Math 203
Differential and integral Calculus

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Office Number: 2A184 Building4

Tel. Number: 4676526 01

Email: bounkhel@ksu.edu.sa

Webpage: http://fac.ksu.edu.sa/bounkhel

Textbook: Calculus by C. H. Edwards and Penney, Sixth Edition, 2002, Prentice-Hall, Inc., NJ.

Reference Books:

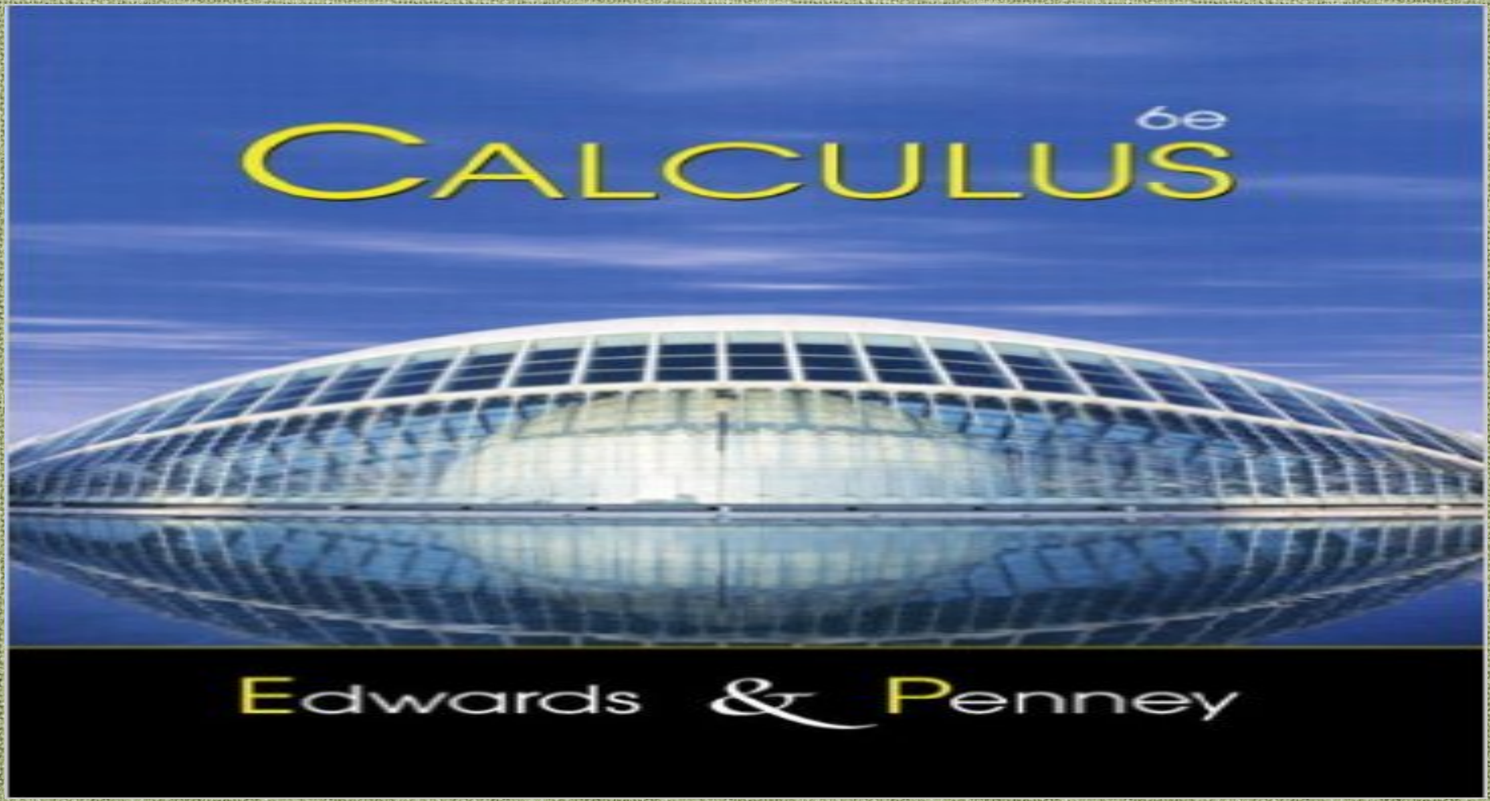
1. Calculus by E. W. Swokowski, M. Olinick, and D. Pence, Sixth Edition, 1994, PWS Publishing Co., Boston.
2. Calculus by R. T. Smith and R. B. Minton, 3rd Edition, 2007, McGraw Hill, Boston.
3. Calculus by James Stewart, Sixth Edition, 2009, Thomson Brooks/Cole, Canada.

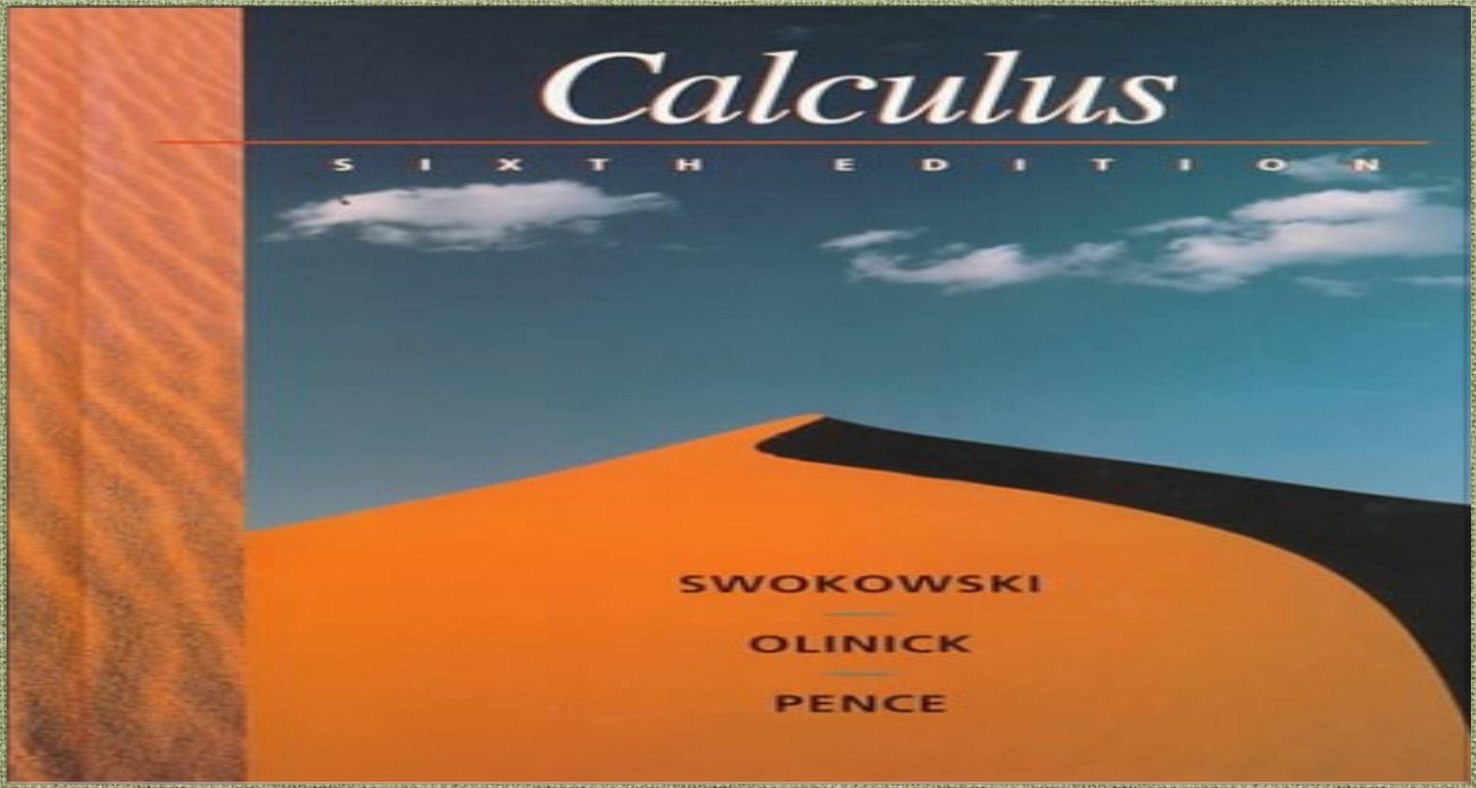
Grading:

MidTerm1: 25 + MidTerm2: 25 + Final Exam: 40 + Tutorial: 10 =100

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University





Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Topics:

Chapter 1: Infinite Sequences and Series

- 1.1 Infinite sequences;
- 1.2 Infinite series and convergence;
- 1.3 Positive Term Series;
- 1.4 The n^{th} term test;
- 1.5 Comparison and limit tests;
- 1.6 The integral Test;
- 1.7 Absolute Ratio Test;
- 1.8 Root Test;
- 1.9 Alternating series and its convergence test;
- 1.10 Power series and interval of convergence;
- 1.11 Taylor series and MacLaurin series.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Chapter 2: Multiple Integrals

- 2.1 Double integrals;
- 2.2 Double integrals over general regions;
- 2.3 Area and volume by double integration;
- 2.4 Double integrals in polar coordinates;
- 2.5 Applications of double integrals: volume, moment and centre of mass;
- 2.6 Triple integrals;
- 2.7 Applications of triple integrals: volume, moment and center of mass;
- 2.8 Integration in cylindrical and spherical coordinates;
- 2.9 Surface area.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Chapter 3: Vector Calculus

- 3.1 Vector fields;
- 3.2 Line integrals;
- 3.3 The fundamental theorem and independence of path;
- 3.4 Green's Theorems;
- 3.5 Surface integrals;
- 3.6 The divergence theorem;
- 3.7 Stock's Theorem.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Chapter 1: Infinite Series

• 1.2 Infinite sequence

Definition:

*A **infinite sequence** is a real function whose domain is the set of all positive integers. A sequence **u** can be displayed in the form*

$$u(1),\ u(2),\ \ldots.,\ u(n),\ \ldots.$$

The value $u(n)$ is called the n^{th} term of the sequence and is usually written $\boldsymbol{u_n}$.

Example 1: If the sequence is simple enough one can look at the first few terms and guess the general rule for computing the n^{th} term. For instance:

$1,1,1,1,1,1,\ldots\ldots$	$u_n = 1$
$-1,0,1,2,3,\ \ldots\ldots$	$u_n = n-2$
$-2,-4,-6,-8,-10,\ \ldots\ldots$	$u_n = -2n$
$1,-1,1,-1,1,\ldots\ldots$	$u_n = (-1)^{n-1}$
$1,1\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\ldots$	$u_n = \frac{1}{n}$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

An infinite sequence u_n *is said to* **converge** *to a real number* $\mathbf{L}$ *if* $\lim_{n\to\infty} u_n = L$.

A sequence which does not converge to any real number is said to **diverge**.

If $\lim_{x\to\infty} f(x) = L$, *then* $\lim_{n\to\infty} f(n) = L$.

Example 4: $\lim_{n\to\infty} \frac{4n^2+1}{n^2+3n} = \lim_{x\to\infty} \frac{4x^2+1}{x^2+3x} = 4$

Similarly, if $\lim_{x\to\infty} f(x) = \infty$, *then* $\lim_{n\to\infty} f(n) = \infty$.

Example 5: $\lim_{n\to\infty} \ln(n) = \lim_{x\to\infty} \ln(x) = \infty$

Example 6: *Evaluate the limits*

(a) $\lim_{n\to\infty} \left(1+\frac{1}{c}\right)^n$, *where* $c > 0$,

(b) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^c$, *where* $c > 0$,

(c) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.

Sol.:

(a) $\lim_{n\to\infty} \left(1+\frac{1}{c}\right)^n = \lim_{x\to\infty} \left(1+\frac{1}{c}\right)^x = \infty$,

(b) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^c = \lim_{x\to 0} (1+x)^c = 1$,

(c) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Theorem 1: *Each of the following sequences approaches ∞.*

$$\lim_{n\to\infty} n! = \infty,$$
$$\lim_{n\to\infty} b^n = \infty \text{ (if } b>1),$$
$$\lim_{n\to\infty} n^c = \infty \text{ (if } c>0),$$
$$\lim_{n\to\infty} \ln(n) = \infty.$$

$n! = 1 * 2 * 3 * \cdots * n$, (if $n>0$),
By convention $0!$ is defined by $0! = 1$.

In Problems 1-8, find the n^{th} term of the sequence

1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
2. $-1, 2, -3, 4, -5, 6, \ldots$
3. $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots$
4. $2, 4, 16, 256, \ldots.$
5. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
6. $2, 5, 10, 17, 26, 37, \ldots$
7. $1, 3, 6, 10, 15, \ldots$
8. $0.6, 0.61, 0.616, 0.6161, \ldots$

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Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Determine whether the following sequences converge, and find the limits when they exist.

9. $u_n = \sqrt{n}$;
10. $u_n = n - \frac{n^2}{n+1}$;
11. $u_n = \frac{(-1)^n}{\sqrt{n}}$;
12. $u_n = \frac{n+2}{n}$;
13. $u_n = n(-1)^n$;
14. $u_n = \frac{n!}{n^3}$;
15. $u_n = \sqrt[n]{n}$;
16. $u_n = \ln(\ln(n))$;
17. $u_n = \sqrt{n^2+n} - n$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

1.3 Infinite series and convergence:

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Given an infinite sequence u_n, each finite sum

$$u_1 + u_2 + \cdots + u_n$$

is defined . This sum is called the ***n*^th^ partial sum** of the series. Thus, with each infinite series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \ldots + u_n + \cdots,$$

there are associated two sequences, the sequence of terms,

$$u_1, u_2, \ldots, u_n, \ldots,$$

and the sequence of partial sums,

$$u_1, u_2, \ldots, u_n, \ldots,$$

$$S_1, S_2, \ldots, S_n, \ldots, \quad \text{where } S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n.$$

*Definition: The **sum** of an infinite series is defined as the limit of the sequence of partial sums if the limit exists,*

$$\sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} \left(\sum_{k=1}^{n} u_k\right) = \lim_{n\to\infty} S_n.$$

*The series is said to **converge** to a real number **S**, **diverge**, or **diverge to** ∞, if the sequence of partial sums S_n converges to **S**, diverges, or diverges to ∞, respectively.*

Given an infinte series, we often wish to answer two questions.
Does the series converges? what is the sum of the series?

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1: Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solutions:

We compute the nth partial sum S_n

$$u_1 = 1-\frac{1}{2};$$

$$u_2 = \frac{1}{2}-\frac{1}{3};$$

$$u_3 = \frac{1}{3}-\frac{1}{4};$$

$$\vdots$$

$$u_n = \frac{1}{n}-\frac{1}{n+1};$$

$$S_n = \sum_{k=1}^{n} u_k = \left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right).$$

$$= 1-\frac{1}{n+1}.$$

Therefore the sum of the series is

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1-\frac{1}{n+1} = 1.$$

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Department of Mathematics
King Saud University

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Department of Mathematics
King Saud University

Example 2: Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{(n+1)(n+2)}\right)$.

Solutions:

First we decompose the n^{th} term :

$$\frac{3}{(n+1)(n+2)} = \frac{3}{n+1} - \frac{3}{n+2}$$

We compute the n^{th} partial sum S_n

$$u_1 = \frac{3}{2} - \frac{3}{3};$$

$$u_2 = \frac{3}{3} - \frac{3}{4};$$

$$u_3 = \frac{3}{4} - \frac{3}{5};$$

$$\vdots$$

$$u_n = \frac{3}{n+1} - \frac{3}{n+2};$$

$$S_n = \sum_{k=1}^{n} u_k = \frac{3}{2} - \frac{3}{n+2}.$$

Therefore the sum of the series is

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2} - \frac{3}{n+2} = \frac{3}{2}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

For each constant c, the series $\sum_{n=0}^{\infty} c^n = 1 + c + c^2 + \cdots + c^n + \cdots$ is called the ***geometric series*** for c.

Theorem 1: *If* $|c| < 1$, *then the geometric series* ***converges*** *and*

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

Example 1: $1 + 0.1 + 0.01 + 0.001 + \cdots = \frac{1}{1-\frac{1}{10}} = 1\frac{1}{9}$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

If the series $\sum_{n\geq 0} u_n = u_1 + u_2 + \ldots + u_n + \cdots$, *converges, then* $\lim_{n\to\infty} u_n = 0$.

TEST 1: n^{th} term test

If $\lim_{n\to\infty} u_n \neq 0$, *then the series* $\sum_{n=0}^{\infty} u_n = u_1 + u_2 + \ldots + u_n + \cdots$, ***diverges***.

Example 2:

1- *If* $|c| \geq 1$, *the geometric series* $\sum_{n=0}^{\infty} c^n$ ***diverges.***

2- The ***harmonic series*** $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ ***diverges.***

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Theorem 2: The harmonic series

$$\sum_{n=1}^{\infty} u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is the example of series having the property $\lim_{n\to\infty} u_n = 0$ and the series diverges.

Theorem 3: Suppose $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent.

(i) **Constant Rule:** For any constant c, $\sum_{n=1}^{\infty} c u_n = c \sum_{n=1}^{\infty} u_n$;

(ii) **Sum Rule:** $\sum_{n=1}^{\infty} u_n + v_n = \sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n$;

(iii) **Inequality Rule:** If $u_n \le v_n$ for all n then $\sum_{n=1}^{\infty} u_n \le \sum_{n=1}^{\infty} v_n$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Positive Term Series

By a positive term series, we mean a series in which every term is greater than zero. For example, the geometric series

$$\sum_{n=1}^{\infty} c^n = 1 + c + c^2 + \cdots + c^n + \cdots$$

is a positive term series if $c > 0$ but not if $c \leq 0$.

We are going to give several tests for the convergence of a positive term series.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

TEST 2: Basic comparison test

Suppose $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are positive term series and* $a_n \leq b_n$ *for all* n.

(i) *If* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

Example 1: *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{6^n}{7^n + 5^n}$.

Sol. Clearly we have $\frac{6^n}{7^n+5^n} \leq \frac{6^n}{7^n} = \left(\frac{6}{7}\right)^n$ *and since the geometric series is convergent, so the given series converges.*

Example 2: *Test the convergence of the series* $\sum_{n=2}^{\infty} \frac{n^2}{n^3 - 1}$.

Sol. Clearly we have $\frac{n^2}{n^3-1} \geq \frac{n^2}{n^3} = \frac{1}{n}$ *and since the harmonic series* $\sum_{n=2}^{\infty} \frac{1}{n}$ *is divergent, so the given series diverges.*

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

TEST 3: Limit comparison test

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ be positive term series and let $c = \lim_{n\to\infty} \frac{a_n}{b_n} > 0$.
Then $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ have the same behavior, that is,
1- If $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ is convergent,
2- If $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ is divergent.

Example 1 For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + 1\right)^2$;
2. $\sum_{n=1}^{\infty} \frac{n^2}{n^2-3}$;
3. $\sum_{n=1}^{\infty} \frac{n-1}{\sqrt{n^6+1}}$;
4. $\sum_{n=1}^{\infty} \frac{2n^2+7}{n^4 \sin^2(n)}$;
5. $\sum_{n=1}^{\infty} \frac{4}{n^2-2n-3}$;
6. $\sum_{n=1}^{\infty} \frac{7}{n(n+1)}$;
7. $\sum_{n=1}^{\infty} \frac{2^n \sin^2(5n)}{n^4+\cos^2(n)}$;
8. $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2+2n}$;
9. $\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}$;
10. $\sum_{n=1}^{\infty} \frac{4n^2-n}{n^3+9}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

TEST 4: Integral test

Suppose f is a continous decreasing function and $f(x) > 0$ for all $x \geq 1$. Then the improper integral $\int_1^\infty f(x)dx$ and the infinite series $\sum_{n=1}^\infty f(n)$ either both converge or both diverge to ∞.

Example:(p–series) *The p series* $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if p>1 and diverges if p $\leq$ 1.

1- Case 1: p = 1, The p series is just the harmonic series $\sum_{n=1}^\infty \frac{1}{n} = \infty$.

$\int_1^\infty \frac{1}{x} dx = \lim_{n\to\infty} (\ln(n) - \ln(1)) = \lim_{n\to\infty} \ln(n) = \infty$. Therefore the series diverges.

2- Case 2: p > 1, The improper integral converges,

$\int_1^\infty \frac{1}{x^p} dx = \lim_{n\to\infty} \int_1^n x^{-p} dx = \lim_{n\to\infty} \left(\left[\frac{n^{1-p}}{1-p}\right] - \left[\frac{1}{1-p}\right]\right) = \frac{1}{1-p}$. Therefore the series converges.

3- Case 3: p < 1, The improper integral diverges to ∞,

$\int_1^\infty \frac{1}{x^p} dx = \lim_{n\to\infty} \int_1^n x^{-p} dx = \lim_{n\to\infty} \left(\left[\frac{n^{1-p}}{1-p}\right] - \left[\frac{1}{1-p}\right]\right) = \infty$. Therefore the series diverges.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example: (p-series)

The p series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ converges because $\frac{4}{3} > 1$.

The p series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges to ∞ because $\frac{1}{2} < 1$.

Example:

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

Problems:

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{n}{n+4}$;
2. $\sum_{n=1}^{\infty} \frac{2}{4n-3}$;
3. $\sum_{n=1}^{\infty} \frac{n+1}{4n^3}$;
4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$;
5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4}}$;
6. $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$;
7. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example: Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Solution:

The function $f(x) = \frac{\ln x}{x^2}$ is well defined and continuous on $[1, \infty)$ and its derivative is given by $f'(x) = \frac{1-2lnx}{x^3}$.

Clearly for any $x \geq 2$ *we have* $\ln x \geq \ln(2) = 0.69 > \frac{1}{2}$. Then $1-2\ln x<0$ and so $f'(x)<0$, that is, f is decreasing.

Simple computations by parts give

$$\int_2^t \frac{\ln x}{x^2} dx = -\left[\frac{\ln x}{3x^3} + \frac{1}{15\,x^5}\right]_2^t = -\left(\frac{\ln t}{3t^3} + \frac{1}{15\,t}\right) + \left(\frac{\ln 2}{32^3} + \frac{1}{15\,2^5}\right)$$

and so

$$\int_2^\infty \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x^2} dx = \left(\frac{\ln 2}{32^3} + \frac{1}{15\,2^5}\right),$$

that is the improper integral is convergent and hence by the integral test the given series is convergent.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Root Test

Let $\sum_{n=0}^{\infty} a_n$ be a postive term series ($i.e.$ $a_n > 0$) and let

$$L := \lim_{n\to\infty} \sqrt[n]{a_n}.$$

- If $L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges;
- If $L > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges;
- If $L = 1$, then the test gives no information and the series may converge or diverge.

Example: *Test the convergence of the series* $\sum_{n=1}^{\infty} \left(\frac{2n^3+n+7}{n^3+5n^2-1}\right)^n$.

Solution. Clearly we have $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2n^3+n+7}{n^3+5n^2-1} = \lim_{n\to\infty} \frac{2n^3}{n^3} = 2 > 1$ and hence by Root Test the given series diverges.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Alternating Series

An *alternating series* is a series in which the odd numbered terms are positive and the even numbered terms are negative, or vice versa.
An example is the geometric series

$$\sum_{n=1}^{\infty} a^n, \qquad a < 0.$$

Given any positive term series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots,$
the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

and

$$\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 + \cdots$$

are alternating series.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

TEST 5: Alternating Series test

Assume that

i. $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is an alternating series;

ii. The terms (a_n) are decreasing, $a_1 > a_2 > \cdots > a_n > \cdots$;

iii. The terms approach zero, $\lim_{n\to\infty} a_n = 0$.

Then the series converges to a sum $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = S$. Moreover, the sum S is between any two consecutive partial sums,

$$S_{2n} < S < S_{2n+1}.$$

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

converges by the Alternating Series Test, because $\frac{1}{n}$ is decreasing and approaches zero as $n \to \infty$. The partial sums are

$$1, \frac{1}{2}, \frac{5}{6}, \frac{7}{12}, \frac{47}{60}, \frac{37}{60}, \ldots$$

The sum S is between any two consecutive partial sums,

$$\frac{37}{60} < S < \frac{47}{60}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
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Example 2: The alternating series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+1)}{n}=2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots+\frac{(-1)^{n+1}(n+1)}{n}+\cdots$$

diverges. The terms $\frac{n+1}{n}$ are decreasing, but their limit is one instead of zero,

$$\text{Lim}_{n\to\infty}\frac{n+1}{n}=1.$$

Problems: Test the convergence of the following alternating series:

1. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$;
2. $\sum_{n=1}^{\infty}(-1)^{n}\sqrt{n}$;
3. $\sum_{n=1}^{\infty}\frac{(-1)^{n}\sqrt{n}}{n+1}$;
4. $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln n}$;
5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{10n+5}$;
6. $\sum_{n=1}^{\infty}(-1)^{n}n^{-\frac{1}{3}}$;
7. $\sum_{n=1}^{\infty}(-1)^{n}\sqrt[n]{\frac{1}{2}}$;
8. $\sum_{n=1}^{\infty}(-1)^{n}$;
9. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{n}}{n!}$;
10. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n!}{2^{n}}$;
11. $\sum_{n=1}^{\infty}\frac{(-1)^{n}n}{\ln n}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

12. $\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(\ln n)}$;

13. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^3}$;

14. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$;

15. $\sum_{n=2}^{\infty} (-1)^n \frac{2^{n-2}+1}{2^{n+3}+5}$;

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$;

17. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)$;

18. $\sum_{n=0}^{\infty} (-1)^n \frac{2^n+1}{3^n-2}$;

19. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(1 + \frac{1}{n}\right)^{-n}$;

20. Approximate the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-3}$ to two decimal places;

21. Approximate the series $1 - \frac{2}{10} + \frac{3}{100} - \frac{4}{1000} + \cdots$ to four decimal places;

22. Approximate the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ to two decimal places;

23. Approximate the series $\sum_{n=1}^{\infty} (-n)^{-n}$ to three decimal places.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Absolute and conditional convergence

Consider a series $\sum_{n=1}^{\infty} a_n$ which has both positive and negative terms.
We may form a new series whose terms are the absolute values of the terms of the given series. If all the terms a_n are nonzero, then $|a_n| > 0$, so $\sum_{n=1}^{\infty} |a_n|$ is a positive term series.
If $\sum_{n=1}^{\infty} a_n$ is already a positive term series, then $|a_n| = a_n$ and the series is identical to its absolute value series $\sum_{n=1}^{\infty} |a_n|$.
Sometimes it is simpler to study the convergence of the absolute value series $\sum_{n=1}^{\infty} |a_n|$ than of the given series $\sum_{n=1}^{\infty} a_n$.
This is because we have at our disposal all the convergence tests for positive term series from the preceding sections.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Definition

A series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if its absolute value series $\sum_{n=1}^{\infty} |a_n|$ is convergent.
A series which is convergent but not absolutely convergent is called **conditionally convergent**.

Given an arbitrary series $\sum_{n=1}^{\infty} a_n$ the theorem shows that exactly one of the following three things can happen:

1. The series is **absolutely convergent**;
2. The series is **conditionally convergent**;
3. The series is **divergent**.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1 The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots,$$

is absolutely convergent, because its absolute value series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots,$$

Is convergent.

Example 2 The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots,$$

is conditionally convergent. It converges by the Alternating Series Test. But its absolute value series

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots,$$

diverges.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

TEST 6: Absolute Ratio test

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Suppose that the limit of the ratio $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists or is ∞, $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

i. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely,

ii. If $L > 1$, or $L = \infty$, the series diverges.

iii. If $L = 1$, the test gives no information and the series may converge absolutely, converge conditionally, or diverge.

Example 1 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n!\,(n+1)} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0<1,$$

so by the Absolute Ratio Test the series converges.

Example 2 Test the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1} n!}{n^n (n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n (n+1) n!}{n^n n!\,(n+1)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e>1,$$

so by the Absolute Ratio Test the series diverges.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

For both series we have $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$,

Since

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(n+1)} = 1$$

and

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1,$$

So the Absolute Ratio Test does not apply to both series.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

TEST 7: Root test

Suppose that the limit of the nth root of the nth term $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists or is ∞,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L.$$

i. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely,

ii. If $L > 1$, or $L = \infty$, the series diverges.

iii. If $L = 1$, the test gives no information and the series may converge absolutely, converge conditionally, or diverge.

Example 2 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n^2-3n-6)^n}{(8n^2+7)^n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(n^2-3n-6)^n}{(8n^2+7)^n}} = \lim_{n\to\infty} \left(\frac{(n^2-3n-6)^n}{(8n^2+7)^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2-3n-6}{8n^2+7} = \frac{1}{8} < 1.$$

so by the Root Test the series converges.

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Problems: Apply the Ratio test to the following series:

1. $\sum_{n=1}^{\infty} 3^n$;
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$;
3. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$;
4. $\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$;
5. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$;
6. $\sum_{n=1}^{\infty} \frac{1}{2^n}$;
7. $\sum_{n=1}^{\infty} n^2$;
8. $\sum_{n=1}^{\infty} \frac{1}{n^3}$;
9. $\sum_{n=1}^{\infty} \frac{5^n}{6^n-5^n}$;
10. $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$;
11. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$;
12. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n}$;
13. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$;
14. $\sum_{n=1}^{\infty} \frac{e^n(n!)}{n^n}$;
15. $\sum_{n=1}^{\infty} \frac{3^n(n!)^2}{(2n)!}$;
16. $\sum_{n=2}^{\infty} \frac{10^n}{(\ln n)^n}$;
17. $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots\cdot(2n-1)}$;
18. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdots\cdot(2n)}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Power Series

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Definition. A **power series** in x is a series of functions of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

The n^{th} finite partial sum of a power series is just a polynomial of degree n,

$$\sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n .$$

For each power series $\sum_{n=0}^{\infty} a_n x^n$ one of the following happens.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

I. The series converges absolutely at $x = 0$ and diverges everywhere else,
II. The series converges absolutely on the whole real line $(-\infty, \infty)$,
III. The series converges absolutely at every point in an open interval $(-r, r)$ and diverges at every point outside the closed interval $[-r, r]$. At the Endpoints $-r$ and r the series may converge or diverge, so the interval of convergence is one of the sets

$$(-r, r),\ [-r, r)\ ,\ (-r, r]\ ,\ [-r, r].$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

The number r is called the radius of convergence of the power series. In case (I) the radius of convergence is zero, and in case (II) it is ∞. Once the radius of convergence is determined, we need only test the series at $x = r$ and $x = -r$ to find the interval of convergence.

Example 1. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} b^n x^n, \qquad \text{where } b > 0.$$

Solution.

$$\lim_{n\to\infty} \frac{|b^{n+1}x^{n+1}|}{|b^n x^n|} = \lim_{n\to\infty} b|x| = b|x| < 1 \iff |x| < \frac{1}{b} \iff x \in \left(-\frac{1}{b}, \frac{1}{b}\right).$$

At $x = -\frac{1}{b}$ and $x = \frac{1}{b}$ the series becomes $\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty}(1)^n$ respectively which are divergent and so the interval of convergence is $\left(-\frac{1}{b}, \frac{1}{b}\right)$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{x^n}{n}=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots.$$

Solution. We compute the limit $\lim_{n\to\infty}\frac{\left|\frac{x^{n+1}}{n+1}\right|}{\left|\frac{x^n}{n}\right|}=|x|\lim_{n\to\infty}\frac{n}{n+1}=|x|$. By the Absolute Ratio Test the series converges for $|x|<1$ and diverges for $|x|>1$, so the radius of convergence is $r=1$.

- At $x=1$, the power series is the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent.

- At $x=-1$, the power series is the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ which is convergent by the Alternating Series Test.

Then the interval of convergence is $[-1,1)$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3 Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots.$$

Solution. For all x we have

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1.$$

By the Absolute Ratio Test the series converges for all x. It has radius of convergence ∞ and interval of convergence $(-\infty, \infty)$.

Example 4 Find the interval of convergence of

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + \cdots + n!\, x^n + \cdots.$$

Solution. For all $x \neq 0$ we have

$$\lim_{n\to\infty} \frac{|(n+1)!x^{n+1}|}{|n!x^n|} = \lim_{n\to\infty} (n+1)|x| = \infty > 1.$$

By the Absolute Ratio Test the series diverges for all $x \neq 0$ and the radius of convergence is $r = 0$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

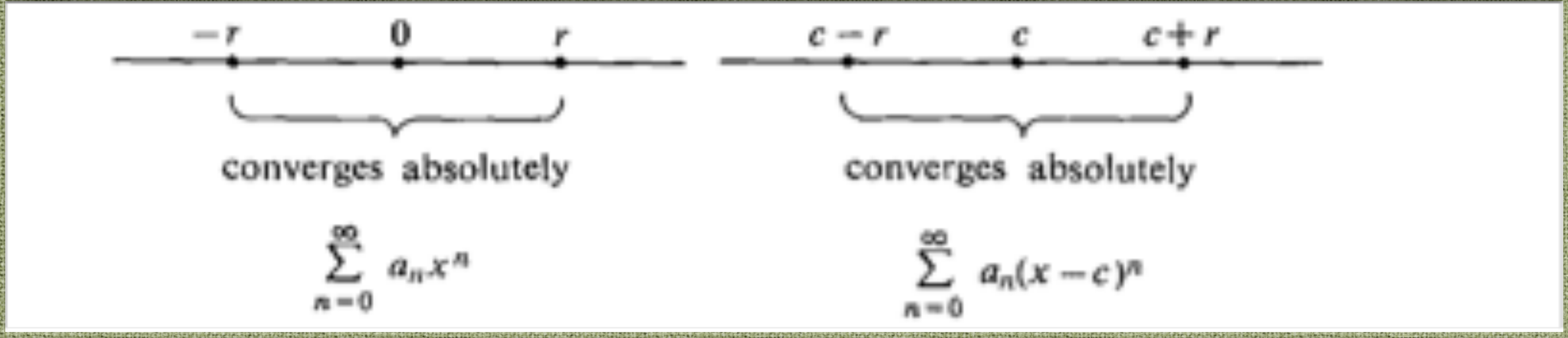
Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

If we replace x by $x - c$ we obtain a power series in $x - c$,

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots,$$

The power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n x^n$ and the interval of convergence is simply moved over so that its center is c instead of 0. For example, if $\sum_{n=0}^{\infty} a_n x^n$ has the interval of convergence $(-r, r]$, then $\sum_{n=0}^{\infty} a_n (x-c)^n$ has the interval of convergence $(c-r, c+r]$.

By the Absolute Ratio Test the series converges for all x. It has radius of convergence ∞ and interval of convergence $(-\infty, \infty)$.



Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 5. Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(x+5)^n = 1 + \frac{1}{2}(x+5) + \frac{(2!)^2}{4!}(x+5)^2 + \cdots.$$

Solution. We have

$$\lim_{n\to\infty} \frac{\left|\frac{(n+1)!\,(n+1)!}{(2n+2)!}(x+5)^{n+1}\right|}{\left|\frac{(n!)^2}{(2n)!}(x+5)^n\right|} = \lim_{n\to\infty}\left|\frac{(n+1)!\,(n+1)!\,(2n)!\,(x+5)}{(2n+2)!\,(n!)^2}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(n+1)^2(x+5)}{(2n+1)(2n+2)}\right| = |x+5| \lim_{n\to\infty}\left(\frac{(n+1)^2}{(2n+1)(2n+2)}\right) = \frac{|x+5|}{4}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

By the Absolute Ratio Test the series converges for $|x+5|<4$ and diverges for $|x+5|>4$.

The radius of convergence is $r=4$ and interval of convergence is centered at -5 , that is , $x\in(-9,-1)$. We note that

$$\frac{(n!)^2}{(2n)!}=\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{n}{2n-1}\cdot\frac{n}{2n}>\left(\frac{1}{2}\cdot\frac{1}{2}\right)^n=\left(\frac{1}{4}\right)^n.$$

Therefore at $x=-9$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}(x+5)^n=\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}(-4)^n>\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n(-4)^n=\sum_{n=0}^{\infty}(-1)^n,$$

and at $x=-1$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}(x+5)^n=\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}(4)^n>\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n(4)^n=\sum_{n=0}^{\infty}(1)^n.$$

By Basic Comparison Test these two series are divergent and hence the inter val of convergence is $(-9,-1)$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Problems: Find the interval and radius of convergence of the following power series:

1. $\sum_{n=0}^{\infty} 5x^n$;
2. $\sum_{n=1}^{\infty} n^n x^n$;
3. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$;
4. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!} x^n$;
5. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x^n$;
6. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$;
7. $\sum_{n=2}^{\infty} \frac{n^n x^n}{(\ln n)^n}$;
8. $\sum_{n=2}^{\infty} \sqrt[n]{n}\, x^n$;
9. $\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n+1)}{n!} x^n$;
10. $\sum_{n=1}^{\infty} \frac{x^n}{3n^2}$;
11. $\sum_{n=1}^{\infty} \frac{x^n}{5^n\sqrt{n}}$;
12. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$;
13. $\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} x^n$;
14. $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$;
15. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n} x^n$;
16. $\sum_{n=2}^{\infty} \frac{x^n}{(\ln(\ln n))^n}$;
17. $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)} x^n$;
18. $\sum_{n=1}^{\infty} \frac{x^n}{5\sqrt{n}}$;
19. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^n}}$;
20. $\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$;
21. $\sum_{n=1}^{\infty} \frac{n! x^n}{\sqrt{n^n}}$;
22. $\sum_{n=1}^{\infty} \frac{x5^{3n}}{n}$;
23. $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$;
24. $\sum_{n=1}^{\infty} \frac{x^{6n}}{n!}$;
25. $\sum_{n=1}^{\infty} 3^n x^{2n}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Problems: Find the interval and radius of convergence of the following power series:

1. $\sum_{n=0}^{\infty} 5x^n$;
2. $\sum_{n=1}^{\infty} n^n x^n$;
3. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$;
4. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!} x^n$;
5. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x^n$;
6. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$;
7. $\sum_{n=2}^{\infty} \frac{n^n x^n}{(\ln n)^n}$;
8. $\sum_{n=2}^{\infty} \sqrt[n]{n}\, x^n$;
9. $\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n+1)}{n!} x^n$;
10. $\sum_{n=1}^{\infty} \frac{x^n}{3n^2}$;
11. $\sum_{n=1}^{\infty} \frac{x^n}{5n\sqrt{n}}$;
12. $\sum_{n=1}^{\infty} \frac{x^n}{3n}$;
13. $\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} x^n$;
14. $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$;
15. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n} x^n$;
16. $\sum_{n=2}^{\infty} \frac{x^n}{(\ln(\ln n))^n}$;
17. $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)} x^n$;
18. $\sum_{n=1}^{\infty} \frac{x^n}{5\sqrt{n}}$;
19. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^n}}$;
20. $\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$;
21. $\sum_{n=1}^{\infty} \frac{n!x^n}{\sqrt{n^n}}$;
22. $\sum_{n=1}^{\infty} \frac{x^{3n}}{5^n}$;
23. $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$;
24. $\sum_{n=1}^{\infty} \frac{x^{6n}}{n!}$;
25. $\sum_{n=1}^{\infty} 3^n x^{2n}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

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Department of Mathematics
King Saud University

26. $\sum_{n=0}^{\infty} n3^n x^n$;

27. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$;

28. $\sum_{n=1}^{\infty} \frac{1}{6\sqrt{n}} x^n$;

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n^2} x^n$;

30. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\sqrt{n}}$;

31. $\sum_{n=2}^{\infty} n!\,(x-3)^n$;

32. $\sum_{n=2}^{\infty} \frac{(x+8)^n}{2^n}$;

33. $\sum_{n=1}^{\infty} (3^n + 4^n)\, x^n$;

34. $\sum_{n=1}^{\infty} 3^n \; x^{2n}$;

35. $\sum_{n=1}^{\infty} \frac{e^n (x-4)^n}{n^2}$;

36. $\sum_{n=1}^{\infty} 2x^n$;

37. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$;

38. $\sum_{n=2}^{\infty} \frac{2^n x^n}{\ln n} x^n$;

39. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n$;

40. $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n}$;

41. $\sum_{n=2}^{\infty} \frac{(x-5)^n}{n!}$;

42. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x^n$;

43. $\sum_{n=1}^{\infty} \frac{4^n x^n}{3^n + 5^n}$;

44. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n5^n}$;

45. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Derivatives and integrals of power series

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \qquad \text{radius of convergence } r = 1$$

Theorem. Suppose that $f(x)$ is the sum of a power series , i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with radius of convergence $r > 0$ and let $-r < x < r$. Then :

a) f has the derivative $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$;

b) f has the integral $\int_0^x f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$;

c) The power series in (a) and (b) both have radius of convergence r;

d) The n^{th} finite partial sum of a power series is just a polynomial of degree n,

$$\sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad x \in (-1,1).$$

We will get several new power series formulas starting from the power series for $\frac{1}{1-x}$. We will use the following methods:

a) Differentiate a power series;
b) Integrate a power series;
c) Substitute bu for x;
d) Substitute u^p for x;
e) Multiply a power series by a constant;
f) Multiply a power series by x^p;
g) Add two power series.

Methods c), d), and g) may change the radius of convergence.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad x \in (-1,1).$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n, \qquad 2x \in (-1,1), \text{i.e.,} \qquad x \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

By integrating $\frac{1}{1-x}$ and multiplying by -1 we get a power series for $\ln(1-x)$, that is, $\int_0^x \frac{1}{1-t}\,dt = -\ln(1-x)$.

$$\ln(1-x) = -x - \frac{x^2}{2} - \cdots - \frac{x^n}{n} - \cdots = \sum_{n=0}^{\infty} -\frac{x^n}{n}, \qquad x \in (-1,1);$$

$$\ln(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^n \frac{x^n}{n} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \qquad x \in (-1,1);$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad x \in (-1,1);$$

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots, \qquad x \in (-1,1)$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Finally let us differentiate the series for $\frac{1}{1-x}$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2},$$

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots+(n+1)x^n+\cdots=\sum_{n=0}^{\infty}(n+1)x^n,\qquad x\in(-1,1)$$

$$x\ln(1-x)=-x^2-\frac{x^3}{2}-\cdots-\frac{x^{n+1}}{n}-\cdots=\sum_{n=1}^{\infty}-\frac{x^{n+1}}{n},\qquad x\in(-1,1);$$

$$x^2\ln(1+x)=x^3-\frac{x^4}{2}+\cdots-\frac{x^{n+2}}{n}-\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+2}}{n},\qquad x\in(-1,1);$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Approximations by Power series

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Power series are one of the most important methods of approximation in mathematics. Consider a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

The partial sums give approximate values for the function,

$$f(x) \approx \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Example 1. Approximate ln(1½) within 0.01.

We use the power series for $\ln(1-x)$,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots + \frac{x^n}{n} + \cdots$$

Setting $1 - x = 1½$ gives $x = -\frac{1}{2}$,

$$\ln(1½) = \frac{1}{2} - \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} - \frac{1}{4\cdot 16} + \frac{1}{5\cdot 32} - \cdots$$

This is an alternating series. The last term shown is less than 0.01,

$$\frac{1}{5\cdot 32} = \frac{1}{160} \approx 0.006.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

By the (A.S.T.), the error in each partial sum is less than the next term. So

$$\ln(1½) \approx \frac{1}{2} - \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} - \frac{1}{4\cdot 16}, \qquad \text{error} \le \frac{1}{5\cdot 32},$$

or This is an alternating series. The last term shown is less than 0.01,

$$\ln(1½) \approx 0.401, \qquad \text{error} \le 0.006.$$

The actual value is $\ln(1½) \approx 0.405.$

Example 2. Approximate $\tan^{-1}\left(\frac{1}{2}\right)$ within 0.001.

The power series for $\tan^{-1} x$ is $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots,$

Setting $x = \frac{1}{2}$, gives $\tan^{-1}(\frac{1}{2}) = \frac{1}{2} - \frac{1}{3\cdot 8} + \frac{1}{5\cdot 32} - \frac{1}{7\cdot 128} + \frac{1}{9\cdot 512} - \cdots$

This is an alternating series. The last term is less than 0.001,

$$\frac{1}{9\cdot 512} \approx 0.0002.$$

Therefore,

$$\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{3\cdot 8} + \frac{1}{5\cdot 32} - \frac{1}{7\cdot 128}, \text{error} \le 0.0002.$$

Adding up, $\tan^{-1}\left(\frac{1}{2}\right) = 0.4635, \quad \text{error} \le 0.0002.$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Taylor Series

Let $f(x)$ have derivatives of all orders at $x = c$.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n .$$

This series is called the ***Taylor series*** for the function $f(x)$ about the point $x = c$. When $c = 0$, ***Taylor series*** is called ***MacLaurin series*** *and becomes* $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

Example 2. Find ***MacLaurin series*** , for the function $f(x) = e^x$.
Sol. The n^{th} derivative is $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$. Then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3. Find ***MacLaurin series*** , for the function $f(x) = \sin x$.
Sol. The n^{th} derivative of f:

$$
\begin{aligned}
f(x) &= \sin x, & f(0) &= 0.\\
f'(x) &= \cos x, & f'(0) &= 1.\\
f''(x) &= -\sin x, & f''(0) &= 0.\\
f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1.\\
f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0.\\
f^{(5)}(x) &= \cos x, & f^{(5)}(0) &= 1.\\
&\ldots.. & &\ldots.
\end{aligned}
$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Here is a review of the power series obtained earlier in this chapter.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad x \in (-1,1),$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad x \in (-1,1),$$

$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + \cdots + 2^n x^n + \cdots = \sum_{n=0}^{\infty} 2^n x^n, x \in \left(-\frac{1}{2}, \frac{1}{2}\right),$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad x \in (-1,1),$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots - \frac{x^n}{n} - \cdots = \sum_{n=0}^{\infty} -\frac{x^n}{n}, \qquad x \in (-1,1),$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad x \in (-1,1),$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots = \sum_{n=0}^{\infty} (n+1)x^n, \qquad x \in (-1,1),$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x \in (-\infty, \infty),$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}, \qquad x \in (-\infty, \infty),$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \qquad x \in (-\infty, \infty),$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \qquad x \in (-\infty, \infty),$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}, \qquad x \in (-\infty, \infty),$$

$$\int_0^x e^{-t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot n!} \qquad x \in (-\infty, \infty),$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}, \qquad x \in (-1, 1),$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad x \in (-\infty, \infty),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad x \in (-\infty, \infty),$$

$$(1+x)^p = 1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{p(p-1)(p-2)\ldots(p-(n-1))x^n}{n!}, \qquad x \in (-1,1),$$

where p is constant. The last series is called the ***binomial series***.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Problems on Chapter 1:

Determine whether the sequences converge and find the limits when they exist.

1- $a_n = \left(1 + \frac{1}{n^2}\right)^n$; 2- $a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$; 3- $a_n = (1+n)^{\frac{1}{n}}$;

4- $a_n = \dfrac{n^n}{n!}$ $(Hint\text{: Show that } a_{n+1} \geq 2a_n)$; 5- $a_n = n! - 10^n$;

Determine whether the series converge and find the sums when they exist.

1- $1 + \frac{3}{7} + \frac{9}{49} + \ldots + \left(\frac{1}{7}\right)^n + \cdots$; 2- $1 - 1.1 + 1.11 - 1.111 + 1.1111 - \cdots$;

3- $\left(1 - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{27}\right) + \left(\frac{1}{27} - \frac{1}{64}\right) + \cdots + \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \cdots$;

4- $\displaystyle\sum_{n=0}^{\infty} \frac{7^n - 6^n}{5^n}$; $\displaystyle\sum_{n=0}^{\infty} \frac{7 \cdot 5^n}{6^n}$; $\displaystyle\sum_{n=0}^{\infty} \frac{2n-3}{5n+6}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Test the convergence of the following series:

1. $\sum_{n=0}^{\infty} \frac{3n-7}{10n+6}$;
2. $\sum_{n=0}^{\infty} \frac{5}{6n^2+n-1}$;
3. $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{1+2\sqrt{n}+3n}$;
4. $\sum_{n=0}^{\infty} ne^{-n}$;
5. $\sum_{n=0}^{\infty} (\ln(n))^{-n}$;
6. $\sum_{n=0}^{\infty} n^{-\ln n}$;
7. $\sum_{n=0}^{\infty} \ln n^{-\ln n}$;
8. $\sum_{n=3}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$;
9. $\sum_{n=1}^{\infty} (-1)^n \left(1-\frac{1}{n^2}\right)$;
10. $\sum_{n=1}^{\infty} (-1)^n n^{-\frac{1}{n}}$;
11. $\sum_{n=1}^{\infty} (-1)^n n^{-\frac{1}{n}}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University