv p$<br>$p \vee F \equiv P$ | Identity laws |
| $p \vee T \equiv T$<br>$p \wedge F \equiv F$ | Domination laws |
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | Idempotent laws |
| $\neg(\neg p) \equiv p$ | Double negation law |
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative laws |
| $(p \vee q) \vee r \equiv p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative laws |

# Logical Equivalences

| Equivalence | Name |
|---|---|
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |
| $p \vee (p \wedge q) \equiv p$<br>$p \wedge (p \vee q) \equiv p$ | Absorption laws |
| $p \vee \neg p \equiv T$<br>$p \wedge \neg p \equiv F$ | Negation laws |

Table: Logical Equivalences.

# Logical Equivalences

| Equivalence |
|---|
| $p \rightarrow q \equiv \neg p \vee q$ |
| $\neg(p \rightarrow q) \equiv p \wedge \neg q$ |
| $p \rightarrow q \equiv \neg q \rightarrow \neg p$ |
| $p \vee q \equiv \neg p \rightarrow q$ |
| $p \wedge q \equiv \neg(p \rightarrow \neg q)$ |
| $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ |
| $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$ |
| $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ |
| $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ |

Table: Logical Equivalences Involving Conditional Statements.

# Logical Equivalences

| Equivalence |
|---|
| $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ |
| $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ |
| $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ |
| $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$ |

Table: Logical Equivalences Involving Biconditional Statements.

# Morgan's Laws

## Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Furthermore, note that Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$$

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv \neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n$$

# Examples

## Examples

1. We can show that $\neg(p \to q)$ and $p \land \neg q$ are logically equivalent.
2. We can show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences.
3. We can show that $(p \land q) \to (p \lor q)$ is a tautology.

# Examples

## Solution

1. $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv \neg(\neg p) \wedge \neg q \equiv p \wedge \neg q$
2. $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q) \equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$
   $\equiv \neg p \wedge (p \vee \neg q) \equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \equiv F \vee (\neg p \wedge \neg q)$
   $\equiv (\neg p \wedge \neg q) \vee F \equiv \neg p \wedge \neg q$
3. $(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q) \equiv$
   $(\neg p \vee \neg q) \vee (p \vee q) \equiv (\neg p \vee p) \vee (\neg q \vee q) \equiv T \vee T \equiv T$

# Propositional function (Predicate)

Consider $P(x) = x < 5$

- $P(x)$ has no truth values ($x$ is not given a value)
- $P(1)$ is true
  - The proposition 1<5 is true
- P(10) is false
  - The proposition 10<5 is false
- Thus, P(x) will create a proposition when given a value

# Propositional function (Predicate)

Consider the following statements:

$$x > 3, x = y + 3, x + y = z$$

The truth value of these statements has no meaning without specifying the values of x, y, z.
Extend propositional logic by the following new features.

- Variables: x, y, z, . . .
- Predicates (i.e., propositional functions): $P(X), Q(X), R(X), M(X, Y), \ldots$
- P(x) denotes the value of propositional function P at x.
- The domain is often denoted by U (the universe).

# Predicates

## Predicate

**Predicate:** is a statements involving variables.

## Examples

1. $x > 3, \quad x = y + 3, \quad x + y = z$
2. "computer x is under attack by an intruder,"
3. "computer x is functioning properly,"
4. Let $P(X)$ denote "$x > 5$" and $U$ be the integers. Then
    - $P(8)$ is true
    - $P(5)$ is false

# propositional functions with multiple variables

## Function with multiple variables

1. $P(x, y)$ denote the statement $x + y = 0$
   $P(1, 2)$ is false, $P(1, -1)$ is true
2. $P(x, y, z)$ denote the statement $x + y = z$
   $P(3, 4, 5)$ is false, $P(1, 2, 3)$ is true.

## Remark 2.3

*In general, a statement involving the n variables $x_1, x_2, ..., x_n$ can be denoted by $P(x_1, x_2, ..., x_n)$.*
*A statement of the form $P(x_1, x_2, ..., x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, ..., x_n)$, and P is also called an n-place predicate or a n-ary predicate.*

# Quantifiers

- A quantifier is "an operator that limits the variables of a proposition"
- Two types
  1. Universal, for all $\forall$.
  2. Existential, there exists $\exists$.

| Statement | when True? | When False? |
|---|---|---|
| $\forall x P(x)$ | $p(x)$ is true for every $x$ | There is an $x$ for which $P(x)$ is false |
| $\exists x P(x)$ | there is an $x$ for which $P(x)$ is true | $P(x)$ is false for every x. |

Table: Quantifires.

# Quantifiers

## Definition 2.11

*The universal quantification of $P(x)$ is the statement $P(x)$ for all values of $x$ in the domain.*

*The notation $\forall x \; P(x)$ denotes the universal quantification of $P(x)$.*
*Here $\forall$ is called the **universal quantifier**. We read $\forall x \; P(x)$ as "for all $x$ $P(x)$" or "for every $x$ $P(x)$." An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.*

## Definition 2.12

*The existential quantification of $P(x)$ is the proposition "There exists an element $x$ in the domain such that $P(x)$."*
*We use the notation $\exists x \; P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.*

# Quantifiers

## Example 2.14

*Let $P(x)$ denote the statement "$x > 3$".*
*What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?*
***Solution:***
*Because "$x > 3$" is sometimes true, for instance, when $x = 4$ the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.*

# Precedence of Quantifiers

## Remark

**Precedence of Quantifiers:**

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.

In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x[P(x) \vee Q(x)]$.

# Logical Equivalences Involving Quantifiers

**Definition 2.13**

*Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements $S$ and $T$ involving predicates and quantifiers are logically equivalent.*

## Example 2.15

*Show that* $\forall x(P(x) \wedge Q(x))$ *and* $\forall xP(x) \wedge \forall xQ(x)$ *are logically equivalent (where the same domain is used throughout).*

## Solution 2

*First, we show that if* $\forall x(P(x) \wedge Q(x))$ *is true,*
*then* $\forall xP(x) \wedge \forall xQ(x)$ *is true.*
*Second, we show that if* $\forall xP(x) \wedge \forall xQ(x)$ *is true, then* $\forall x(P(x) \wedge Q(x))$ *is true.*
*So, suppose that* $\forall x(P(x) \wedge Q(x))$ *is true. This means that if a is in the domain, then* $P(a) \wedge Q(a)$ *is true. Hence, P(a) is true and Q(a) is true. Because P(a) is true and Q(a) is true for every element in the domain, we can conclude that* $\forall xP(x)$ *and* $\forall xQ(x)$ *are both true. This means that* $\forall xP(x) \wedge \forall xQ(x)$ *is true.*

# Precedence of Quantifiers

**Remak A**

When all the elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1)$, $P(x_2)$, $\ldots$, $P(x_n)$ are all true.

# Precedence of Quantifiers

**Remak A**

When all the elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1)$, $P(x_2)$, $\ldots$, $P(x_n)$ are all true.

**Example 2.16**

*What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?*

***Solution:***

*The domain is $\{1, 2, 3, 4\}$,*
*The statement $\forall x P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$.*
*Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.*

# Precedence of Quantifiers

**Remak B**

When all elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ the existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$, because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

# Precedence of Quantifiers

## Remak B

When all elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ the existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$, because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

## Example 2.17

*What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the domain consists of the positive integers not exceeding 4?*

***Solution:***

*The domain is $\{1, 2, 3, 4\}$,*
*The statement $\exists x P(x)$ is the same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4)$.*
*Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true.*

# Negating Quantified Expressions

| Negation | Equivalent Statement | When True? | When False? |
|---|---|---|---|
| $\neg\exists x P(x)$ | $\forall x \neg P(x)$ | For every $x$, $P(x)$ is false | There is an $x$ for which $P(x)$ is true. |
| $\neg\forall x P(x)$ | $\exists x \neg P(x)$ | there is an $x$ for which $P(x)$ is false | $P(x)$ is true for every x. |

Table: De Morgan's Laws for Quantifiers..

# Negating Quantified Expressions

## Example 2.18

*What are the negations of the statements*

1. $\forall x(x^2 > x)$
2. $\exists x(x^2 = 2)$

# Negating Quantified Expressions

## Example 2.18

*What are the negations of the statements*

1. $\forall x(x^2 > x)$
2. $\exists x(x^2 = 2)$

## Solution

1. The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$.
   This can be rewritten as $\exists x(x^2 \leq x)$.
2. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$.
   This can be rewritten as $\forall x(x^2 \neq 2)$.

The truth values of these statements depend on the domain.

# Review

- Recall that $P(x)$ is a propositional function.
- Recall that a proposition is a statement that is either TRUE or FALSE
- $P(x)$ is NOT a proposition
- There are TWO ways to make a propositional function into a proposition:
    1. Supply it with a value
       For example, $P(5)$ is false, $P(0)$ is true
    2. Provide a quantifiaction
       For example, $\forall x P(x)$ is false, and $\exists x P(x)$ is true.

# Introduction to Proofs

**Definition 2.14**

***A proof***

*is a sequence of statements. These statements come in two forms: givens and deductions.*

**Methods of Proving Theorems**

1. Direct Proofs
2. Proof by Contraposition
3. Proofs by Contradiction

# Direct Proof

## Definition

**A Direct Proof:**

is a sequence of statements which are either givens or deductions from previous statements, and whose last statement is the conclusion to be proved.

## Definition 2.15

*The integer $n$ is* ***even*** *if there exists an integer $k$ such that $n = 2k$, and $n$ is* ***odd*** *if there exists an integer $k$ such that $n = 2k + 1$.*

# Example of direct proof

## Example 2.19

*Give a direct proof of the theorem "If n is an odd integer, then $n^2$ is odd."*

## Solution

1. we assume that $n$ is odd
2. By definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer.
3. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses $n^2$.
4. we find that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
   **Conclusion:** By the definition of an odd integer, we can conclude that $n^2$ is an odd integer

# Direct Proof

**Remark 2.4**

*If we writ P(n) is "n is an odd integer" and Q(n) is "$n^2$ is odd." Note that this theorem states:*

$$\forall n P(n) \rightarrow Q(n)$$

.

# Proof by Contraposition

## Definition 2.16

***Proof by Contraposition*** *(indirect proofs):*
*An extremely useful type of indirect proof is known as proof by contraposition.*
*Proofs by contraposition make use of the fact that the conditional statement* $p \to q$ *is equivalent to its contrapositive,* $\neg q \to \neg p$.

## Example 2.20

*Prove that if* $n$ *is an integer and* $3n + 2$ *is odd, then* $n$ *is odd.*

# Proof by Contraposition

## Example 2.21

*Prove that if n is an integer and $3n + 2$ is odd, then n is odd.*

## Solution

In a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then $n$ is odd" is false; namely, assume that n is even.

**Then**, by the definition of an even integer, n $=$ 2k for some integer k. Substituting 2k for n, we find that

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

This tells us that $3n + 2$ is even and therefore not odd.

## Definition 2.17

*The real number $r$ is rational if there exist integers $p$ and $q$ with $q \neq 0$ such that $r = \frac{p}{q}$. A real number that is not rational is called irrational.*

## Example 2.22

*Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number $r$ and every real number $s$, if $r$ and $s$ are rational numbers, then $r + s$ is rational.)*

## Solution

We first attempt a direct proof.To begin, suppose that $r$ and $s$ are rational numbers. From the definition of a rational number, it follows that there are integers $p$ and $q$, with $q \neq 0$, such that $r = \frac{p}{q}$, and integers $t$ and $u$, with $u \neq 0$, such that $s = \frac{t}{u}$. Can we use this information to show that $r + s$ is rational? The obvious next step is to add $r = \frac{p}{q}$ and $s = \frac{t}{u}$, to obtain $r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu+qt}{qu}$. Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$ . Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and $qu$, where $qu \neq 0$ . This means that $r + s$ is rational.We have proved that the sum of two rational numbers is rational; our attempt to find a direct proof succeeded.

# Proofs by Contradiction

**Definition 2.18**

*we can prove that $p$ is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition $r$.*
*Proofs of this type are called* ***proofs by contradiction****.*

**Example 2.23**

*Give a proof by contradiction of the theorem "If $3n+2$ is odd, then $n$ is odd."*

# Proofs by Contradiction

1. Let p be "3n + 2 is odd" and q be "n is odd." To construct a proof by contradiction, assume that both p and $\neg q$ are true.
2. assume that 3n + 2 is odd and that n is not odd, so $n = 2k$.
3. This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.
4. Because 3n + 2 is 2t , where t = 3k + 1, 3n + 2 is even.
5. Because both p and $\neg p$ are true, we have a contradiction.