INTEGRAL CALCULUS (MATH 106)

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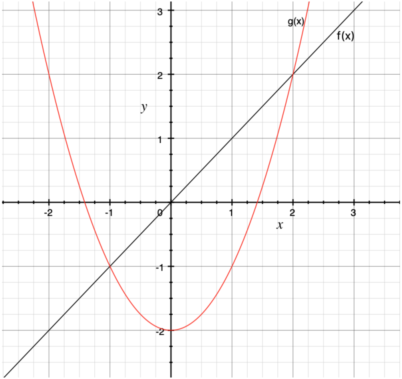
Area Between Two Curves

In this section we are going to look at finding the area between two curves.
we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$
We are also going to assume that $f(x) \geq g(x)$.

$$A = \int_a^b \text{(upper function) - (lower function) } dx, \qquad a \le x \le b$$



$$A = \int_a^b f(x) - g(x) \; dx$$

Area Between Two Curves (Example)

Example 2.1

Find the area enclosed between the graphs $y = x$ and $y = x^2 - 2$.

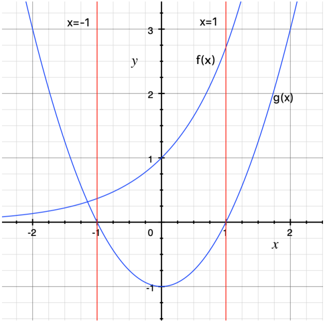
Note that upper function is $y = x$ and lower function is $y = x^2 - 2$
Note that $y = x^2 - 2$ is a parabola opens upward with vertex $(0, -2)$, and $y = x$ is a straight line passing through the origin.
Points of intersection between $y = x^2 - 2$ and $y = x$ is:
$x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0$
$\Rightarrow x = -1$ and $x = 2$

$$A = \int_{-1}^{2} x - (x^2 - 2)\, dx = \int_{-1}^{2} x - x^2 + 2\, dx = [\frac{x^2}{2} - \frac{x^3}{3} + 2x]_{-1}^{2} = \frac{27}{6}$$

Area Between Curves (Example)

Example 2.2

Find the area enclosed between the graphs $y = e^x, y = x^2 - 1, x = -1,$ *and* $x = 1$



Area Between Curves (Example)

Note that upper function is $y = e^x$ and lower function is $y = x^2 - 1$

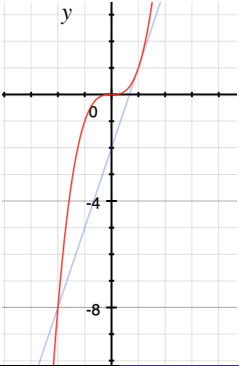
$$A = \int_{-1}^{1} e^x - (x^2 - 1)\ dx = \int_{-1}^{1} e^x - x^2 + 1\ dx = [e^x - \tfrac{1}{3}x^3 + x]_{-1}^{1}$$

$$= e - \frac{1}{e} + \frac{4}{3}$$

Area Between Curves (Example)

Example 2.3

Compute the area oh the region bounded by the curves $y = x^3$ *and* $y = 3x - 2$



Area Between Curves (Example)

Note that upper function is $y = x^3$ and lower function is $y = 3x - 2$

Points of intersection between $y = x^3$ and $y = 3x - 2$

$x^3 - 3x + 2 = 0 \Rightarrow (x-1)(x^2 + x - 2) = 0 \Rightarrow x = -2$ and $x = 1$

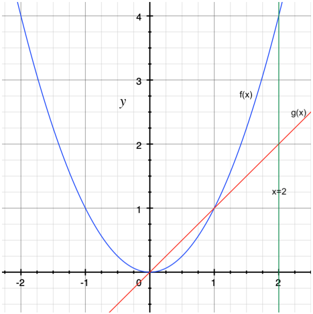
$$A = \int_{-2}^{1} x^3 - (3x-2)\ dx = \int_{-2}^{1} x^3 - 3x + 2\ dx = [\tfrac{x^4}{4} - \tfrac{3}{2}x^2 + 2x]_{-2}^{1}$$

$$= \tfrac{3}{4} + 6 = \tfrac{27}{4}$$

Area Between Curves (Example)

Example 2.4

Find the area enclosed between the graphs $f(x) = x^2$ *and* $g(x) = x$ *between* $x = 0$, *and* $x = 2$.



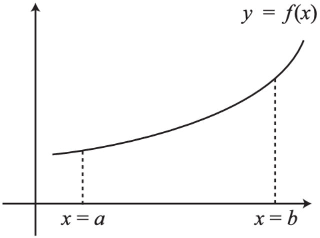
Area Between Curves (Example)

we see that the two graphs intersect at $(0,0)$ and $(1,1)$.
In the interval $[0,1]$, we have $g(x) = x \geq f(x) = x^2$,
and in the interval $[1,2]$, we have $f(x) = x^2 \geq g(x) = x$
Therefore the desired area is:

$$\int_0^1 (x - x^2)\ dx + \int_1^2 (x^2 - x)\ dx = [\tfrac{x^2}{2} - \tfrac{x^3}{0}]_0^1 + [\tfrac{x^3}{3} - \tfrac{x^2}{2}]_1^2$$
$$= \tfrac{1}{6} + \tfrac{5}{6} = 1$$

Volume Of A Solid Revolution (The Disk Method)

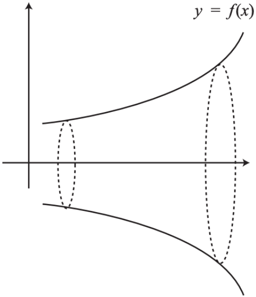
Suppose we have a curve $y = f(x)$



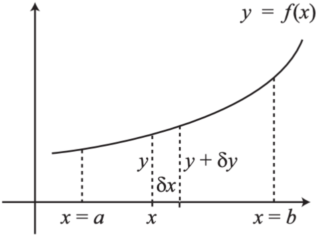
Imagine that the part of the curve between the ordinates $x = a$ and $x = b$ is rotated about the x-axis through 360 degree.

Volume Of A Solid Revolution (The Disk Method)

Now if we take a cross-section of the solid, parallel to the y-axis, this cross-section will be a circle.



But rather than take a cross-section, let us take a thin disc of thickness δx, with the face of the disc nearest the y-axis at a distance x from the origin.



The radius of this circular face will then be y. The radius of the other circular face will be $y + deltay$, where δy is the change in y caused by the small positive increase in $x, \delta x$.

The volume δV of the disc is then given by the volume of a cylinder, $\pi r^2 h$, so that

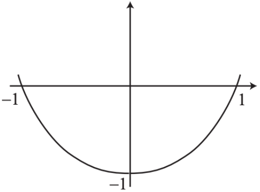
$$\delta V = \pi r^2 \delta x$$

So the volume V of the solid of revolution is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \delta V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \pi \int_a^b [f(x)]^2 \, dx$$

Example 3.1

The curve $y = x^2 - 1$ is rotated about the x-axis through 360 degree. Find the volume of the solid generated when the area contained between the curve and the x-axis is rotated about the x-axis by 360 degree.



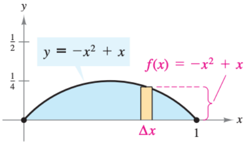
The graph of $y = x^2 - 1$

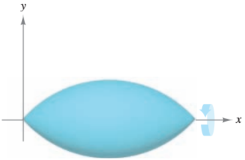
$$\pi \int_a^b [f(x)]^2 \, dx = \pi \int_{-1}^{1} [x^2 - 1]^2 \, dx = \pi \int_{-1}^{1} (x^4 - 2x^2 + 1) \, dx$$

$$= [\frac{x^5}{5} - \frac{2x^3}{3} + x]_{-1}^{1} = \frac{16\pi}{15}$$

Example 3.2

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + x$ and the x-axis about the x-axis.





Using the Disk Method, you can find the volume of the solid of revolution.

$$V = \pi \int_0^1 [f(x)]^2 \, dx = \pi \int_0^1 [(-x^2 + x)^2 \, dx = \pi \int_0^1 (x^4 - 2x^3 + x^2) \, dx$$

$$= \pi [\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3}]_0^1 = \frac{\pi}{30}$$

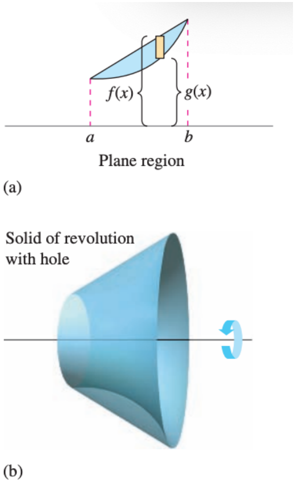
Volume Of A Solid Revolution (The Washer Method)

The Washer Method

Let f and g be continuous and nonnegative on the closed interval $[a, b]$, if $f(x) \geq g(x)$ for all x in the interval, then the volume of the solid formed by revolving the region bounded by the graphs of $f(x)$ and $g(x)$ $(a \leq x \leq b)$, about the x-axis is:

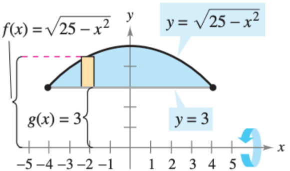
$$V = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}\, dx$$

$f(x)$ is the **outer radius** and $g(x)$ is the **inner radius**.



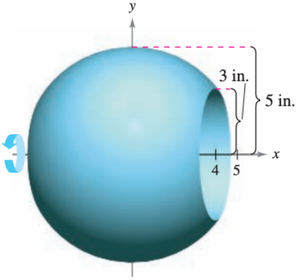
Example 3.3

Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = \sqrt{25 - x^2}$ *and* $g(x) = 3$



Plane region

(a)



Solid of revolution

(b)

First find the points of intersection of f and g, by setting $f(x)$ equal to $g(x)$ and solving for x.

$\sqrt{25-x^2} = 3 \Rightarrow 25 - x^2 = 9 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Using $f(x)$ as the outer radius and $g(x)$ as the inner radius, you can find the volume of the solid as shown.

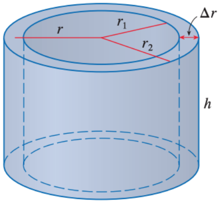
$$V = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}\ dx = \pi \int_{-4}^{4} (\sqrt{25-x^2})^2 - (3)^2\ dx$$

$$= \pi \int_{-4}^{4} (16 - x^2)\ dx = \pi[16x - \frac{x^3}{3}]_{-4}^{4} = \frac{256\pi}{3}$$

Volume Of A Solid Revolution (Cylindrical shells method)

The method of cylindrical shells

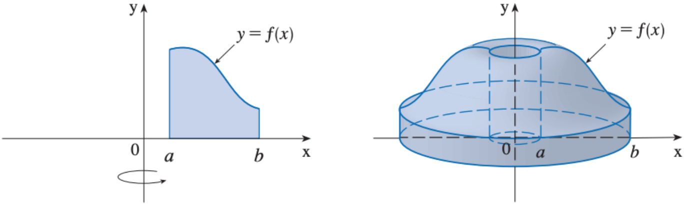
the cylindrical shell with inner radius r_1, outer radius r_2 , and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:



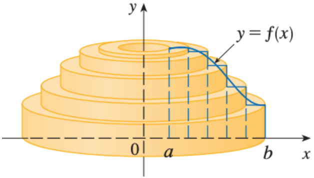
$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h = \pi(r_2 - r_1)(r_2 + r_1)h$$
$$= 2\pi \frac{r_2+r_1}{2} h(r_2 - r_1) \Rightarrow V = 2\pi rh\Delta r$$

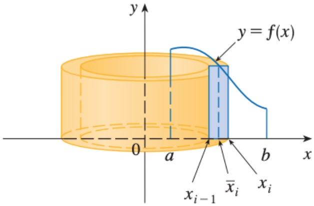
Volume Of A Solid Revolution (Cylindrical shells method)

let be the solid obtained by rotating about the -axis the region bounded by $y = f(x)$,
where $f(x) \geq 0$, $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$.



Volume Of A Solid Revolution (Cylindrical shells method)





We divide the interval into n subintervals $[x_{i-1}, x_{i+1}]$ of equal width and let $\overline{x_i}$ be the midpoint of the i th subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\overline{x}_i)$ is rotated about the $y-$ axis then the result is a cylindrical shell with average radius $\overline{x}_i$ height $f(\overline{x}_i)$ and thickness Δx so its volume is:

$$V_i = (2\pi)\overline{x}_i[f(\overline{x}_i)]\Delta x$$

Volume Of A Solid Revolution (Cylindrical shells method)

An approximation to the volume of is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x$$

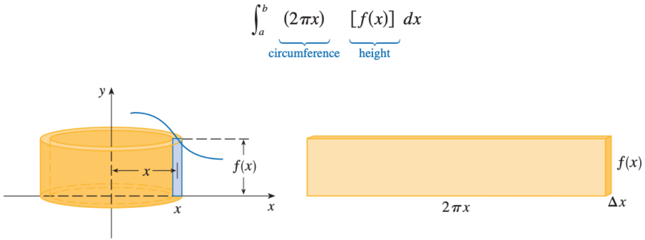
This approximation appears to become better as $n \to \infty$ But, from the definition of an integral, we know that

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x = \int_a^b 2\pi x f(x)\, dx$$

The volume of the solid, obtained by rotating about the $y-$axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)dx \qquad \textit{where} \quad 0 \leq a < b$$

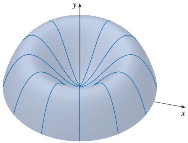
The best way to remember the last Formula is to think of a typical shell, cut and flattened as in Figure with radius x, circumference $2\pi x$, height $f(x)$ and thickness Δx or dx :

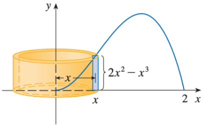


Cylindrical shells method (Examples)

Example 3.4

Find the volume of the solid obtained by rotating about the $y-$axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$





by the shell method, the volume is

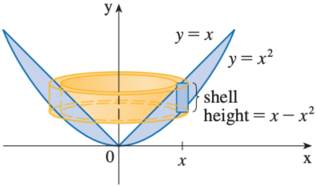
$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\ dx = 2\pi \int_0^2 (2x^3 - x^4)\ dx = 2\pi[\frac{x^4}{2} - \frac{x^5}{5}]_0^2$$

$$= 2\pi(8 - \frac{32}{5}) = \frac{16}{5}\pi$$

Example 3.5

Find the volume of the solid obtained by rotating about the $y-$axis the region between $y = x$ and $y = x^2$.

$$V = \int_0^1 (2\pi x)(x - x^2)\ dx$$
$$= 2\pi \int_0^1 (x^2 - x^3)\ dx$$
$$= 2\pi [\frac{x^3}{3} - \frac{x^4}{4}]_0^1 = \frac{\pi}{6}$$

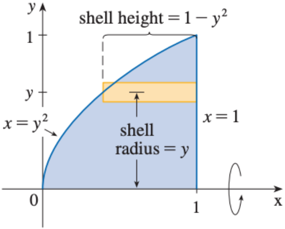


Example 3.6

Use cylindrical shells to find the volume of the solid obtained by rotating about the $x-$axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

For rotation about the $x-$axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

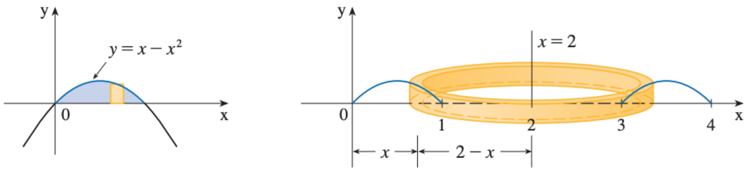
$$V = \int_0^1 (2\pi y)(1 - y^2)\, dy$$
$$= 2\pi \int_0^1 (y - y^3)\, dy$$
$$= 2\pi[\frac{y^2}{2} - \frac{y^4}{4}]_0^1 = \frac{\pi}{2}$$



Example 3.7

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ *and* $y = 0$ *about the line* $x = 2$.

the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.



$$V = \int_0^1 2\pi(2 - x)(x - x^2)\ dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\ dx$$
$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$

Arc Length

Definition 4.1

1. *If* $f(x)$ *is continuous function on the interval* $[a,b]$, *then the arc length of* $f(x)$ *from* $x=a$ *to* $x=b$ *is:*

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

2. *If* $g(y)$ *is continuous function on the interval* $[c,d]$, *then the arc length of* $g(y)$ *from* $y=c$ *to* $y=d$ *is:*

$$L=\int_c^d \sqrt{1+[g'(y)]^2}\,dy$$

Arc Length (Example)

Example 4.1

Determine the length of $y = \ln(\sec x)$ *between* $0 \leq x \leq \frac{\pi}{4}$

$$f'(x) = \frac{\sec x \tan x}{\sec c} = \tan x \Rightarrow [f'(x)]^2 = \tan^2 x$$

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

The arc length is then,

$$\int_0^{\frac{\pi}{4}} \sec x \; dx = [\ln|\sec x + \tan x|]_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1)$$

Arc Length (Example)

Example 4.2

Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ *between* $1 \leq y \leq 4$

$\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \Rightarrow \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y - 1} = \sqrt{y}$

The arc length is then,

$$
\begin{aligned}
L &= \int_1^4 \sqrt{y}\, dy \\
&= \frac{2}{3} y^{\frac{3}{2}} \Big|_1^4 \\
&= \frac{14}{3}
\end{aligned}
$$

Arc Length (Example)

Example 4.3

Determine the length of $x = \frac{1}{2}y^2$ *between* $0 \leq x \leq \frac{1}{2}$. *Assume that* y *is positive.*

$\frac{dx}{dy} = y \quad \Rightarrow \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$

Before writing down the length notice that we were given x limits and we will need y limits. $0 \leq y \leq 1$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

Arc Length (Example)

$$L = \int_0^1 \sqrt{1+y^2}\,dy$$

This integral will require the following trig substitution.

$y = \tan\theta \qquad dy = \sec^2\theta\, d\theta$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$

$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1+y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

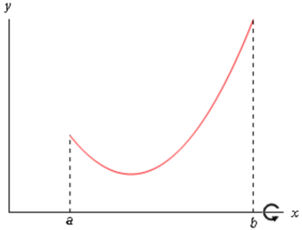
Arc Length (Example)

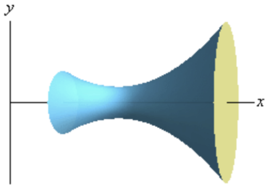
The length is then,

$$\begin{aligned}
L &= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta \\
&= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right)
\end{aligned}$$

Area of a Surface of Revolution

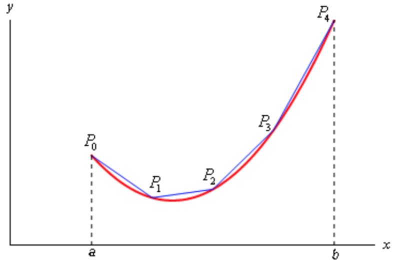
Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the $x-$axis as shown in the following figure.





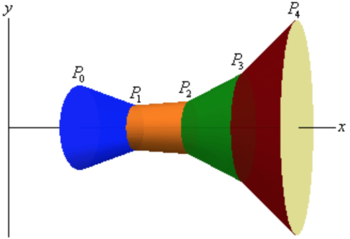
Area of a Surface of Revolution

1. We'll start by dividing the interval into n equal subintervals of width Δx
2. On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of each interval.
3. Here is a sketch of that for our representative function using $n = 4$



Area of a Surface of Revolution

Now, rotate the approximations about the $x-$axis and we get the following solid.



The approximation on each interval gives a distinct portion of the solid and to make this clear each portion is colored differently.

Area of a Surface of Revolution

The area of each of these is:

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad r_1 = \text{radius of right end}$$

$$r_2 = \text{radius of left end}$$

and l is the length of the slant of each interval.

Area of a Surface of Revolution

We know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1+[f'(x_i^*)]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

Area of a Surface of Revolution

So, the surface area of each interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi \left(\frac{f(x_i) + f(x_{i-1})}{2} \right) |P_{i-1}\ P_i|$$
$$\approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

Area of a Surface of Revolution

and we can get the exact surface area by taking the limit as n goes to infinity.

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \; \Delta x$$
$$= \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

If we wanted to we could also derive a similar formula for rotating $x = h(y)$ on $[c, d]$ about the $y-$axis. This would give the following formula.

$$S = \int_c^d 2\pi \, h(y) \sqrt{1 + [h'(y)]^2} \, dy$$

Area of a Surface of Revolution (Example)

Example 4.4

Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}, -2 \leq x \leq 2$ *about the* $x-$*axis.*

$$S = \int_c^d 2\pi \, h(y) \sqrt{1 + \left[h'(y)\right]^2} \, dy$$

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{(9 - x^2)^{\frac{1}{2}}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

Here's the integral for the surface area,

$$S = \int^{2} 2\pi y \frac{3}{\quad} \, dx$$

Area of a Surface of Revolution (Example)

There is a problem however. The dx means that we shouldn't have any y's in the integral. So, before evaluating the integral we'll need to substitute in for y as well.

$$\begin{aligned} S &= \int_{-2}^{2} 2\pi\sqrt{9-x^2}\frac{3}{\sqrt{9-x^2}}\,dx \\ &= \int_{-2}^{2} 6\pi\,dx \\ &= 24\pi \end{aligned}$$

Area of a Surface of Revolution (Example)

Example 4.5

Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}, 1 \leq y \leq 2$ *about the* $y-$*axis.*

Solution

$$x = y^3 \qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9y^4}$$

The surface area is then,

$$S = \int_1^2 2\pi x \sqrt{1 + 9y^4}\, dy$$

we'll need to substitute in for the x. Doing that gives,

$$S = \int_1^2 2\pi y^3 \sqrt{1+9y^4}\,dy \qquad\qquad u = 1+9y^4$$
$$= \frac{\pi}{18}\int_{10}^{145} \sqrt{u}\,du$$
$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$

Parametric equations

To this point we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$
It is easy to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

Parametric equations

$y = \sqrt{r^2 - x^2}$	(top)	$x = \sqrt{r^2 - y^2}$	(right side)
$y = -\sqrt{r^2 - x^2}$	(bottom)	$x = -\sqrt{r^2 - y^2}$	(left side)

There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations.**

Parametric equations

Instead of defining y in terms of x, $y = f(x)$ or x in terms of y $x = h(y)$ we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad y = g(t)$$

This third variable is usually denoted by t.
Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve.**

Parametric equations (Example)

Example 4.6

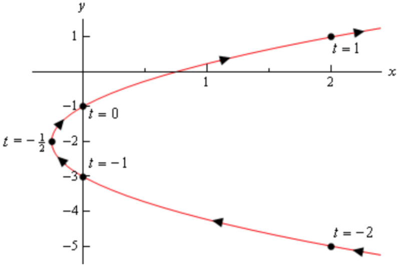
Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \quad -2 \leq t \leq 2$$

At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

Parametric equations (Example)

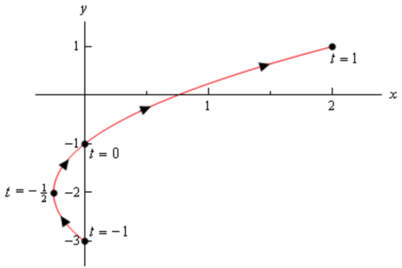
t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1



Example 4.7

Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \leq t \leq 1$$



The slope of the tangent line to a parametric curve

If $C : x = x(t), y = y(t); a \leq t \leq b$ is a differentiable parametric curve then the slope of the tangent line to C at $t_0 \in [a, b]$ is:

$$m = \frac{dy}{dx}|_{t=t_0} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}|_{t=t_0}$$

Remark

1. The tangent line to the parametric curve is horizontal if the slope equals zero, which means that $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$
2. The tangent line to the parametric curve is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

The slope of the tangent line to a parametric curve (Example)

Example 4.8

Find the slope of the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad y = t^2 \qquad at \;\; (0, 4)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

$$0 = t^5 - 4t^3 = t^3\left(t^2 - 4\right) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

1. at $t = -2$:

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

2. at $t = 2$

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

Example 4.9

Find the equation of the tangent line to
$C : x = t^3 - 3t,\ \ y = t^2 - 5t$ *at* $t = 2$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t-5}{3t^2-3}$$

The slope of the tangent line is $\frac{dy}{dx}|_{t=2} = -\frac{1}{9}$
At $t = 2 : x = 2$ and $y = -7$
The tangent line to C at $t = 2$ passes through the point $(2, -7)$ and its slope is $-\frac{1}{9}$
therefore its equation is $\frac{y+7}{x-2} = -\frac{1}{9}$

Example 4.10

Find the points on $C : x = e^t,\ y = e^{-t}$ *at which the slope of the tangent line to* C *equals* $-e^{-2}$

$$m = \frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{-e^{-t}}{e^t} = -e^{-2t}$$

$\Rightarrow m = e^{-2t} \Rightarrow e^{-2t} = -e^{-2} \Rightarrow t = 1.$

At $t = 1 : x = e^1 = e$ and $y = e^{-1} = \frac{1}{e}.$

Hence, the point at which the slope of the tangent line to C equals $-e^{-2}$ is $(e, \frac{1}{e})$

Arc Length of a Parametric Equations

Definition 4.2

If $C : x = x(t),\ y = y(t); a \leq t \leq b$ *is a differentiable parametric curve ,then its arc length equals*

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Example 4.11

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \leq t \leq 2\pi$$

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(3t) + 81\cos^2(3t)}\ dt \\ &= \int_0^{2\pi} 9\ dt \\ &= 18\pi \end{aligned}$$

Example 4.12

Determine the length of the parametric curve given by the following set of parametric equations.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + (8-t)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad \frac{dy}{dt} = -\frac{3}{2}(8-t)^{\frac{1}{2}}$$

$$L = \int_0^4 \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\frac{3}{2}(8-t)^{\frac{1}{2}}\right]^2}\, dt = \int_0^4 \sqrt{144t + \frac{9}{4}(8-t)}\, dt$$

$$= \int_0^4 \sqrt{\frac{567}{4}t + 18}\, dt = \frac{4}{567}\left(\frac{2}{3}\right)\left(\frac{567}{4}t + 18\right)^{\frac{3}{2}}\Bigg|_0^4$$

$$= \frac{8}{1701}\left(585^{\frac{3}{2}} - 18^{\frac{3}{2}}\right) = 66.1865$$

Surface Area Generated By Revolving A Parametric Curve

If $C : x = x(t),\ y = y(t); a \leq t \leq b$ is a differentiable parametric curve ,then the surface area generated by revolving C around the $x-$axis is

$$SA = 2\pi \int_a^b |y(t)| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

The surface area generated by revolving C around the $y-$axis is

$$SA = 2\pi \int_a^b |x(t)| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Example 4.13

Determine the surface area of the solid obtained by rotating the following parametric curve about the $x-$axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{d\theta} = -3\cos^2\theta \sin\theta \qquad \frac{dy}{d\theta} = 3\sin^2\theta \cos\theta$$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\, d\theta$$
$$= 3\left|\cos\theta \sin\theta\right| \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 3\cos\theta \sin\theta$$

$$
\begin{aligned}
SA &= 2\pi \int_0^{\frac{\pi}{2}} \sin^3\theta \left(3\cos\theta \sin\theta\right) \, d\theta \\
&= 6\pi \int_0^{\frac{\pi}{2}} \sin^4\theta \cos\theta \, d\theta \qquad\qquad u = \sin\theta \\
&= 6\pi \int_0^{1} u^4 \, du \\
&= \frac{6\pi}{5}
\end{aligned}
$$

Example 4.14

Determine the surface area of the object obtained by rotating the parametric curve about the y−axis.

$$x = 3\cos(\pi t) \qquad y = 5t + 2 \qquad 0 \le t \le \frac{1}{2}$$

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -3\pi\sin(\pi t) \qquad \frac{dy}{dt} = 5$$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{[-3\pi\sin(\pi t)]^2 + [5]^2} = \sqrt{9\pi^2\sin^2(\pi t) + 25}$$

$$SA = \int_0^{\frac{1}{2}} 2\pi\left(3\cos(\pi t)\right)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$

$$= 6\pi \int_0^{\frac{1}{2}} \cos(\pi t)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$

$$u = \sin(\pi t) \quad \rightarrow \quad \sin^2(\pi t) = u^2 \qquad du = \pi\cos(\pi t)$$

$$t = 0: \quad u = \sin(0) = 0 \qquad t = \frac{1}{2}: \quad u = \sin\left(\frac{1}{2}\pi\right) = 1$$

$$SA = 6\int_0^1 \sqrt{9\pi^2 u^2 + 25}\,du$$

$$u = \frac{5}{3\pi}\tan\theta \qquad du = \frac{5}{3\pi}\sec^2\theta\, d\theta$$

$$\sqrt{9\pi^2 u^2 + 25} = \sqrt{25\tan^2\theta + 25} = 5\sqrt{\tan^2\theta + 1} = 5\sqrt{\sec^2\theta} = 5\left|\sec\theta\right|$$

$$u = 0 : 0 = \frac{5}{3\pi}\tan\theta \qquad \to \tan\theta = 0 \qquad \to \qquad \theta = 0$$

$$u = 1 : 1 = \frac{5}{3\pi}\tan\theta \qquad \to \tan\theta = \frac{3\pi}{5} \to \theta = \tan^{-1}\left(\frac{3\pi}{5}\right) = 1.0830$$

$$
\begin{aligned}
SA &= \int_0^{\frac{1}{2}} 2\pi \left(3\cos\left(\pi t\right)\right) \sqrt{9\pi^2 \sin^2\left(\pi t\right) + 25}\, dt \\
&= 6\int_0^{1} \sqrt{9\pi^2 u^2 + 25}\, du \\
&= 6\int_0^{1.0830} \left(5\sec\theta\right)\left(\frac{5}{3\pi}\sec^2\theta\right)\, d\theta \\
&= 6\int_0^{1.0830} \frac{25}{3\pi}\sec^3\theta\, d\theta \\
&= \frac{25}{\pi}\left(\sec\theta\, tan\theta + ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{1.0830} = 43.0705
\end{aligned}
$$