INTEGRAL CALCULUS (MATH 106)

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1. Half-Angle Substitution

2. Improper Integrals

Half-Angle Substitution

It is used to solve integrals of rational functions involving $\sin x$ or $\cos x$

Example 2.1

$\int \frac{1}{2+\cos x} dx$, *and* $\int \frac{1}{1-\sin x} dx$

Question

How to solve an integral using half angle trigonometric substitution?

To solve this type of integral we have to concentrate on:

1. $u = \tan\frac{x}{2}$
2. $\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$
3. $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-u^2}{1+u^2}$
4. $\frac{du}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(\tan^2\frac{x}{2} + 1) = \frac{1}{2}(u^2 + 1) \Rightarrow dx = \frac{2}{(u^2+1)}du$

Example 2.2

Evaluate $\int \frac{1}{2+\cos x}dx$ *and* $\int \frac{1}{1-\sin x}dx$

to solve $\int \frac{1}{2+\cos x}dx$

we put $u = \tan\frac{x}{2}$ So $\cos x = \frac{1-u^2}{1+u^2}$, and $dx = \frac{2}{(u^2+1)}du$

$\int \frac{1}{2+\cos x}dx = \int \frac{2}{(2+\frac{1-u^2}{1+u^2})(u^2+1)}du = \int \frac{2}{u^2+3}du = \frac{2}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{\frac{x}{2}}{\sqrt{3}} + c$

To solve $\int \frac{1}{1-\sin x}dx$

We put $u = \tan\frac{x}{2}$, $\sin x = \frac{2u}{1+u^2}$ and $dx = \frac{2}{(u^2+1)}du$

$\int \frac{1}{1-\sin x}dx = \int \frac{2}{1-\frac{2u}{1+u^2}(1+u^2)}du = -\frac{2}{(u-1)} + c = \frac{-2}{tan\frac{x}{2}-1} + c$

Example 2.3

How we can integrate using tangent half angle substitution.
$\int \frac{1}{3-5\sin x} dx$

We put $u = \tan \frac{x}{2}, \sin x = \frac{2u}{u^2+1}$ and $dx = \frac{2}{u^2+1}$
Hence, the given integral becomes:

$$\int \frac{1}{3-5\sin x} dx = \int \frac{\frac{2}{u^2+1}}{3-5(\frac{2u}{u^2+1})} du = \int \frac{2}{3u^2-10u+3} du$$

Now, we need to do partial fraction decomposition.

$$\frac{2}{3u^2-10u+3} = \frac{2}{(u-3)(3u-1)} = \frac{A}{u-3} + \frac{B}{3u-1}$$

$2 = A(3u-1) + B(u-3) \Rightarrow 2 = (3A+B)u - A - 3B$

$3A + B = 0$

$-A - 3B = 2$

$A = \frac{1}{4}$, and $B = -\frac{3}{4}$

$$\int \frac{1}{3-5\sin x}dx = \int \frac{2}{3u^2-10u+3}du = \int \frac{1}{4(u-3)}\ du - \int \frac{3}{4(3u-1)}\ du$$

$$= \frac{1}{4}\ln|u-3| - \frac{3}{4}\frac{1}{3}\ln|3u-1| + c$$

$$\int \frac{1}{3-5\sin x}dx = \frac{1}{4}\ln|\tan^{-1}\frac{x}{2}-3| - \frac{1}{4}\ln|3\tan^{-1}\frac{x}{2}-1| + c$$

1. Integrals involving fraction powers of x.
   $\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx$, to solve this integral we put:
   $u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5du$
2. Integrals involving a square root of a linear factor.
   $\int \frac{1}{(x+1)\sqrt{x-2}}dx$, to solve this integral we put:
   $u = \sqrt{x-2} \Rightarrow x = u^2 + 2 \Rightarrow dx = 2udu$

Improper Integrals with a discontinuous integrand

Definition 3.1

1. *If f is continuous on $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$ then*
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$
2. *If f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$ then*
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

If the limit exists (and equals a value L) then the improper integral **converges** *(to L).*
If the limit does not exist then the improper integral **diverges**.

Remark

If f is continuous on $[a, b]$ except at a point $c \in (a, b)$ and $|f(x)| \to \infty$ as $x \to c$ then

$$\int_a^b f(x)\ dx = \lim_{t\to c^-} \int_a^t f(x)\ dx + \lim_{t\to c^+} \int_t^b f(x)\ dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$).

If at least one of the limits does not exist then the improper integral **diverges**.

Improper Integrals with an infinite limit of integration

Definition 3.2

1. *If f is continuous on* $[a, \infty)$ *then*
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$
2. *If f is continuous on* $(-\infty, a$ *then*
$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$$

if the limit exists (and equals a value L) then the improper integral **converges** *(to L).*
If the limit does not exist then the improper integral **diverges**.

Remark

If f is continuous on $(-\infty, \infty)$ then for any constant a

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\ dx + \lim_{t \to \infty} \int_{a}^{t} f(x)\ dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$).

If at least one of the limits does not exist then the improper integral **diverges**.

Examples

1. $\int_1^\infty x^{-2}\,dx$ is an improper integral.
   Some such integrals can sometimes be computed by replacing infinite limits with finite values
   $\int_1^\infty x^{-2}\,dx = \lim_{y\to\infty}\int_1^y x^{-2}\,dx = \lim_{y\to\infty}[-\frac{1}{x}]_1^y = \lim_{y\to\infty}(-\frac{1}{y}+1) = 1$
2. $$\int_1^\infty \frac{1}{x}\,dx = \lim_{c\to\infty}\int_1^c \frac{1}{x}\,dx = \lim_{c\to\infty}[\ln|x|]_1^c$$
   $$= \lim_{c\to\infty}(\ln|c| - \ln|1|) = \infty$$

Examples

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\, dx$$

1. Split the integral in two.
$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+X^2}\, dx + \int_{0}^{\infty} \frac{1}{1+X^2}\, dx$$
2. Turn each part into a limit.
$$= \lim_{c\to-\infty} \int_{c}^{0} \frac{1}{1+X^2}\, dx + \lim_{c\to\infty} \int_{0}^{c} \frac{1}{1+X^2}\, dx$$
3. Evaluate each part and add up the results.
$$= \lim_{c\to-\infty} [\tan^{-1} x]_c^0 + \lim_{c\to\infty} [\tan^{-1} x]_0^c$$
$$= \lim_{c\to-\infty} [\tan^{-1} 0 - \tan^{-1} c] + \lim_{c\to\infty} [\tan^{-1} c - \tan^{-1} 0]$$
$$= (0 - (-\tfrac{\pi}{2})) + (\tfrac{\pi}{2} - 0) = \pi$$

Examples

$\int_1^\infty \frac{1}{x^p}\,dx$, where p is a real number.
We have to consider every possible value of p.

First, for $p = 1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \,|_1^b = \lim_{b\to\infty} \ln b = \infty$$

so the integral diverges when p = 1.

Now, for $p \neq 1$, the power rule applies:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{1}{1-p}x^{1-p}\,|_1^b =$$

$$\lim_{b\to\infty}\left(\frac{1}{1-p}b^{1-p} - \frac{1}{1-p}\right) = \frac{1}{1-p}$$

This means that for $p \leq 1$, the integral diverges, and for $p > 1$, it equals $\frac{1}{1-p}$

Examples

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+16} dx$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+16} dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{x^2+16} dx + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{x^2+16} dx$$

$$= \lim_{a \to -\infty} (0 - \frac{1}{4} tan^{-1} \frac{a}{4}) + \lim_{b \to \infty} (\frac{1}{4} tan^{-1} \frac{b}{4} - 0)$$

$$= \frac{1}{4} \frac{\pi}{2} + \frac{1}{4} \frac{\pi}{2} = \frac{\pi}{4}$$