Math 316

Second Midterm Exam 1440, 1st semester

Name: ID:

Q1 Prove or disprove each of the following statements:

(a) $\|L_{100}\|^2 = \frac{2}{201}$, where L_n refers to Laguerre polynomial.

(b) $\langle H_n, H_m \rangle = 0$ for all $n, m \in \mathbb{N}_0, n \neq m$, where H_n refers to Hermite polynomial.

Q2 Consider the function f defined by

$$f(x) = \begin{cases} 1, & -1 \leq x < 0 \\ \frac{1}{2}, & x = 0 \\ x, & 0 < x \leq 1 \end{cases},$$

and

$$f(x+2) = f(x), \ x \notin [-1, 1].$$

(a) Sketch the function f on the interval $[-3, 3]$. What is the period of f?

(b) Find the Fourier series representation for f.

(c) Find the sum of the Fourier series at $x = -\frac{1}{4}$.

(d) Show that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Q3. Consider the identity

$$(n+1) P_{n+1}(x) + nP_{n-1}(x) = (2n+1) xP_n(x), \quad n \in \mathbb{N}.$$

where P_n is Legendre polynomial.

(a) Show that

$$\begin{aligned} n \|P_n\|^2 &= (2n-1) \langle xP_{n-1}, P_n \rangle, \\ n \|P_{n-1}\|^2 &= (2n+1) \langle xP_n, P_{n-1} \rangle. \end{aligned}$$

(b) Use part (a) to prove

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

Q4 Solve the heat equation

$$u_t = u_{xx}, \qquad 0 < x < \pi,\ t > 0,$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= u(\pi,t) = 0, \qquad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < \pi. \end{aligned}$$

Good Luck
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