INTEGRAL CALCULUS (MATH 106)

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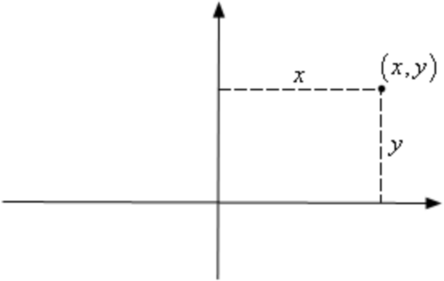
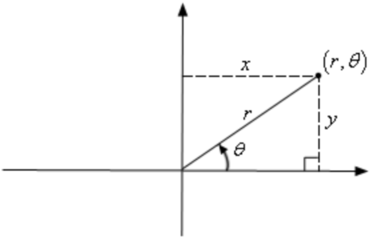
1. Polar Coordinates

Cartesian coordinate system (or Rectangular, or x-y)

the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically.



Cartesian coordinate is not the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive $x-$axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point.

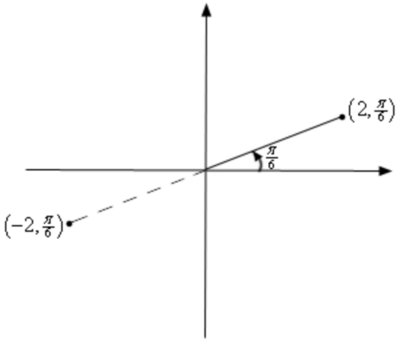


Coordinates in this form are called **polar coordinates.**

Polar Coordinates

Example 2.1

The two points $(2, \frac{\pi}{6})$ *and* $(-2, \frac{\pi}{6})$



Polar Coordinates

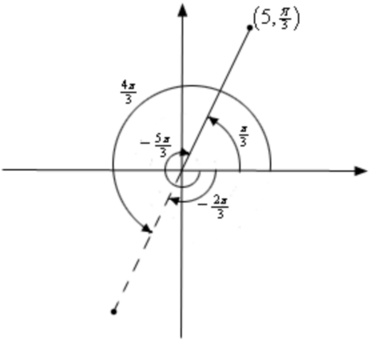
Remark

The polar coordinates of a point is not unique, if $P = (r, \theta)$ then other representations are:

1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$
2. $P = (-r, \theta + \pi)$
3. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$
4. $P = (-r, \theta - \pi)$
5. $P = (-r, \theta - \pi + 2n\pi)$, where $n \in \mathbb{Z}$

Example 2.2

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$$



Polar Coordinates

Relationship between the polar and the Cartesian coordinates
the following equations that will convert polar coordinates into Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \end{aligned}$$

$$r = \sqrt{x^2 + y^2}, \quad \textit{and} \ \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Example 2.3

1. *Convert* $\left(-4, \frac{2\pi}{3}\right)$ *into Cartesian coordinates.*
2. *Convert* $(-1, -1)$ *into polar coordinates.*

1. This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

   So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$

2. Let's first get r
   $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$
   Now, let's get θ
   $\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Example 2.4

Convert each of the following into an equation in the given coordinate system.

1. *Convert* $2x - 5x^3 = 1 + xy$ *into polar coordinates.*
2. *Convert* $r = -8\cos\theta$ *into Cartesian coordinates.*

1.

$$2\left(r\cos\theta\right) - 5(r\cos\theta)^3 = 1 + \left(r\cos\theta\right)\left(r\sin\theta\right)$$
$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

2. $r^2 = -8r\cos\theta \Rightarrow x^2 + y^2 = -8x$

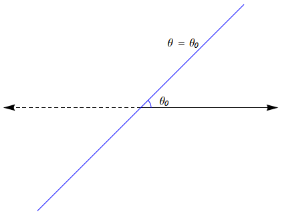
Polar Curves

First - Straight Lines:

1-Lines passing through the pole : Any straight line passing through the pole has the form $\theta = \theta_0$ where θ_0 is the angle between the straight line and the polar axis .

$\theta = \theta_0 \Rightarrow \tan\theta = \tan\theta_0 \Rightarrow \frac{y}{x} = \tan\theta_0 \Rightarrow y = x\tan\theta_0$

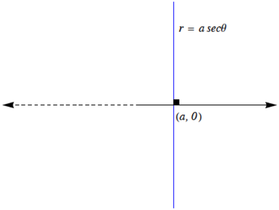
The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan\theta_0$.



Polar Curves

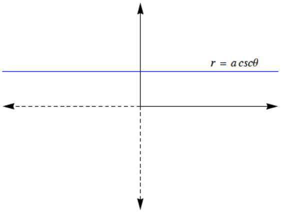
2-Lines perpendicular to the polar axis : Any straight line perpendicular to the polar axis has the form $r = a\sec\theta$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$r = a\sec\theta \Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r\cos\theta = a \Rightarrow x = a$



Polar Curves

3-Lines parallel to the polar axis : Any straight line parallel to the polar axis has the form $r = a\csc\theta$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$
$r = a\csc\theta \Rightarrow r = \frac{a}{\sin\theta} \Rightarrow r\sin\theta = a \Rightarrow y = a$



Polar Curves

Second- Circles

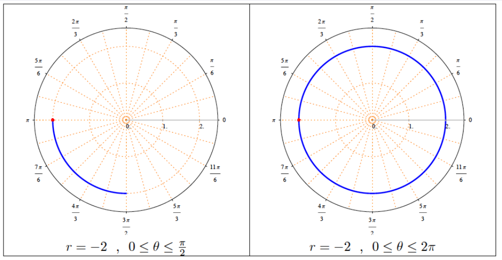
1-Circles of the form $r = a$, where $a \in \mathbb{R}^*$

$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$

Therefore, $r = a$ represents a circle with center $= (0,0)$ and radius equals $|a|$

Example 2.5

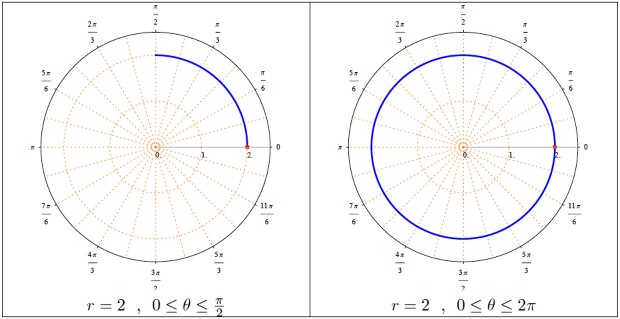
$r = -2$ represents a circle with center $= (0,0)$ and radius to 2.



Polar Curves

Example 2.6

$r = 2$ represents a circle with center $= (0,0)$ and radius to 2.



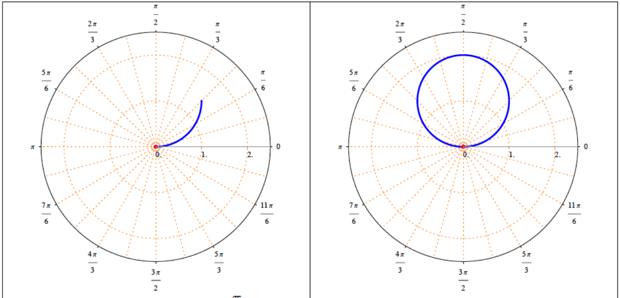
Polar Curves

2-Circles of the form $r = a\sin\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$
$r = a\sin\theta \Rightarrow r^2 = a\ r\sin\theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0 \Rightarrow x^2 + (y^2 - ay + \frac{a^2}{4}) = \frac{a^2}{4} \Rightarrow x^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4}$
Therefore, $r = a\sin\theta$ represents a circle with center $= (0, \frac{a}{2})$ and radius equals to $\frac{|a|}{2}$

Polar Curves

Example 2.7

$r = 2\sin\theta$ *represents a circle with center* $= (0,1)$ *and radius equals to 1*

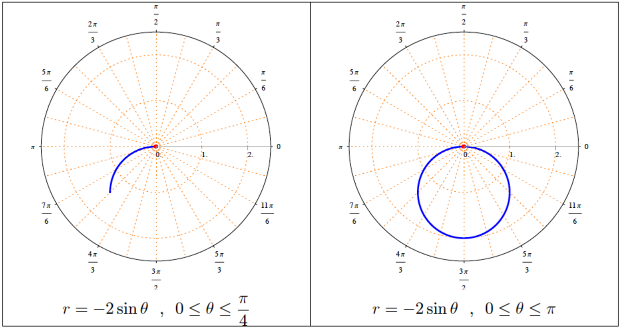


$r = 2\sin\theta \;,\; 0 \le \theta \le \frac{\pi}{4}$ $\qquad$ $r = 2\sin\theta \;,\; 0 \le \theta \le \pi$

Polar Curves

Example 2.8

$r = -2\sin\theta$ represents a circle with center $= (0, -1)$ and radius equals to 1.



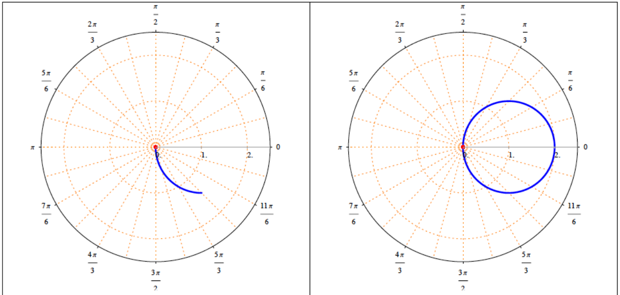
Polar Curves

3-Circles of the form $r = a\cos\theta$, where $a \in \mathbb{R}^*$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$r = a\cos\theta \Rightarrow r^2 = a\ r\cos\theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0 \Rightarrow (x^2 - ax + \frac{a^2}{4}) + y^2 = \frac{a^2}{4} \Rightarrow (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$
Therefore, $r = a\cos\theta$ represents a circle with center $= (\frac{a}{2}, 0)$ and radius equals to $\frac{|a|}{2}$

Polar Curves

Example 2.9

$r = 2\cos\theta$ represents a circle with center $= (1,0)$ and radius equals to 1.

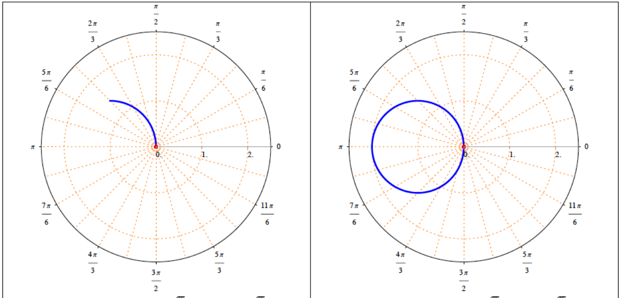


$r = 2\cos\theta \ , \ -\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{4}$ $\qquad$ $r = 2\cos\theta \ , \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Polar Curves

Example 2.10

$r = -2\cos\theta$ *represents a circle with center* $= (-1, 0)$ *and radius equals to 1.*



$r = -2\cos\theta \;,\; -\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{4}$ $\qquad$ $r = -2\cos\theta \;,\; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Polar Curves

Third - Limacon curves:

The general form of a Limacon curve is
$r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

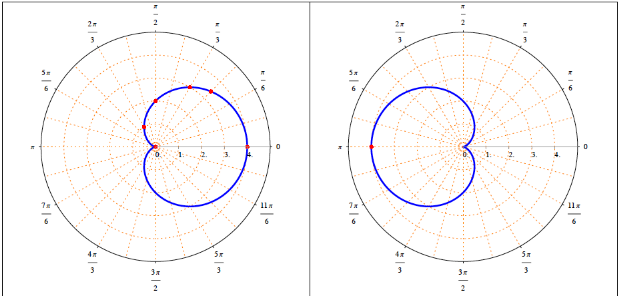
1-Cardioid (Heart-shaped): It has the form $r(\theta) = a + a\sin\theta$ or $r(\theta) = a + a\cos\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

Polar Curves

Example 2.11

$r(\theta) = 2 + 2\cos\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	$2+\sqrt{2}$	3	2	1	0

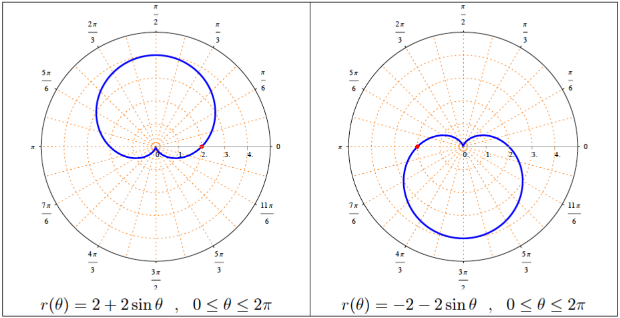


$r(\theta) = 2 + 2\cos\theta \;,\; 0 \le \theta \le 2\pi$ $\qquad$ $r(\theta) = -2 - 2\cos\theta \;,\; 0 \le \theta \le 2\pi$

Polar Curves

Example 2.12

$r(\theta) = 2 + 2\sin\theta$ *and* $r(\theta) = -2 - 2\sin\theta$

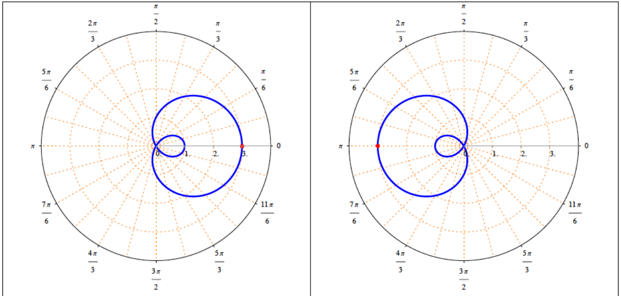


Polar Curves

2-Limacon with inner loop: It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \le \theta \le 2\pi$

Example 2.13

$r(\theta) = 1 + 2\cos\theta$ *and* $r(\theta) = -1 - 2\cos\theta$



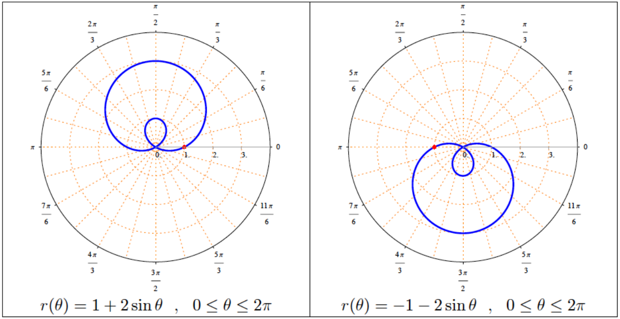
$r(\theta) = 1 + 2\cos\theta \ , \ 0 \le \theta \le 2\pi$

$r(\theta) = -1 - 2\cos\theta \ , \ 0 \le \theta \le 2\pi$

Polar Curves

Example 2.14

$r(\theta) = 1 + 2\sin\theta$ *and* $r(\theta) = -1 - 2\sin\theta$

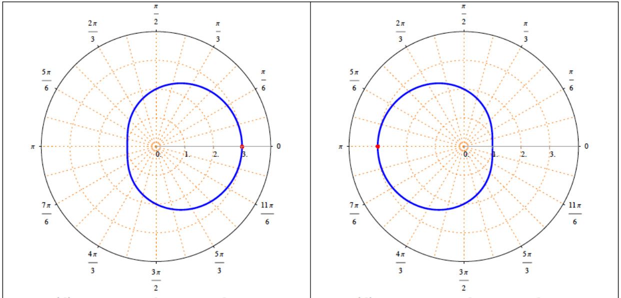


Polar Curves

3-Dimpled Limacon : It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta)a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$

Example 2.15

$r(\theta) = 2 + \cos\theta$ *and* $r(\theta) = -2 - \cos\theta$

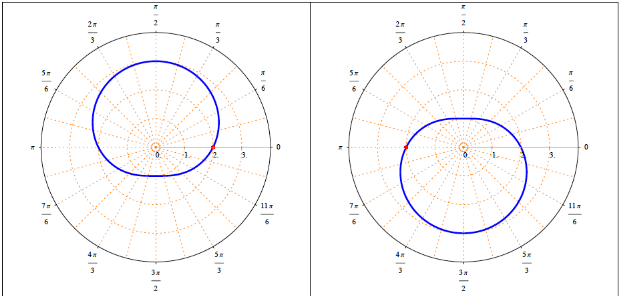


$r(\theta) = 2 + \cos\theta \;,\; 0 \leq \theta \leq 2\pi$ $\qquad$ $r(\theta) = -2 - \cos\theta \;,\; 0 \leq \theta \leq 2\pi$

Polar Curves

Example 2.16

$r(\theta) = 2 + \sin\theta$ *and* $r(\theta) = -2 - \sin\theta$



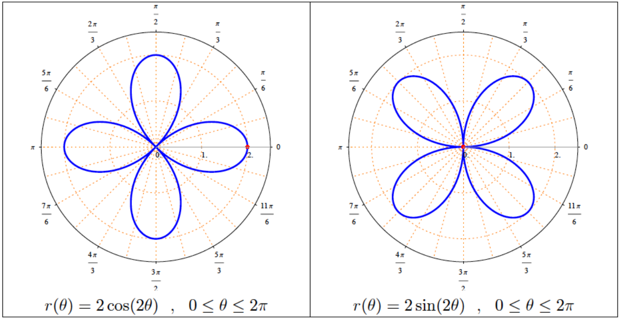
$r(\theta) = 2 + \sin\theta \quad , \quad 0 \le \theta \le 2\pi$ $\qquad$ $r(\theta) = -2 - \sin\theta \quad , \quad 0 \le \theta \le 2\pi$

Polar Curves

Fourth - Rose curves:
It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*, n \in \mathbb{N}$ and $n \geq 2$
1-n is even: In this case the number of loops (or leaves) is 2n.

Example 2.17

$r(\theta) = 2\cos(2\theta)$ *or* $r(\theta) = 2\sin(2\theta), 0 \leq \theta \leq 2\pi$

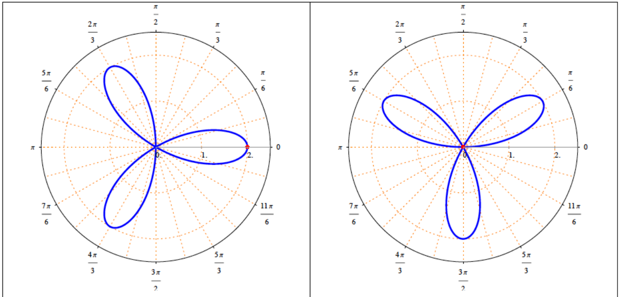


Polar Curves

n is odd: In this case the number of loops (or leaves) is n .

Example 2.18

$r(\theta) = 2\cos(3\theta)$ *or* $r(\theta) = 2\sin(3\theta), 0 \leq \theta \leq \pi$



$r(\theta) = 2\cos(3\theta)$, $0 \leq \theta \leq \pi$ $\quad$ $r(\theta) = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$

Slope Of The Tangents Line With Polar Coordinates

If $r = r(\theta)$ is a smooth polar curve, then the slope of the tangent line to $r = r(\theta)$ is $m = \frac{dy}{dx}$ where $(x = r\cos\theta, \quad y = r\sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example

Example 2.19

Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ *at* $\theta = \frac{\pi}{6}$

We'll first need the following derivative. $\frac{dr}{d\theta} = 8\cos\theta$
The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

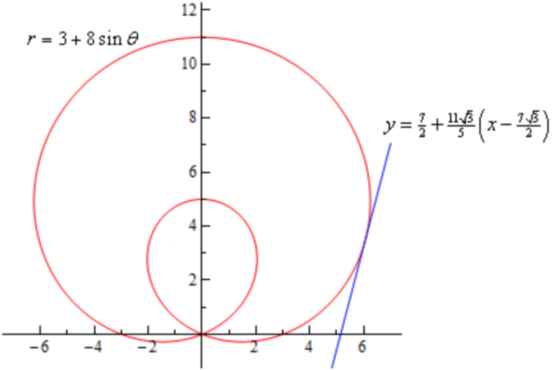
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$ We'll need to get the corresponding $x - y$ coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.



Example 2.20

Find the points on the polar curve $r(\theta) = 1 + \cos\theta,\ 0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal.

The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$
$x = r(\theta)\cos\theta \Rightarrow x = \cos\theta(1 + \cos\theta) = \cos\theta + \cos^2\theta$
$y = r(\theta)\sin\theta \Rightarrow y = \sin\theta(1 + \cos\theta) = \sin\theta + \sin\theta\cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$
$\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta = -\sin\theta - \sin 2\theta$
$\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$
$\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta + \cos 2\theta = 0 \Rightarrow 2\cos^2\theta - 1 + \cos\theta = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = -1$ or $\cos\theta = \frac{1}{2}$
$\Rightarrow \theta = \pi$ or $\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$

For $\theta = \pi$, $\frac{dx}{d\theta} = 0$.
For $\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3} \in [0, 2\pi]$ and $\frac{dx}{d\theta} \neq 0$.
At $\theta = \frac{\pi}{3}$: $r(\pi 3) = 1 + \frac{1}{2} = \frac{3}{2}$
At $\theta = \frac{5\pi}{3}$: $r(\frac{5\pi}{3}) = 1 + \frac{1}{2} = \frac{3}{2}$
The points on $r(\theta) = 1 + \cos\theta, 0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal are $(\frac{3}{2}, \frac{\pi}{3}), (\frac{3}{2}, \frac{5\pi}{3})$

Area Inside-Between Polar Curves

The area of the region bounded by the graphs of the polar curves $r = r(\theta), \theta = \theta_1$ and $\theta = \theta_2$ is

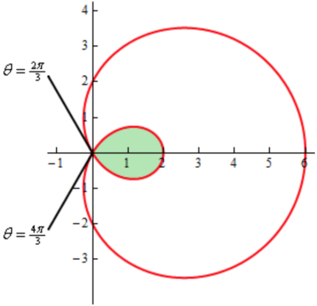
$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$$

Example 2.21

Determine the area of the inner loop of $r = 2 + 4\cos\theta$

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



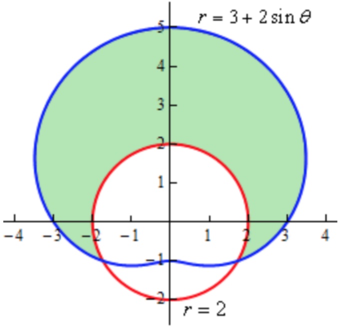
$$
\begin{aligned}
A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2 \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4+16\cos\theta+16\cos^2\theta\right) \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2+8\cos\theta+4\left(1+\cos\left(2\theta\right)\right) \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6+8\cos\theta+4\cos\left(2\theta\right) \, d\theta \\
&= \left(6\theta+8\sin\theta+2\sin\left(2\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \\
&= 4\pi-6\sqrt{3} = 2.174
\end{aligned}
$$

Example 2.22

Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$



$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left((3+2\sin\theta)^2 - (2)^2\right)\, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(5+12\sin\theta + 4\sin^2\theta\right)\, d\theta$$

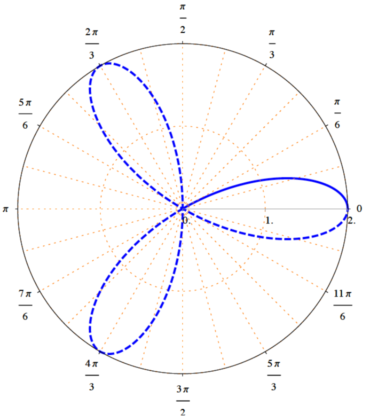
$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(7+12\sin\theta - 2\cos(2\theta)\right)\, d\theta$$

$$= \frac{1}{2}\left(7\theta - 12\cos\theta - \sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187$$

Example 2.23

Find the area inside one leaf of the rose curve $r = 2\cos 3\theta$



The rose curve $r = 2\cos 3\theta, 0 \leq \theta \leq \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$
$r = 0 \Rightarrow 2\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$ Since the desired area is symmetric with respect to the polar axis , then

$$
\begin{aligned}
A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos 3\theta)^2 d\theta\right) \\
&= 4\int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta \\
&= 4\int_0^{\frac{\pi}{6}} \frac{1}{2}(1+\cos 6\theta)\, d\theta \\
&= 2\int_0^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta \\
&= 2\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{6}} = \frac{\pi}{3}
\end{aligned}
$$

Arc Length Of A Polar Curve

The arc length of the polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 2.24

Determine the length of the following polar curve.
$r = -4\sin\theta,\ 0 \leq \theta \leq \pi$

$\frac{dr}{d\theta} = -4\cos\theta$

$$L == \int_0^{\pi} \sqrt{[-4\sin\theta]^2 + [-4\cos\theta]^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{16\sin^2\theta + 16\cos^2\theta}\, d\theta = 4\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_0^{\pi} 4d\theta$$

$$L = \int_0^{\pi} 4\, d\theta = [4\theta]_0^{\pi} = 4\pi$$

Example 2.25

Find the arc length of the following polar curve: $r = e^{-\theta}$

$\frac{dr}{d\theta} = -e^{-\theta}$

$$L = = \int_0^{\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta = \int_0^{\pi} \sqrt{2e^{-2\theta}}\, d\theta = \sqrt{2} \int_0^{\pi} e^{-\theta}\, d\theta$$

$$L = \sqrt{2}\left[-e^{-\theta}\right]_0^{\pi} = \sqrt{2}[-e^{-\pi} + e^0] = \sqrt{2}(1 - e^{-\pi})$$

Surface Area Generated By Revolving A Polar Curve

The surface area generated by revolving the polar curve $r = r(\theta),\ \theta_1 \le \theta \le \theta_2$ around the polar axis is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The surface area generated by revolving the polar curve $r = r(\theta),\ \theta_1 \le \theta \le \theta_2$ around the line $\theta = \frac{\pi}{2}$ is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 2.26

Find the surface area generated by revolving the following polar curve: $r = 2 + 2\cos\theta, 0 \le \theta \le \frac{\pi}{2}$ *around the polar axis.*

$\frac{dr}{d\theta} = -2\sin\theta$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |(2 + 2\cos\theta)\sin\theta| \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)\sin\theta \sqrt{4\,(2 + 2\cos\theta)}\, d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)^{\frac{3}{2}} \sin\theta\, d\theta$$

$$= -2\pi \int^{\frac{\pi}{2}} (2 + 2\cos\theta)^{\frac{3}{2}} \left(-2\sin\theta\right)\, d\theta$$

$$SA = -2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)^{\frac{3}{2}} \, (-2\sin\theta) \, d\theta$$

$$= -2\pi \left[\frac{2}{5} (2 + 2\cos\theta)^{\frac{5}{2}} \right]_0^{\frac{\pi}{2}}$$

$$= -2\pi \frac{2}{5} \left[4\sqrt{2} - 32 \right] = \frac{16\pi}{5} \left(8 - \sqrt{2} \right)$$

Example 2.27

Find the surface area generated by revolving the following polar curve: $r = 2\sin\theta, 0 \leq \theta \leq \frac{\pi}{2}$ *around the line* $\theta = \frac{\pi}{2}$

$\frac{dr}{d\theta} = 2\cos\theta$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |2\sin\theta\cos\theta| \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin 2\theta \sqrt{4}\, d\theta$$

$$SA = 4\pi \left[-\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 4\pi$$