Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Chapter 3: Vector Calculus

3.1 Vector fields

Definition Vector Fields in Two Dimensions
Let f and g be defined on a region R of $\mathbf{R}^2$. A vector field in $\mathbf{R}^2$ is a function F that assigns to each point in R a vector $(f(x,y), g(x,y))$. The vector field is written as

$$F(x,y) = (f(x,y), g(x,y)) \text{ or } F(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}.$$

A vector field $F = (f, g)$ is continuous or differentiable on a region R of $\mathbf{R}^2$ if f and g are continuous or differentiable on R, respectively.

Example 1 Vector Fields Sketch representative vector of the following fields.

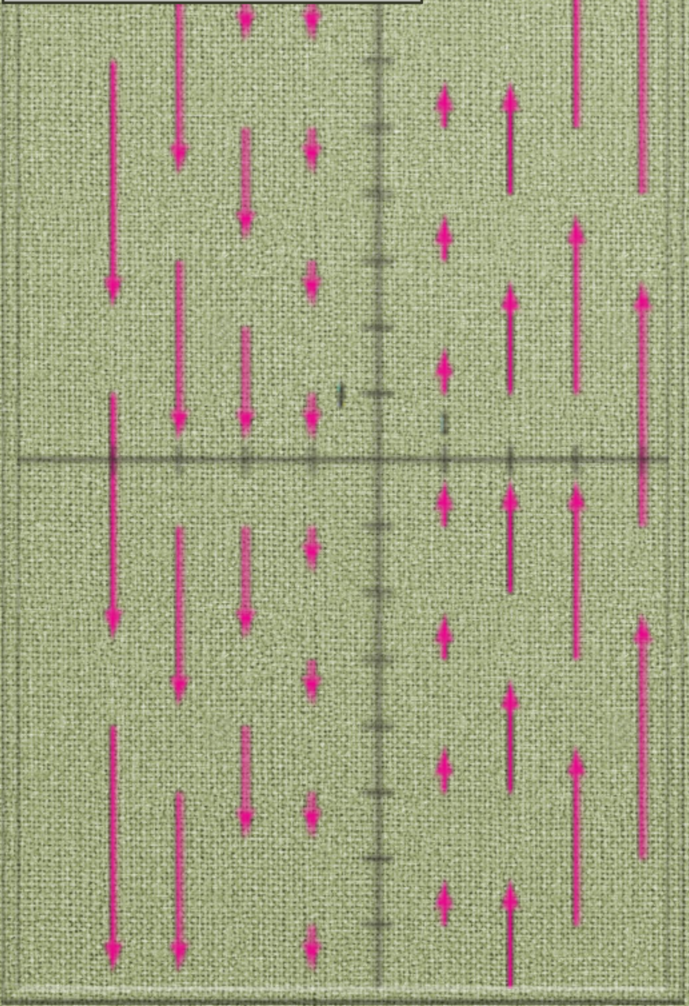
a) $F(x,y) = (0, x) = x\mathbf{j}$; (a shear field).

b) $F(x,y) = (1-y^2, 0) = (1-y^2)\mathbf{i}$, for $|y| \le 1$; (channel flow).

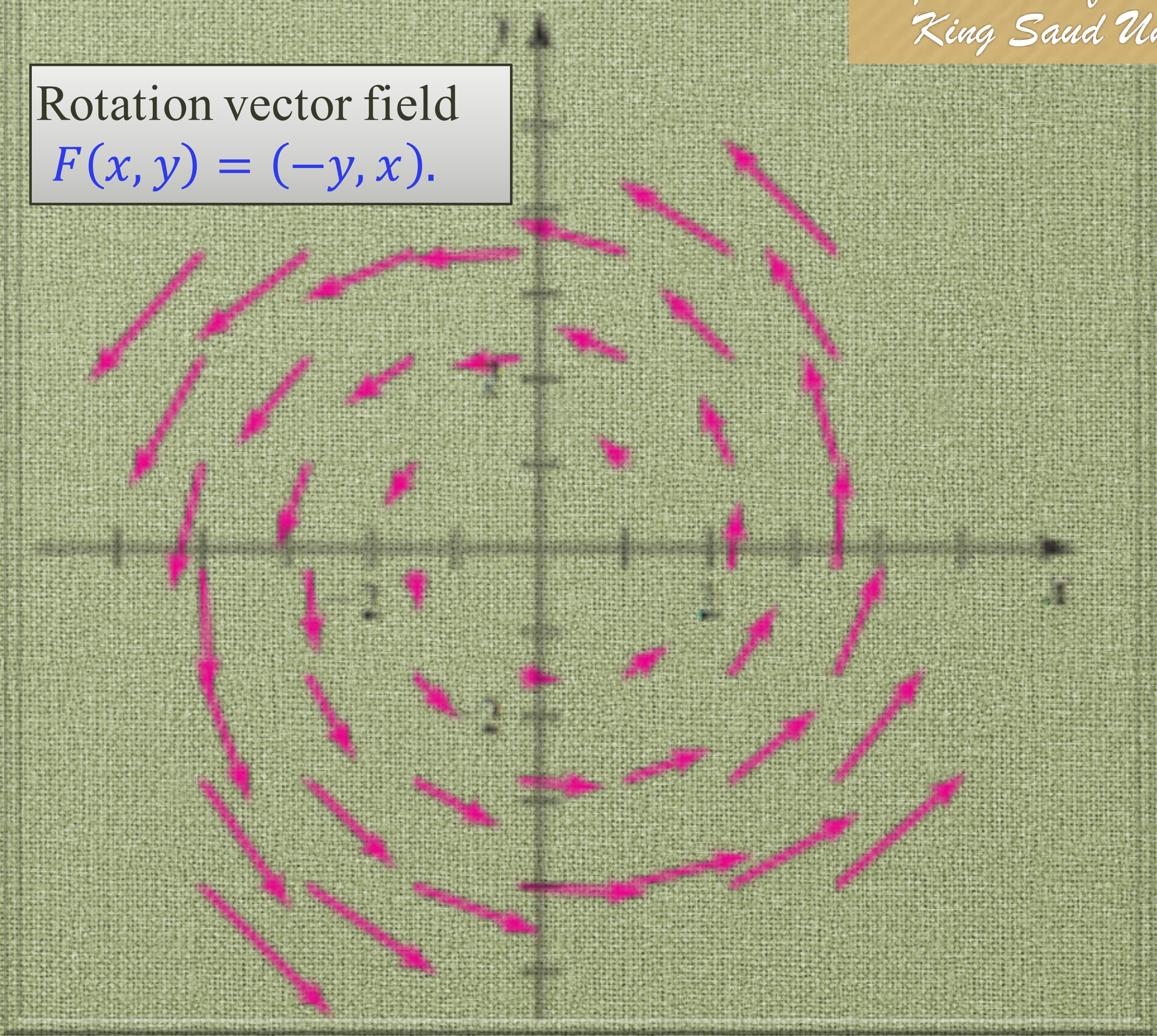
c) $F(x,y) = (-y, x) = -y\mathbf{i} + x\mathbf{j}$; (a rotation field).

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

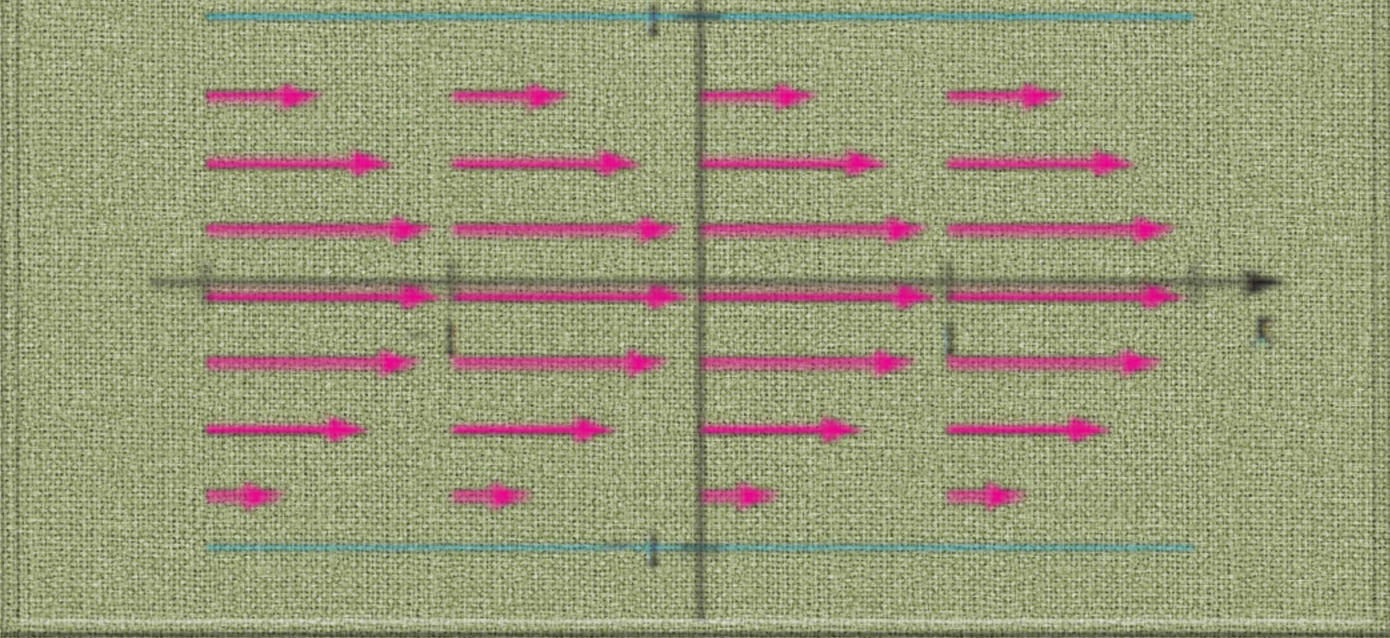
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University



Shear vector field
$F(x,y) = (0,x).$



Rotation vector field
$F(x,y) = (-y,x).$



Channel flow
$F(x,y) = (1-y^2,0).$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Definition Vector Fields in Three Dimensions

Let f, g, and h be defined on a region D of $\mathbf{R}^3$. A vector field in $\mathbf{R}^3$ is a functi on $\mathbf{F}$ that assigns to each point in D a vector $(f(x,y,z), g(x,y,z), h(x,y,z),)$. The vector field is written as

$$\mathbf{F}(x,y,z) = \big(f(x,y,z), g(x,y,z), h(x,y,z)\big) \text{ or}$$

$$\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}.$$

Example 2 Vector Fields in $\mathbf{R}^3$.

a) $\mathbf{F}(x,y,z) = (x, y, e^{-z})$, for $z \geq 0$;

b) $\mathbf{F}(x,y,z) = (0,0,1 - x^2 - y^2)$, for $x^2 + y^2 \leq 1$.

Definition of Inverse Square Field

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an inverse vector field if

$$\mathbf{F}(x,y,z) = \frac{k}{\|\mathbf{r}\|^2}\,\mathbf{u},$$

where k is a real number and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$ is a unit vector in the direction of $\mathbf{r}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Definition Gradient Fields and Potential Functions

Let $z = f(x,y)$ and $w = f(x,y,z)$ be differentiable functions on regions of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is a ***gradient field***, and the function φ a is a ***potential function*** of $\mathbf{F}$.

Definition Conservative vector Fields

A vector field $\mathbf{F}$ is called ***conservative*** if there exists a differentiable function φ such that $\mathbf{F} = \nabla\varphi$. The function φ is called the a ***potential function*** for $\mathbf{F}$.

Test for Conservative Vector Field in the Plane

Let f and g have continuous first partial derivatives on an open disk R.
The vector field given by $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ is conservative if and only if

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Definition of Curl of a vector field

The curl of $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ is

$$\mathbf{curl}\ \mathbf{F}(x,y,z) = \nabla\times\mathbf{F}(x,y,z) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

Test for Conservative Vector Field in Space

Suppose that f, g and h have continuous first partial derivatives on an open sphere Q in space. The vector field given by $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ is conservative if and only if

$$\mathbf{curl}\ \mathbf{F}(x,y,z) = 0,$$

that is, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial h}{\partial y} = \frac{\partial g}{\partial z}, \qquad \frac{\partial h}{\partial x} = \frac{\partial f}{\partial z}, \text{ and } \qquad \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1 **Testing for conservative fields.** Determine whether the following vector fields are conservative on $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively.

a) $\mathbf{F}(x,y) = (e^x \cos y, -e^x \sin y)$;

b) $\mathbf{F}(x,y,z) = (2xy - z^2, x^2 + 2z, 2y - 2xz)$.

Solution **a.** Letting $f(x,y) = e^x \cos y$ and $f(x,y) = -e^x \sin y$, we see that

$$\frac{\partial g}{\partial x} = -e^x \sin y = \frac{\partial f}{\partial y}.$$

Then **F** is conservative.

b. Letting $f(x,y,z) = 2xy - z^2, g(x,y,z) = x^2 + 2z,$ and $h(x,y,z) = 2y - 2xz,$ we see that

$$\frac{\partial g}{\partial x} = 2x = \frac{\partial f}{\partial y};\ \frac{\partial h}{\partial x} = -2z = \frac{\partial f}{\partial z};\ \frac{\partial g}{\partial z} = 2 = \frac{\partial h}{\partial y}.$$

Then **F** is conservative.

Example 2 **Finding potential functions.** Find a potential function for the concervative vector fields in Example 1.

a) $\mathbf{F}(x,y) = (e^x \cos y, -e^x \sin y)$;

b) $\mathbf{F}(x,y,z) = (2xy - z^2, x^2 + 2z, 2y - 2xz)$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boumkhel
Department of Mathematics
King Saud University

Solution a. A potential function φ for $\mathbf{F} = (f, g)$ has the property that $\mathbf{F} = \nabla\varphi$ and satisfies the conditions see that

$$\varphi_x = f(x,y) = e^x \cos y \text{ and } \varphi_y = g(x,y) = -e^x \sin y.$$

The first equation is integrated with respect to x (holding y fixed) to obtain

$$\int \varphi_x dx = \int e^x \cos y dx,$$ which implies that

$$\varphi(x,y) = e^x \cos y + c(y).$$

In this case the "*constant of integration*" $c(y)$ is an arbitrary function of y.
To find the arbitrary function $c(y)$, we differentiate $\varphi(x,y) = e^x \cos y + c(y)$ with respect to y and equate the result to $g(x,y)$:

$$\varphi_y = -e^x \sin y + c'(y) \quad \text{and} \quad g(x,y) = -e^x \sin y.$$

We conclude that $c'(y)=0$, which implies that $c(y)$ is any real number, which ensures that

$$\varphi(x,y) = e^x \cos y + c.$$

Prof. Messaoud Boumkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

b. Let φ be a potential function of $\mathbf{F} = (f, g, h)$, that is, $\mathbf{F} = \nabla\varphi$. Then
$\varphi_x = f(x,y,z) = 2xy - z^2$, $\varphi_y = g(x,y,z) = x^2 + 2z$, and $\varphi_z = h(x,y,z) = 2y - 2xz$.
Integrating the first equation with respect to x (holding y and z fixed) we obtain

$$\varphi(x,y,z) = \int 2xy - z^2 dx = x^2y - xz^2 + c(y,z).$$

Because the integration is with respect to x, the arbitrary "constant" is a function of y and z.
To find $c(y,z)$, we differentiate φ with respect to y, which results in

$$\varphi_y = x^2 + c_y(y,z).$$

Equating φ_y and $g(x,y,z) = x^2 + 2z$, we see that $c_y(y,z) = 2z$.
To obtain $c(y,z)$, we integrate $c_y(y,z) = 2z$ with respect to y (holding z fixed), which results in $c(y,z) = 2yz + d(z)$. The constant of integration is now a function of z, which we call $d(z)$. At this point, a potential functions looks like

$$\varphi(x,y,z) = x^2y - xz^2 + 2yz + d(z).$$

To determine $d(z)$, we differentiate φ with respect to z :

$$\varphi_z(x,y,z) = -2xz + 2y + d'(z).$$

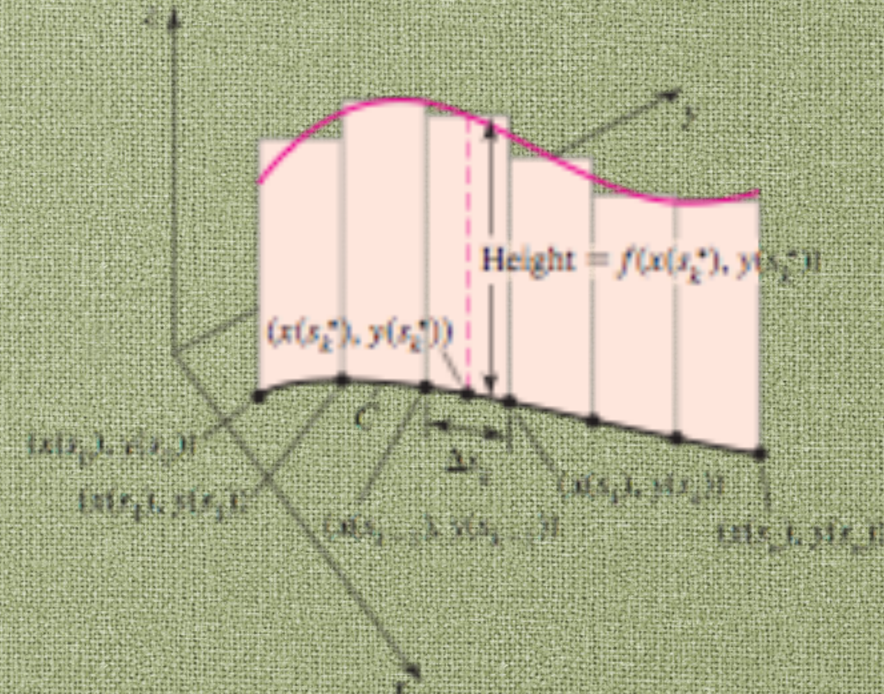
Equating φ_z and $h(x,y,z) = 2y - 2xz$, we see that $d'(z) = 0$, that is, $d(z) = d$ (constant).
So

$$\varphi(x,y,z) = x^2y - xz^2 + 2yz + d.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

3.2 Line integrals

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University



Line Integral in the plane.

C: $\mathbf{r}(\mathbf{s}) = \big(x(s), y(s)\big)$, for $a \le s \le b$;

$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$.

$$\text{area} \approx \sum_{k=1}^{n} f\big(x(s_k^*), y(s_k^*)\big)\Delta s_k \,.$$

Suppose that the scalar-valued function f is defined on the smooth curve

C: $\mathbf{r}(\mathbf{s}) = \big(x(s), y(s)\big)$, for $a \le s \le b$.

The ***line integral*** of f over C is defined to be

$$\int_C f\big(x(s), y(s)\big) ds = \lim_{n\to\infty} \sum_{k=1}^{n} f\big(x(s_k^*), y(s_k^*)\big)\Delta s_k \,,$$

provided this limit exists over all partitions of C. When the limit exists f is said to be integrable on C.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Evaluating Line Integrals in the plane.

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \big(x(t), y(t)\big)$, for $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(x(t), y(t))|r'(t)|dt = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}dt \ ,$$

Procedure Evaluating Line Integrals $\int_C f\,ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \big(x(t), y(t)\big)$, for $a \le t \le b$.
2. T Compute $|r'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y and evaluate an ordinary integral

$$\int_C f\,ds = \int_a^b f(x(t), y(t))|r'(t)|dt \ .$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Line Integral in $\mathbf{R}^3$

Suppose that the scalar-valued function $f(x,y,z)$ is defined on the smooth curve

$$C: \quad \mathbf{r}(\mathbf{s}) = (x(t), y(t), z(t)), \qquad \text{for} \quad a \le t \le b.$$

Evaluating Line Integral in $\mathbf{R}^3$

The line integral f of over C is

$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))|r'(t)|dt$$

$$= \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt\ .$$

If $f(x,y,z) = 1$, then the line integral gives the length of C. Then

$$\text{Length of } C = \int_C ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt\ .$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3 **Line integrals in $\mathbf{R}^3$.** Evaluate $\int_C (xy + 2z)\,ds$ on the following line segments.

a) The line segment from $P(1,0,0)$ to $Q(0,1,1)$;

b) The line segment from $Q(0,1,1)$ to $P(1,0,0)$.

Solution **a.** A parametric description of the line segment from $P(1,0,0)$ to $Q(0,1,1)$ is

$$\mathbf{r}(t) = (1,0,0) + t(-1,1,1) = (1-t, t, t), \qquad \text{for } 0 \le t \le 1.$$

The speed factor is

$$|r'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}.$$

Substituting $x(t) = 1 - t, y(t) = t,$ and $z(t) = t$, the value of the line integral is

$$\int_C xy + 2z\, ds = \int_0^1 \big((1-t)(t) + 2(t)\big)\sqrt{3}\,dt = \sqrt{3}\int_0^1 (t - t^2 + 2t)\,dt$$

$$= \sqrt{3}\int_0^1 (3t - t^2)\,dt = \sqrt{3}\left[\frac{3t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{7\sqrt{3}}{6}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

b. The line segment from $Q(0,1,1)$ to $P(1,0,0)$ may be described parametrically by

$$\mathbf{r}(t)=(0,1,1)+t(1,-1,-1)=(t,1-t,1-t),\qquad \text{for } 0\le t\le 1.$$

The speed factor is

$$|r'(t)|=\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}=\sqrt{(-1)^2+1^2+1^2}=\sqrt{3}\,.$$

We substitute $x(t)=t,\ y(t)=1-t,$ and $z(t)=1-t$, the value of the line integral is

$$\int_C xy+2z\,ds=\int_0^1\big((t)(1-t)+2(1-t)\big)\sqrt{3}\,dt=\sqrt{3}\int_0^1(t-t^2+2-2t)dt$$

$$=\sqrt{3}\int_0^1(2-t-t^2)dt=\sqrt{3}\left[2t-\frac{t^2}{2}-\frac{t^3}{3}\right]_0^1=\frac{7\sqrt{3}}{6}.$$

Conclusion. The line integral is independent of the orientation and parametrization of the curve.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Line Integrals of Vector fields

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C. Let $\mathbf{T}$ be the unit tangent vector at each point of C aconsistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$$ds = |\mathbf{r}'(t)|dt$$

This integral may be written in several different forms. If $\mathbf{F} = (f, g, h)$, then the line integral may be evaluated in component form as

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b \left(fx'(t) + gy'(t) + hz'(t)\right)dt,$$

where f stands for $f(x(t), y'(t), z(t))$ with analogous expression for g and h.

Another useful form is obtained by using that

$$dx = x'(t)dt, \qquad dy = y'(t)dt, \qquad dz = z'(t)dt.$$

Making these replacements in the previous integral results in the form

This integral may be written in several different forms. If $\mathbf{F} = (f, g, h)$, then the line integral may be evaluated in component form as

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C f\,dx + g\,dy + h\,dz.$$

Finally, if we let $d\mathbf{r} = (dx, dy, dz)$, then $f\,dx + g\,dy + h\,dz = \mathbf{F}\cdot d\mathbf{r}$, and we have

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ may be expressed in the following forms, where vector $\mathbf{F} = (f, g, h)$ and C has a parameterization $\mathbf{r}(t) = (x(t), y(t), z(t))$, for $a \le t \le b$.

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f x'(t) + g y'(t) + h z'(t))\, dt = \int_C f dx + g dy + h dz = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

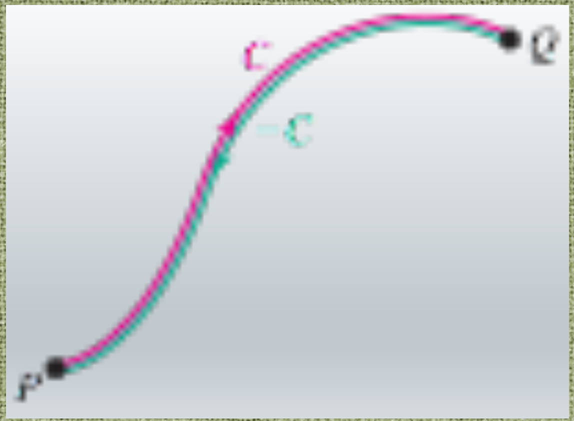
Example 5 Different paths.

Evaluate $\int_C \mathbf{F} \cdot \mathbf{T} ds$ with $\mathbf{F} = (y - x, x)$ on the following oriented paths in $\mathbf{R}^2$:

a) The quarter circle C_1 from $P(0,1)$ to $Q(1,0)$;

b) Type equation here.The quarter circle $-C_1$ from $Q(1,0)$ to $P(0,1)$;

c) The path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$.



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Solution

a) Working in $\mathbf{R}^2$, a parametric description of the curve with the required (clockwise) orientation is $\mathbf{r}(t) = (\sin t, \cos t)$, for $0 \le t \le 2\pi$. Along C_1 the vector field is

$$\mathbf{F} = (y - x, x) = (\cos t - \sin t, \sin t).$$

The velocity vector is $\mathbf{r}'(t) = (\cos t, -\sin t)$,

so the integrand of the line line integral is

$$\mathbf{F} \cdot \mathbf{r}'(t) = (\cos t - \sin t, \sin t) \cdot (\cos t, -\sin t) = \cos^2 t - \sin^2 t - \sin t \cos t.$$

The value of the line integral of $\mathbf{F}$ over C_1 is expressed in the following forms, where vector

$\mathbf{F} = (f, g, h)$ and C has a parameterization $\mathbf{r}(t) = (x(t), y(t), z(t))$, for $a \le t \le b$.

$$\int_0^{\frac{\pi}{2}} \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_0^{\frac{\pi}{2}} (\cos^2 t - \sin^2 t - \sin t \cos t)\, dt = \int_0^{\frac{\pi}{2}} \left(\cos 2t - \frac{1}{2}\sin 2t\right) dt$$

$$= \left[\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right]_0^{\frac{\pi}{2}} = -\frac{1}{2}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

b) A parametrization of the curve $-C_1$ from Q to P is $\mathbf{r}(t) = (\cos t\,, \sin t)$, for $0 \leq t \leq \frac{\pi}{2}$.

The vector field along the curve is

$$\mathbf{F} = (y - x, x) = (\sin t - \cos t\,, \cos t).$$

The velocity vector is $\mathbf{r}'(t)$ $= (-\sin t\,, \cos t)$. so the integrand of the line line integral is

$$\mathbf{F}\cdot\mathbf{r}'(t) = (\sin t - \cos t\,, \cos t)\cdot(-\sin t\,, \cos t)$$
$$= -\sin^2 t + \sin t \cos t + \cos^2 t\,.$$

Then

$$\int_0^{\frac{\pi}{2}} \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\frac{\pi}{2}} (\sin t\cos t + \cos^2 t - \sin^2 t)\,dt$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2t + \cos 2t\right) dt$$
$$= \left[\frac{-1}{4}\cos 2t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}}$$
$$= \left(\left(\frac{-1}{4}\cos\pi + \frac{1}{2}\sin\pi\right) - \left(\frac{-1}{4}\cos 0 + \frac{1}{2}\sin 0\right)\right)$$
$$= \frac{1}{2}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

The results of parts (a)and (b)illustrate the important fact that reversing the orientation of a curve reverses the sign of the line integral of a vector field.

The path C_2 consists of two line segments:

1. The segment from P to O is parameterized by $\mathbf{r}(t) = (0,1-t)$, for $0 \le t \le 1$.

Therefore, $\mathbf{r}'(t) = (0,-1)$ and $\mathbf{F} = (y-x, x) = (1-t, 0)$.

2. The segment from O to Q is parameterized by $\mathbf{r}(t) = (t, 0)$, for $0 \le t \le 1$.

Therefore, $\mathbf{r}'(t) = (1,0)$ and $\mathbf{F} = (y-x, x) = (-t, t)$.

The line integral is split into two parts and evaluated as follows:

$$\int_{C_2} \mathbf{F} \cdot \mathbf{T} ds = \int_{PO} \mathbf{F} \cdot \mathbf{T} ds + \int_{OQ} \mathbf{F} \cdot \mathbf{T} ds$$

$$= \int_0^1 (1-t, 0) \cdot (0,-1)\, dt + \int_0^1 (-t, t) \cdot (1,0)\, dt$$

$$= \int_0^1 0\, dt + \int_0^1 -t\, dt$$

$$= -\frac{1}{2}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 6

Evaluate $\int_C \mathbf{F} \cdot \mathbf{T} ds$ with $\mathbf{F} = (-y + x, x)$ on the following oriented paths in $\mathbf{R}^2$:

a) The quarter circle C_1 from $P(0,1)$ to $Q(1,0)$;

b) The path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments through $O(0,0)$.

Solution

a) Working in $\mathbf{R}^2$, a parametric description of the curve with the required (clockwise) orientation is $\mathbf{r}(t) = (\sin t, \cos t)$, for $0 \le t \le 2\pi$. Along C_1 the vector field is

$$\mathbf{F} = (-y + x, x) = (-\cos t + \sin t, \sin t).$$

The velocity vector is $\mathbf{r}'(t) = (\cos t, -\sin t)$,

so the integrand of the line line integral is

$$\mathbf{F} \cdot \mathbf{r}'(t) = (-\cos t + \sin t, \sin t) \cdot (\cos t, -\sin t) = -\cos^2 t - \sin^2 t + \sin t \cos t .$$

The value of the line integral of $\mathbf{F}$ over C_1 is expressed in the following forms, where vector

$\mathbf{F} = (f, g, h)$ and C has a parameterization $\mathbf{r}(t) = (x(t), y(t), z(t))$, for $a \le t \le b$.

$$\int_0^{\frac{\pi}{2}} \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_0^{\frac{\pi}{2}} (-\cos^2 t - \sin^2 t + \sin t \cos t)\, dt = \int_0^{\frac{\pi}{2}} \left(-1 + \frac{1}{2}\sin 2t\right) dt$$

$$= \left[-t - \frac{1}{4}\cos 2t\right]_0^{\frac{\pi}{2}} = \left(-\frac{\pi}{2} + \frac{1}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{1-\pi}{2}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

The path C_2 consists of two line segments:

1. The segment from P to O is parameterized by $\mathbf{r}(t) = (0,1-t)$, for $0 \le t \le 1$.

Therefore, $\mathbf{r}'(t) = (0,-1)$ and $\mathbf{F} = (-y+x, x) = (t-1, 0)$.

2. The segment from O to Q is parameterized by $\mathbf{r}(t) = (t,0)$, for $0 \le t \le 1$.

Therefore, $\mathbf{r}'(t) = (1,0)$ and $\mathbf{F} = (-y+x, x) = (t,t)$.

The line integral is split into two parts and evaluated as follows:

$$\int_{C_2} \mathbf{F} \cdot \mathbf{T} ds = \int_{PO} \mathbf{F} \cdot \mathbf{T} ds + \int_{OQ} \mathbf{F} \cdot \mathbf{T} ds$$

$$= \int_0^1 (t-1,0)\cdot(0,-1)\,dt + \int_0^1 (t,t)\cdot(1,0)\,dt = \int_0^1 0\,dt + \int_0^1 t\,dt = \frac{1}{2}.$$

The line integral in parts (a) and (c) in Example 5 have the same value and run on different paths from P to Q and the line integral in parts (a) and (b) in Example 6 have different value s and run on different paths from P to Q. *Therefore we might ask: For what vector fields are the values of a line integral independent of path? The answer is : the class of all conservative vector fields, that is, the line integral of conservative vector fields does not dependent on the path.*

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Definition Work Done in a Force Field

Let be a continuous force field in a region of and let , be a smooth curve in with a unit tangent vector consistent with the orientation. The work done in moving an object along in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Example Work Done in a Force Field

Find the work done by the force field $\mathbf{F} = (x, y+2)$ in moving an object along the smooth curve $\mathbf{r}(t) = (t - \sin t, 1 - \cos t),$ for $0 \le t \le 2\pi.$

Solution Clearly we have

$$\mathbf{F} = (x, y+2) = (t - \sin t, 3 - \cos t) \text{ and } \mathbf{r}'(t) = (1 - \cos t, \sin t)$$

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} (t - \sin t, 3 - \cos t) \cdot (1 - \cos t, \sin t)\, dt$$

$$= \int_0^{2\pi} (t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\, dt$$

$$= \int_0^{2\pi} (t + 2\sin t - t\cos t)\, dt = \left[\frac{t^2}{2} - 2\cos t - t\sin t - \cos t\right]_0^{2\pi} = 2\pi^2.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector force field on an open connected region R in $\mathbf{R}^2$ (or D in $\mathbf{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$W = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

For all points A and B in R and all smooth oriented curves C from A to B .

Example 3 **Verifying path independence**

Consider the vector field $\mathbf{F} = (x, -y)$ with potential function $\varphi(x,y) = \frac{1}{2}(x^2 - y^2)$. Let C_1 be th e quarter circle $\mathbf{r}(t) = (\cos t, \sin t)$, for $0 \le t \le \frac{\pi}{2}$, from $A(1,0)$ to $B(0,1)$. Let C_2 be the line $\mathbf{r}(t) = (1-t, t)$, for $0 \le t \le 1$, also from $A(1,0)$ to $B(0,1)$. Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

Solution On C_1 we have $\mathbf{r}'(t) = (-\sin t, \cos t)$ and $\mathbf{F} = (x, -y) = (\cos t, -\sin\ t)$.
The line integral on C_1 is

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{r}'(t)dt = \int_0^{\frac{\pi}{2}} (\cos t, -\sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} -\sin 2t\, dt = \left[\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = -1.$$

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

On C_2 we have $\mathbf{r}'(t) = (-1,1)$ and $\mathbf{F} = (x,-y) = (1-t,-t)$. Therefore

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{r}'(t)dt = \int_0^1 (1-t,-t)\cdot(-1,1)\,dt$$

$$= \int_0^1 -1\,dt = [-t]_0^1 = -1.$$

The two line integrals have the same value, which is equal to

$$\varphi(0,1) - \varphi(1,0) = -\frac{1}{2} - \frac{1}{2} = -1.$$

Example 4 **Line integral of a conservative vector field.**

Evaluate

$$\int_C \big((2xy - z^2)\mathbf{i} + (x^2 + 2z)\mathbf{j} + (2y - 2xz)\mathbf{k}\big) \cdot d\mathbf{r}$$

where C is a simple curve from $A(-3,-2,-1)$ to $B(1,2,3)$.

Solution This vector field is conservative and has a potential function $\varphi(x,y,z) = x^2y - xz^2 + 2yz$. Bv the Fundamental Theorem for line integrals,

$$\int_C \big((2xy - z^2)\mathbf{i} + (x^2 + 2z)\mathbf{j} + (2y - 2xz)\mathbf{k}\big) \cdot d\mathbf{r} = \int_C \nabla(x^2y - xz^2 + 2yz) \cdot d\mathbf{r} =$$

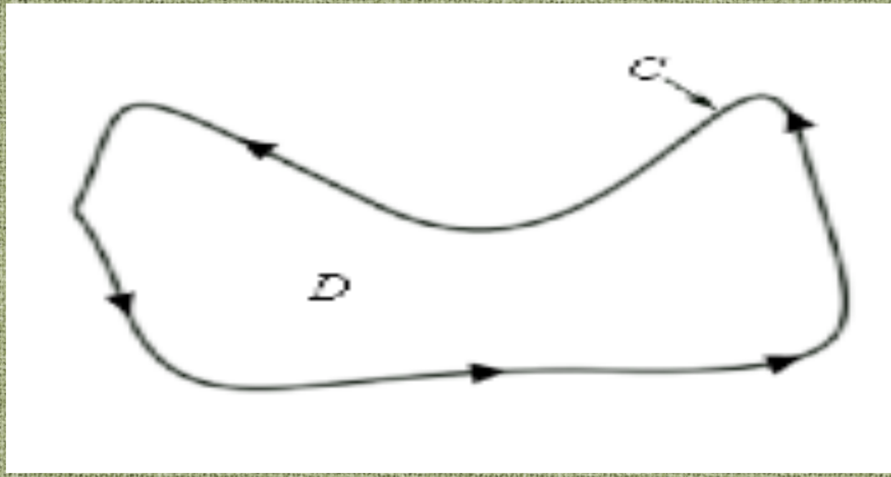
$$\varphi(1,2,3) - \varphi(-3,-2,-1) = 16.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Green's Theorem

Let's start with a simple closed curve C and let D be the region enclosed by the curve. Here is a sketch of such a curve and



We will say that the curve C has a positive orientation if it is traced out in a counter-clockwise direction.

Green's Theorem

Let C be a positively oriented, piecewise smooth simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then,

$$\int_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA .$$

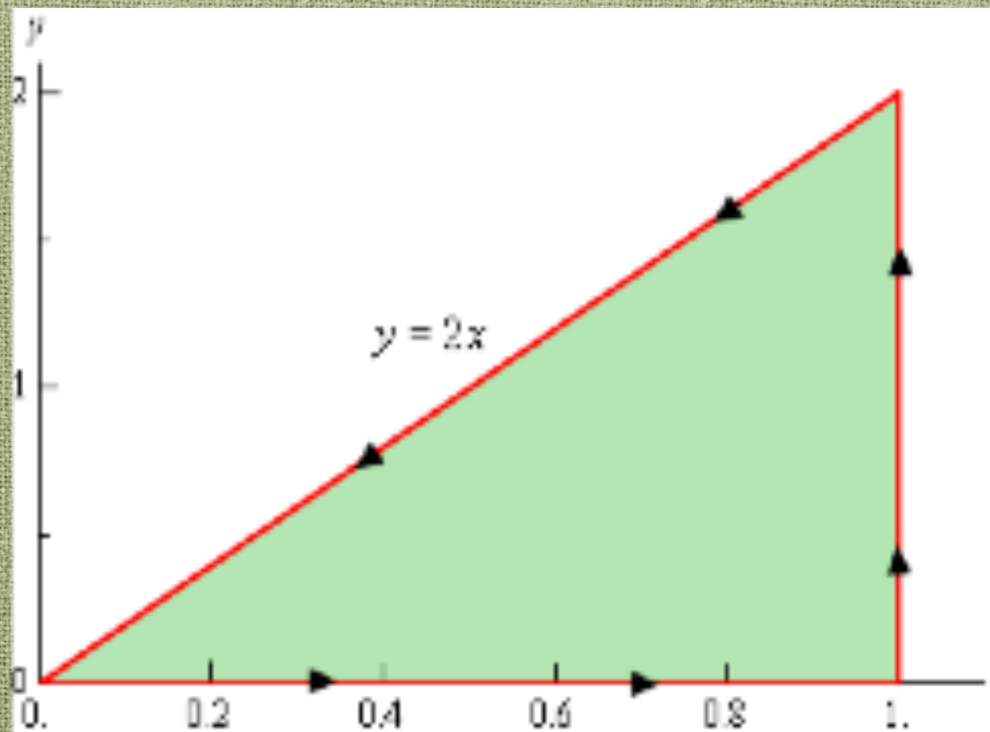
where C is a simple curve from $A(-3,-2,-1)$ to $B(1,2,3)$.

Example 1 Use Green's Theorem to evaluate $\int_C xydx + x^2y^3dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Solution Here we have $P(x,y) = xy$ and $Q(x,y) = x^2y^3$. Clearly the region D is given by
$0 \le x \le 1$ and $0 \le y \le 2x$.



So, using Green's Theorem the line integral becomes,

$$\int_C xydx + x^2y^3dy = \int_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx$$

$$= \int_0^1 \left[\frac{2xy^4}{4} - xy\right]_0^{2x} dx = \int_0^1 8x^5 - 2x^2\, dx = \left[\frac{8x^6}{6} - \frac{2x^3}{3}\right]_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 Evaluate $\int_C y^3\,dx - x^3\,dy$ where C is the positively oriented circle of radius 2 centered at the origin.

Solution Here we have $P(x,y) = y^3$ and $Q(x,y) = -x^3$. Since D is a disk of radius 2 and centered at the origin, so it is given by

$$0 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le 2\pi.$$

So, using Green's Theorem the line integral becomes,

$$\int_C y^3\,dx - x^3\,dy = \int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int\int_D (-3x^2 - 3y^2)\,dA$$

$$= -3\int_0^{2\pi}\int_0^{2} r^3\,dr\,d\theta$$

$$= -3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta$$

$$= -3\int_0^{2\pi} 4\,d\theta$$

$$= -24\pi.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Application of Green's Theorem

We will use Green's Theorem to determine the area of a region D with the following double integral

$$\text{Area} = \int\int_D dA.$$

We have to take P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

There are many functions that will satisfy this equation. Here are some of the more common functions.

1. $P = 0$ and $Q = x$;
2. $P = -y$ and $Q = 0$;
3. $P = -\frac{y}{2}$ and $Q = \frac{x}{2}$.

Then, if we use Green's Theorem in reverse we see that the area of the region D can also be computed by evaluating any of the following line integrals

$$\text{Area} = \int_C x dy = -\int_C y dx = \frac{1}{2}\int_C x dy - y dx,$$

where C is the boundary of the region D.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1 Use Green's Theorem to find the area of a disk of radius a.

Solution We use either of the integrals above.

$$\text{Area} = \frac{1}{2}\int_C xdy - ydy,$$

where C is the circle of radius a. So, to do this we will need a parameterization of C.

$$x = \text{a} \cos t; \qquad y = a \sin t \qquad \text{for all} \quad 0 \le t \le 2\pi.$$

The area is then

$$\text{Area} = \frac{1}{2}\int_C xdy - ydy$$

$$= \frac{1}{2}\left(\int_0^{2\pi} a\cos t(a\cos t)\,dt - \int_0^{2\pi} a\sin t(-a\sin t)\,dt\right)$$

$$= \frac{1}{2}\left(\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,dt\right)$$

$$= \frac{1}{2}\left(\int_0^{2\pi} a^2 dt\right)$$

$$= \pi a^2.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

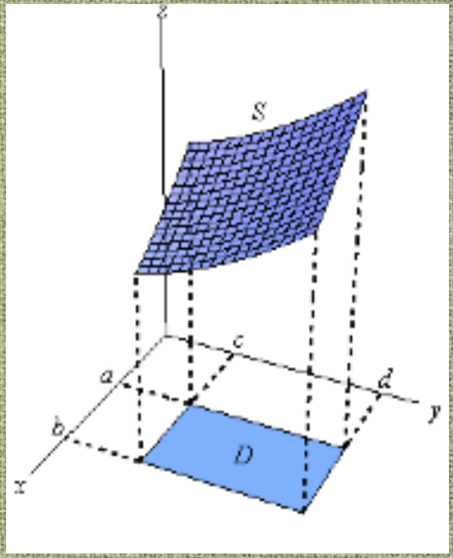
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Surface Integrals

It is now time to think about integrating functions over some surface S, in three-dimension. First, let's look at the surface integral in which the surface S is given by $z = g(x, y)$. In this case the surface integral is

$$\int\int_S f(x,y,z)dS = \int\int_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA,$$

where D is the projection of S on $xy -$ plane.



Example 1 Evaluate $\int\int_S 6xydS$ where S is the portion of the plane $x + y + z = 1$ that lies in the first octant.

Solution Here we have $f(x,y,z) = 6xy$ and $z = g(x,y) = 1 - x - y$. So

$$\frac{\partial g}{\partial x} = -1 \text{ and } \frac{\partial g}{\partial y} = -1.$$

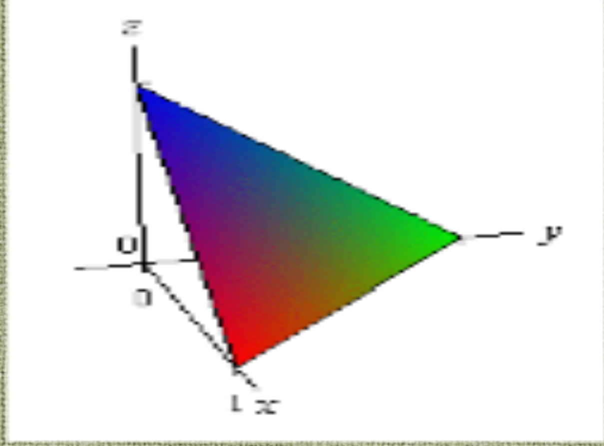
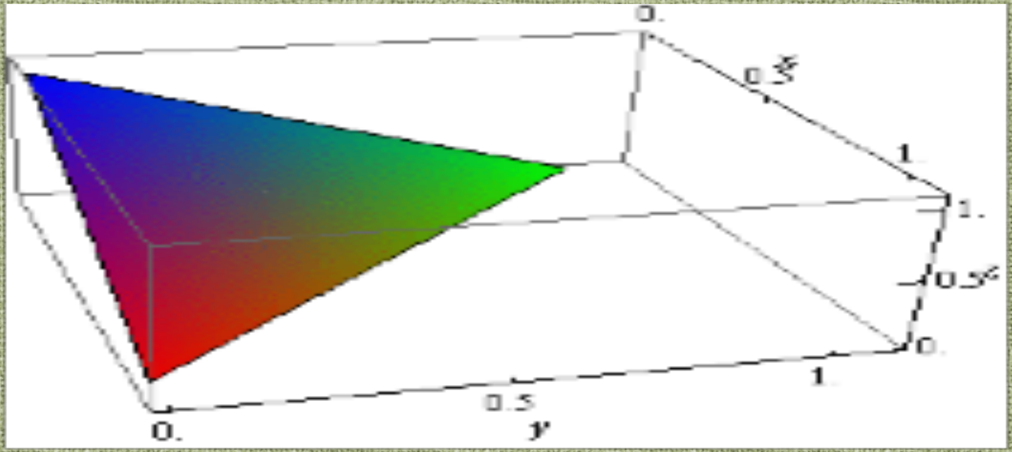
Then

$$\int\int_S 6xydS = \int\int_D 6xy\sqrt{(-1)^2 + (-1)^2 + 1}\, dA.$$

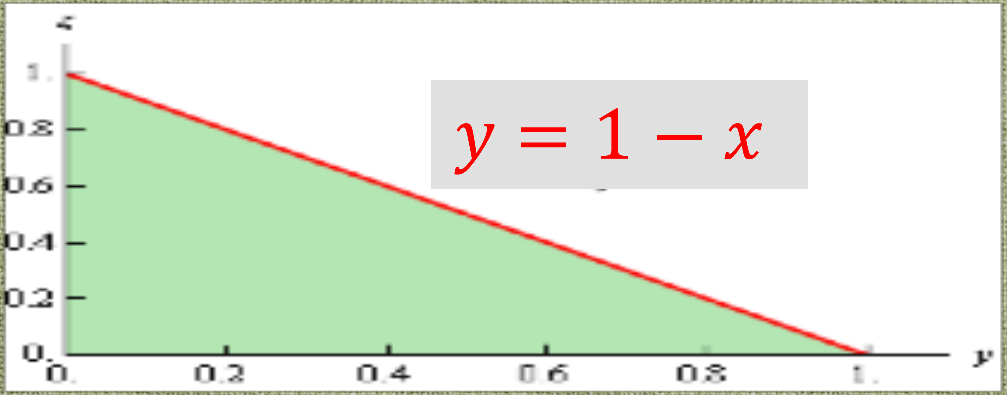
We need to determine the region D.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Here is a sketch of the region D.



The region D is given by $0 \le x \le 1$ and $0 \le y \le 1 - x$. Then

$$\iint_S 6xy\,dS = \iint_D 6xy\sqrt{(-1)^2 + (-1)^2 + 1}\,dA = 6\sqrt{3}\int_0^1\int_0^{1-x} xy\,dy\,dx$$

$$= 6\sqrt{3}\int_0^1 \left[\frac{xy^2}{2}\right]_0^{1-x} dx = 3\sqrt{3}\int_0^1 x(1-x)^2 dx = 3\sqrt{3}\int_0^1 x - 2x^2 + x^3 dx$$

$$= 3\sqrt{3}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = 3\sqrt{3}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{\sqrt{3}}{4}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Surface Integrals of Vector Fields

Given a vector field **F** with unit normal vector *n*, then the surface integral of **F** over the surf
ace *S* is given by

$$\int\int_S \mathbf{F}\, dS = \int\int_S \mathbf{F}\cdot n \ dS,$$

where the right hand integral is a surface integral of a scalar function. This integral is called the ***flux of F across** S*.

If the surface S is given by $z = g(x, y)$ and that the orientation is upward. Assume that the v
ector field is given by $\mathbf{F} = (M,N,P)$. In this case the unit vector normal is

$$n = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial x}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Clearly we have

$$\mathbf{F}\cdot n = (M\mathbf{i}+N\mathbf{j}+P\mathbf{k})\cdot\left(\frac{-\frac{\partial g}{\partial x}\mathbf{i}-\frac{\partial g}{\partial x}\mathbf{j}+\mathbf{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}\right) = \frac{-M\frac{\partial g}{\partial x}-N\frac{\partial g}{\partial x}+P}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}$$

and so

$$\mathbf{F}\cdot n\, dS = \frac{-M\frac{\partial g}{\partial x}-N\frac{\partial g}{\partial x}+P}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\,dA = \left(-M\frac{\partial g}{\partial x}-N\frac{\partial g}{\partial x}+P\right)dA$$

Therefore

surface integral becomes

$$\int\int_S \mathbf{F}\, dS = \int\int_S \mathbf{F}\cdot n\ dS = \int\int_D\left(-M\frac{\partial g}{\partial x}-N\frac{\partial g}{\partial x}+P\right)dA.$$

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Similarly, if the surface is oriented downward, then the unit normal vector is given by

$$n = \frac{\frac{\partial g}{\partial x}\,\mathbf{i} + \frac{\partial g}{\partial x}\,\mathbf{j} - \mathbf{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}.$$

In this case the surface integral becomes

$$\int\int_S \mathbf{F}\, dS = \int\int_D \left(M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y} - P\right) dA.$$

Example 1 Evaluate $\int\int_S \mathbf{F}\, dS$ where $\mathbf{F} = (0, y, -z)$ and S is the surface oriented outward and given by the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$ and the disk $x^2 + z^2 \leq 1$ at $y = 1$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

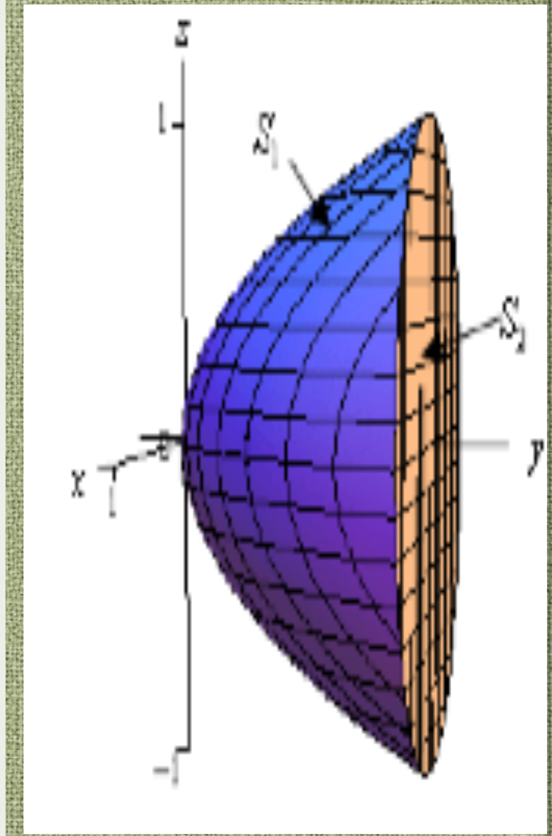
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Solution As noted in the sketch we will denote the paraboloid by S_1 and the disk by S_2 and S is composed of the two surfaces. We will evaluate the s urface integral on each and then add the results to get the overall surface int egral. Let's start with the paraboloid. In this case we have $y = g(x,z) = x^2 + z^2$ and the surface integral has the form

$$\iint_{S_1} \mathbf{F}\, dS = \iint_{D_1} \left(M\frac{\partial g}{\partial x} - N + P\frac{\partial g}{\partial z} \right) dA$$

$$= \iint_{D_1} (0 \cdot 2x - y + (-z)2z)\, dA = -\iint_{D_1} (y + 2z^2)\, dA$$

$$= -\iint_{D_1} (x^2 + 3z^2)\, dA.$$



Here D_1 is the disk in the xz-plane with radius 1 and centered at the origin so it is given by $D_1 = \{(r,\theta): 0 \le r \le 1$ and $0 \le \theta \le 2\pi\}$. In this case we have $x = r\cos\theta \text{ and } z = r\sin\theta$ Therefore,

$$\iint_{S_1} \mathbf{F}\, dS = -\iint_{D_1} (x^2 + 3z^2)\, dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^1 (1 + 2\sin^2\theta)\, r^3\,dr\,d\theta = -\frac{1}{4}\int_0^{2\pi} (2 - \cos 2\theta)\, d\theta = -\frac{1}{4}\left[2\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = -\pi.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

S_2: The cap of the paraboloid. and S is composed of the two surfaces. We will evaluate the surface integral on each and then add the results to get the overall surface integral. Let's start with the paraboloid. In this case we h
ave $y = g(x,z) = x^2 + z^2$ and the surface integral has the form

$$\int\int_{S_1} \mathbf{F}\, dS = \int\int_{D_1} \left(M\frac{\partial g}{\partial x} - N + P\frac{\partial g}{\partial z} \right) dA$$

$$= \int\int_{D_1} (0 \cdot 2x - y + (-z)2z)\, dA = -\int\int_{D_1} (y + 2z^2)\, dA$$

$$= -\int\int_{D_1} (x^2 + 3z^2)\, dA.$$

Here D_1 is the disk in the xz-plane with radius 1 and centered at the origin so it is given by $D_1 = \{(r,\theta): 0 \le r \le 1$ and $0 \le \theta \le 2\pi\}$. In this case we have $x = r\cos\theta \; and \; z = r\sin\theta$
Therefore,

$$\int\int_{S_1} \mathbf{F}\, dS = -\int\int_{D_1} (x^2 + 3z^2)\, dA = -\int_0^{2\pi}\int_0^{1} (r^2 + 2r^2\sin^2\theta)\, r dr d\theta$$

$$= -\int_0^{2\pi}\int_0^{1} (1 + 2\sin^2\theta)\, r^3 dr d\theta = -\frac{1}{4}\int_0^{2\pi} (2 - \cos 2\theta)\, d\theta = -\frac{1}{4}\left[2\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = -\pi.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

S_2: The cap of the paraboloid. In this case we obviously have $n = \mathbf{j}$.
Then

$$\iint_{S_2} \mathbf{F}\, dS = \iint_{S_2} \mathbf{F} \cdot n \; dS = \iint_{S_2} (0, y, -z) \cdot (0,1,0) \; dS = \iint_{S_2} y \; dS.$$

In this case we have $y = g(x,z) = 1$ and so

$$dS = \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2 + 1} \right) dA = dA.$$

Thus

$$\iint_{S_2} \mathbf{F}\, dS = \iint_{S_2} y \; dS = \iint_{D_2} 1 \, dA = \iint_{D_2} dA = \int_{\mathbf{0}}^{\mathbf{2\pi}} \int_{\mathbf{0}}^{\mathbf{1}} r dr d\theta = \frac{1}{2} \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta = \pi.$$

Finally, we obtain

$$\iint_{S} \mathbf{F}\, dS = \iint_{S_1} \mathbf{F}\, dS + \iint_{S_2} \mathbf{F}\, dS = (-\pi) + (\pi) = 0.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Divergence Theorem

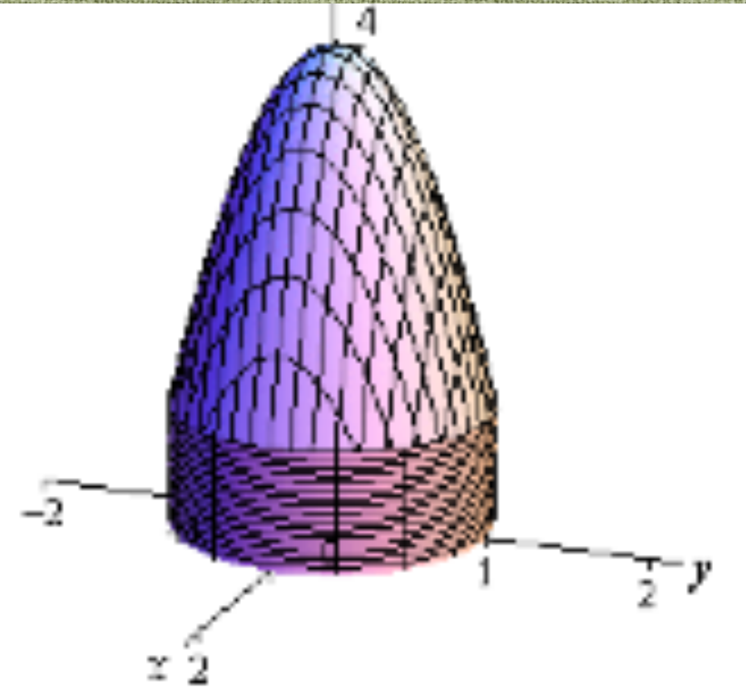
In this section we are going to relate surface integrals to triple integrals. We will do this with the Divergence Theorem.

Divergence Theorem Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous first order partial derivatives. Then

$$\int\int_S \mathbf{F}\, dS = \iiint_E \mathbf{div}\,\mathbf{F} dV .$$

Example 1 Use the divergence theorem to evaluate $\int\int_S \mathbf{F}\, dS$, where $\mathbf{F} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j} + z\mathbf{k}$ and the surface consists of the three surfaces, $z = 4 - 3x^2 - 3y^2, 1 \le z \le 4$ on the top, $x^2+y^2 = 1,\ 1 \le z \le 1$ on the sides and $z = 0$ on the bottom.

Solution. Let's start with a sketch of the surface.



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

The region E for the triple integral is then the region enclosed by these surfaces. So the region E is given by (in cylindrical coordinates system):
$E = \{(r,\theta,z): 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 4 - 3r^2\}$. We will also need the divergence of the vector field

$$\textbf{div}\,\mathbf{F} = \frac{\partial(xy)}{\partial x} + \frac{\partial\left(-\frac{1}{2}y^2\right)}{\partial y} + \frac{\partial(z)}{\partial z} = y - y + 1 = 1.$$

would be be a simple solid region and S is the boundary surface of E with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous first order partial derivatives.

The integral is then,

$$\int\int_S \mathbf{F}\,dS = \iiint_E \textbf{div}\,\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{1}\int_0^{4-3r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} 4r - 3r^3\,dr\,d\theta = \int_0^{2\pi}\left[2r^2 - \frac{3}{4}r^4\right]_0^1 d\theta = \int_0^{2\pi}\frac{5}{4}\,d\theta = \frac{5\pi}{2}.$$

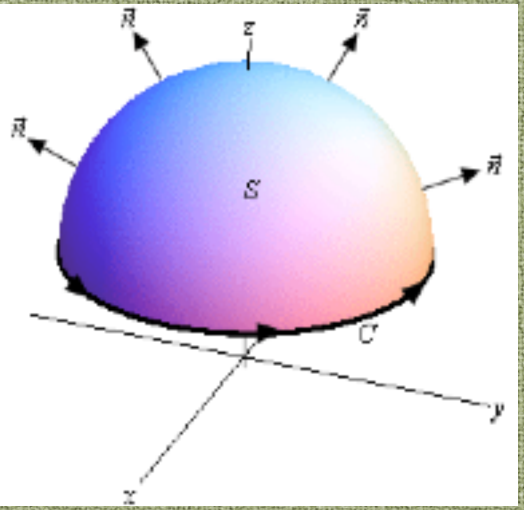
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Stokes Theorem

In this section we are going to take a look at a theorem that is a higher dimensional version of Green's Theorem. In Green's Theorem we related a line integral to a double integral over some region. In this section we are going to relate a line integral to surface integral

Around the edge of this surface we have a curve C. This curve is called the **boundary curve**. The orientation of the surface S will induce the **positive orientation of C**. To get the positive orientation of C think of yourself as walking along the curve. While you are walking along the curve if your head is pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on C.



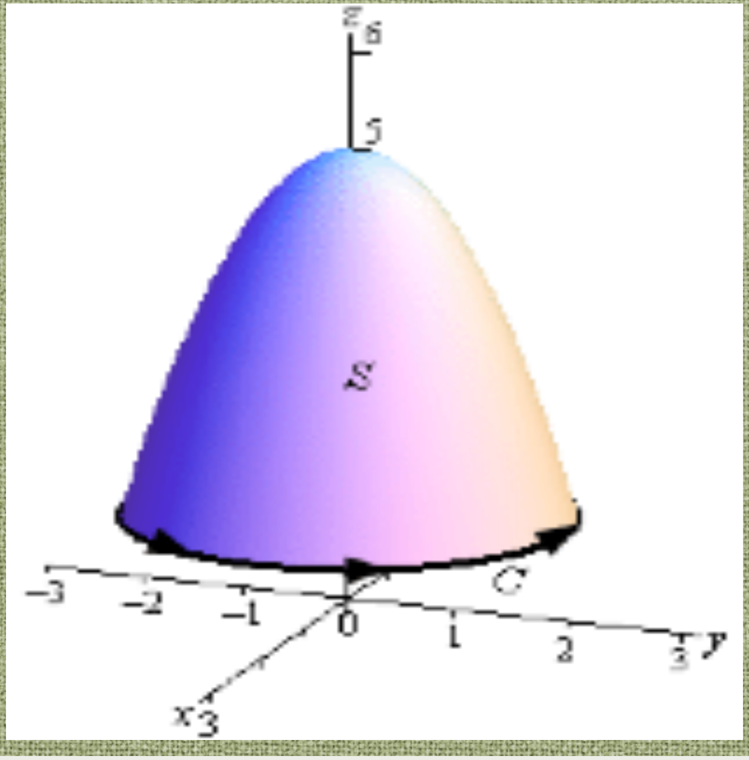
Stokes Theorem Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field then

$$\int_C \mathbf{F}\, d\mathbf{r} = \int\int_S \mathbf{curl\, F}\, dS\,.$$

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1 Use Stokes Theorem to evaluate $\int\int_S \mathbf{curl}\,\mathbf{F}\,dS$, where $\mathbf{F} = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}$ and the surface S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$ and oriented upward.



Solution In this case the boundary curve C is the intersection of the surface and the plane $z = 1$ and so $1 = 5 - x^2 - y^2$, that is, $x^2 + y^2 = 4$ at $z = 1$.
So the boundary curve will be the circle of radius 2 which is in the plane $z = 1$.
The parameterization of the curve is

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}, \quad \text{for} \quad 0 \le t \le 2\pi.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Using Stokes Theorem we can write the surface integral as the following line integral

$$\int\int_S \textbf{curl}\,\mathbf{F}\,dS = \int_C \mathbf{F}\,d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\,(\mathbf{r}(t)) \cdot \mathbf{r}(t)dt\,.$$

where

$$\begin{aligned}\mathbf{F}\,(\mathbf{r}(t)) &= 1\cdot\mathbf{i} - 3(2\cos t\,)(2\sin t)\mathbf{j} + (2\cos t\,)^3(2\sin t)^3\mathbf{k}\\ &= \mathbf{i} - 12\cos t\sin t\,\mathbf{j} + 64\cos^3 t\ \sin^3 t\,\mathbf{k},\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}\,(\mathbf{r}(t))\cdot\mathbf{r}(t) &= (\mathbf{i} - 12\cos t\sin t\,\mathbf{j} + 64\cos^3 t\ \sin^3 t\,\mathbf{k})\cdot(2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k})\\ &= 2\cos t - 24\cos t\sin^2 t + 64\cos^3 t\ \sin^3 t\end{aligned}$$

$$\int\int_S \textbf{curl}\,\mathbf{F}\,dS = \int_C \mathbf{F}\,d\mathbf{r} = \int_0^{2\pi}(2\cos t - 24\cos t\sin^2 t + 64\cos^3 t\ \sin^3 t)dt$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$\iint_S \mathbf{curl\,F}\, dS = \int_C \mathbf{F}\, d\mathbf{r} = \int_0^{2\pi} (2\cos t - 24\cos t\sin^2 t + 64\cos^3 t\ \sin^3 t)dt$$

$$= 2\int_0^{2\pi} \cos t\, dt - 24\int_0^{2\pi} \cos t \sin^2 t\, dt + 64\int_0^{2\pi} \cos^3 t\ \sin^3 t\, dt =$$

$$= 2\int_0^{2\pi} \cos t\, dt - 24\int_0^{2\pi} \sin^2 t\cos t\, dt + 64\int_0^{2\pi} \cos^2 t\ \sin^3 t\cos t\, dt$$

$$= 2\int_0^{2\pi} \cos t\, dt - 24\int_0^{2\pi} \sin^2 t\cos t\, dt + 64\int_0^{2\pi} (1-\sin^2 t)\sin^3 t\cos t\, dt$$

$$= 2\int_0^{2\pi} \cos t\, dt - 24\int_0^{2\pi} \sin^2 t\cos t\, dt + 64\int_0^{2\pi} (\sin^3 t - \sin^5 t)\cos t\, dt$$

$$= 2[\sin t]_0^{2\pi} - 24\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} + 64\left[\frac{1}{4}\sin^4 t - \frac{1}{6}\sin^6 t\right]_0^{2\pi} = 0.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 Use Stokes Theorem to evaluate $\int_C \mathbf{F}\, d\mathbf{r}$, where $\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and C is the triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ with counter-clockwise direction.

Solution By Stokes Theorem we have

$$\int_C \mathbf{F}\, d\mathbf{r} = \int\int_S \mathbf{curl}\ \mathbf{F}\, dS$$

where S is any surface having C as a boundary. We need first to compute the curl of the given vector field

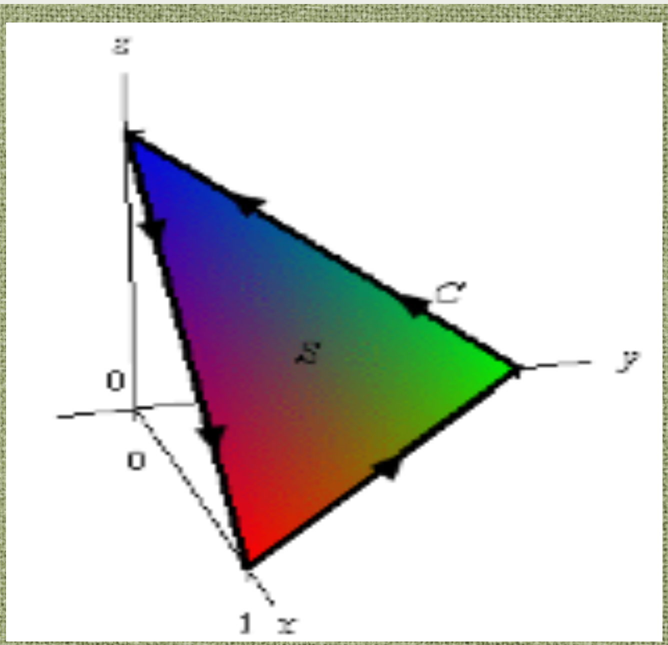
$$\text{curl}\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = (\mathbf{0})\mathbf{i} - (1 - 2z)\mathbf{j} + (0)\mathbf{k} = (2z - 1)\mathbf{j}.$$

The surface is shown in the following figure:

Obviously, since C has counter-clockwise direction, then the surface S is oriented upward.
The equation of the plane containing S is given by

$$x + y + z = 1$$

and so S is given by $z = g(x, y) = 1 - x - y$.



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

The unit upward normal vector is given by

$$n=\frac{-\frac{\partial g}{\partial x}\mathbf{i}-\frac{\partial g}{\partial x}\mathbf{j}+\mathbf{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}=\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}.$$

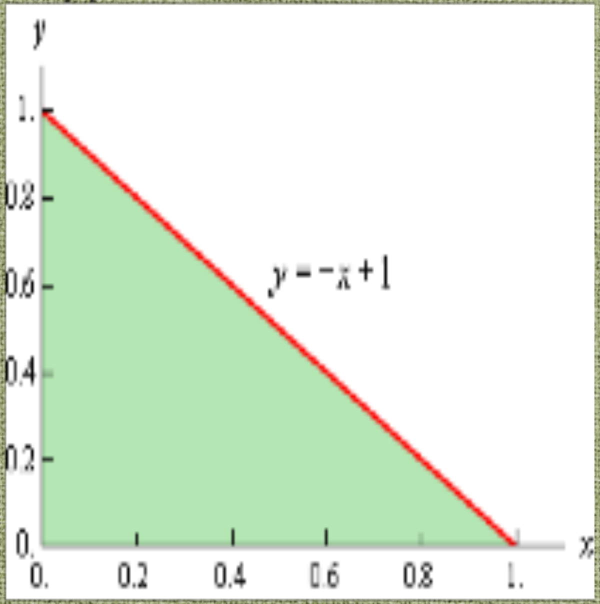
Therefore,

$$\int_C \mathbf{F}\,d\mathbf{r}=\int\int_S \mathbf{curl\ F}\,dS=\int\int_S \mathbf{curl\ F}\cdot n\,dS=\int\int_S ((2z-1)\mathbf{j})\cdot\left(\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}\right)dS$$

$$=\frac{\mathbf{1}}{\sqrt{3}}\int\int_S (2z-1)\,dS=\frac{\mathbf{1}}{\sqrt{3}}\int\int_D (2z-1)\sqrt{3}\,dA,$$

where D is the projection of the surface on the xy-plane which is given by

$$D=\{(x,y): 0\le x\le 1 \text{ and } 0\le y\le 1-x\}.$$



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Then,

$$\int_C \mathbf{F}\, d\mathbf{r} = \iint_D (2z-1)\,dA = \int_0^1 \int_0^{1-x} (2g(x,y)-1)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-2x-2y)\,dy\,dx = \int_0^1 [(1-2x)y - y^2]_0^{1-x}\,dx$$

$$= \int_0^1 [(1-2x)(1-x) - (1-x)^2]\,dx$$

$$= \int_0^1 [(1-x-2x+2x^2) - (1-2x+x^2)]\,dx$$

$$= \int_0^1 [(1-x-2x+2x^2-1+2x-x^2)]\,dx$$

$$= \int_0^1 [(x^2-x)]\,dx = \left[\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right]_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University