INTEGRAL CALCULUS (MATH 106)

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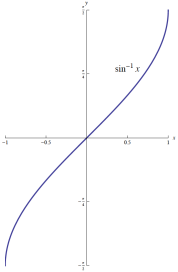
Definition 2.1

The inverse sine function is denoted by $\sin^{-1}$ *and it is defined as* $y = \sin^{-1} x \Leftrightarrow x = \sin y$, *where* $x \in [-1, 1]$ *and* $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

The domain of the inverse sine function is $[-1, 1]$

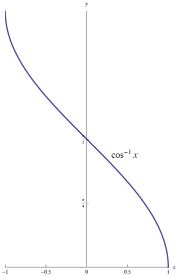
The range of the inverse sine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



Graph of $\sin^{-1} x$

Definition 2.2

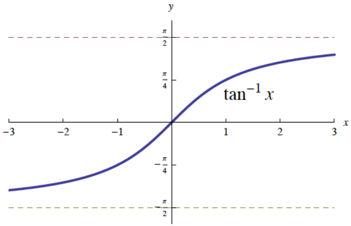
The inverse cosine function is denoted by $\cos^{-1}$ *and it is defined as*
$y = \cos^{-1} x \Leftrightarrow x = \cos y$, *where* $x \in [-1, 1]$ *and* $y \in [0, \pi]$
The domain of the inverse cosine function is $[-1, 1]$
The range of the inverse cosine function is $[0, \pi]$.



Graph of $\cos^{-1} x$

Definition 2.3

The inverse tangent function is denoted by $\tan^{-1}$ *and it is defined as* $y = \tan^{-1} x \Leftrightarrow x = \tan y$, *where* $x \in \mathbb{R}$ *and* $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$
The domain of the inverse tangent function is $\mathbb{R}$
The range of the inverse tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$.



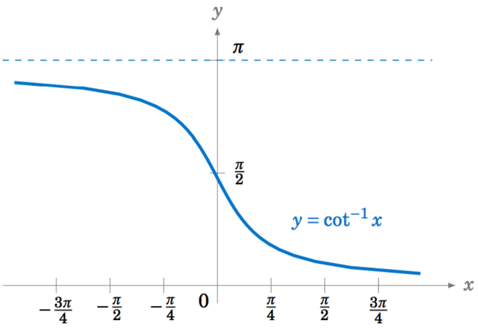
Graph of $\tan^{-1} x$

Definition 2.4

The inverse cotangent function is denoted by $\cot^{-1}$ *and it is defined as* $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, *where* $x \in \mathbb{R}$

The domain of the inverse cotangent function is $\mathbb{R}$

The range of the inverse cotangent function is $(0, \pi)$.

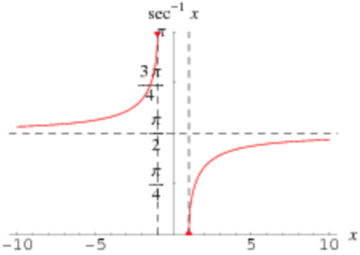


Definition 2.5

The inverse secant function is denoted by $\sec^{-1}$ *and it is defined as* $y = \sec^{-1} x \Leftrightarrow x = \sec y$, *where* $y \in [0, \frac{\pi}{2})$ *if* $x \geq 1$, *and* $y \in [\pi, \frac{3\pi}{2})$ *if* $x \leq -1$

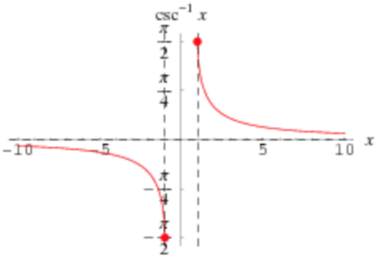
The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

The range of the inverse secant function is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.



Definition 2.6

The inverse cosecant function is denoted by $\csc^{-1}$ *and it is defined as* $\csc^{-1} x = \frac{\pi}{2} - sec^{-1}x$, *where* $|x| \geq 1$
The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$
The range of the inverse cosecant function is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.



Derivatives of the inverse trigonometric functions

1. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$
2. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$
3. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{1-x^2}}$, where $|x| > 1$
6. $\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$, where $|x| > 1$

Integration of the inverse trigonometric functions

1. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}(\frac{x}{a}) + c, \ \ (|x| < a)$
   $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}(\frac{f(x)}{a}) + c, \ \ (|f(x)| < a)$
2. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$
   $\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}(\frac{f(x)}{a}) + c$
3. $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}(\frac{x}{a}) + c, \ \ (|x| > a)$
   $\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}}dx = \frac{1}{a}\sec^{-1}(\frac{f(x)}{a}) + c, \ \ (|f(x)| > a)$

Integration of the inverse trigonometric functions (Examples)

1. $\int \frac{x^2}{5+x^6} dx = \frac{1}{3} \int \frac{3x^2}{(\sqrt{5})^2+(x^3)^2} dx = \frac{1}{3} \frac{1}{\sqrt{5}} \tan^{-1}(\frac{x^3}{\sqrt{5}}) + c$
2. $\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx = \int \frac{(\frac{1}{x})}{\sqrt{(1)^2-(\ln x)^2}} dx = \sin^{-1}(\ln x) + c$
3. $\int \frac{1}{\sqrt{e^{2x}-36}} dx = \int \frac{e^x}{e^x\sqrt{(e^x)^2-(6)^2}} dx = \frac{1}{6} \sec^{-1}(\frac{e^x}{6}) + c$

Integration of the inverse trigonometric functions (Exercises)

Exercise 1

Solve the following integrals :

1. $\int \frac{x+\sin^{-1} x}{\sqrt{1-x^2}} dx$
2. $\int \frac{x+1}{x^2+1} dx$

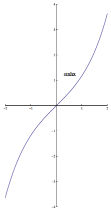
The hyperbolic sine function

Definition 3.1

It is denoted by $\sinh x$ *and it is defined as* $\sinh x = \frac{e^x - e^{-x}}{2}$

Notes:

1. The domain of $\sinh x$ is $\mathbb{R}$ and the range of $\sinh x$ is $\mathbb{R}$.
2. It is an odd function and $\sinh(0) = 0$



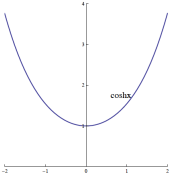
The hyperbolic cosine function

Definition 3.2

It is denoted by $\cosh x$ *and it is defined as* $\cosh x = \frac{e^x + e^{-x}}{2}$

Notes:

1. The domain of $\cosh x$ is $\mathbb{R}$ and the range of $\cosh x$ is $[1, \infty]$.
2. It is an even function and $\cosh(0) = 1$



Definitions :

1. The hyperbolic tangent function is denoted by $\tanh x$ and it is defined as $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$
2. The hyperbolic cotangent function is denoted by $\coth x$ and it is defined as $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^x}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$
3. The hyperbolic secant function is denoted by $sech\ x$ and it is defined as $sech\ x = \frac{1}{\cosh x} = \frac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R}$
4. The hyperbolic cosecant function is denoted by $csch\ x$ and it is defined as $csch\ x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$

Notes:

1. $cosh^2 x - sinh^2 x = 1$ for every $x \in \mathbb{R}$
2. $1 - tanh^2 x = sech^2 x$ for every $x \in \mathbb{R}$
3. $coth^2 x - 1 = csch^2 x$ for every $x \in \mathbb{R} - \{0\}$

Derivatives of the hyperbolic functions

1. $\frac{d}{dx}\sinh x = \cosh x$, and $\frac{d}{dx}\sinh(f(x)) = \cosh(f(x))f'(x)$
2. $\frac{d}{dx}\cosh x = \sinh x$, and $\frac{d}{dx}\cosh(f(x)) = \sinh(f(x))f'(x)$
3. $\frac{d}{dx} tanh\ x = sech^2\ x$ and $\frac{d}{dx} tanh\ (f(x)) = sech^2\ (f(x))f'(x)$
4. $\frac{d}{dx} coth\ x = -csch^2\ x$ and
   $\frac{d}{dx} coth\ (f(x)) = -csch^2\ (f(x))f'(x)$
5. $\frac{d}{dx} sech\ x = -sech\ x\ tanh\ x$ and
   $\frac{d}{dx} sech\ (f(x)) = -sech\ (f(x))\ tanh\ (f(x))f'(x)$
6. $\frac{d}{dx} csch\ x = -csch\ x\ coth\ x$ and
   $\frac{d}{dx} csch\ (f(x)) = -csch\ (f(x))\ coth\ (f(x))f'(x)$

Integration of the hyperbolic functions

- $\int \sinh x \; dx = \cosh x + c,$
  $\int \sinh(f(x))f'(x)dx = \cosh(f(x)) + c$
- $\int \cosh x \; dx = \sinh x + c,$
  $\int \cosh(f(x))f'(x)dx = \sinh(f(x)) + c$
- $\int sech^2 x \; dx = \tanh x + c$
  $\int sech^2(f(x))f'(x)dx = \tanh(f(x)) + c$
- $\int csch^2 x \; dx = -\coth x + c$
  $\int csch^2(f(x))f'(x)dx = -\coth(f(x)) + c$

Integration of the hyperbolic functions

- $\int sech\ x \tanh x\ dx = -sech\ x + c$
  $\int sech(f(x)) \tanh(f(x)) f'(x) dx = -sech\ (f(x)) + c$
- $\int csch\ x\ coth\ x dx = -csch\ x + c$
  $\int csch\ (f(x))\ coth\ (f(x)) f'(x) dx = -csch\ f(x) + c$
- $\int \tanh x\ dx = \ln|\cosh x| + c$
  $\int \tanh(f(x))\ dx = \ln|\cosh(f(x))| + c$
- $\int \coth x\ dx = \ln|\sinh x| + c$
  $\int \coth(f(x)) f'(x)\ dx = \ln|\sinh(f(x))| + c$

Integration of the hyperbolic functions (Examples)

- $\int x^2 \cosh x^3 \ dx = \frac{1}{3} \int \cosh x^3 \ (3x^2) dx = \frac{1}{3} \sinh x^3 + c$
- $\int (e^x - e^{-x}) sech^2 (e^x + e^{-x}) dx = \tanh(e^x + e^{-x}) + c$
- $\int \frac{\sinh x}{1+\sinh^2 x} dx = \int \frac{\sinh x}{\cosh^2 x} dx = \int \frac{1}{\cosh x} \frac{\sinh x}{\cosh x} dx =$
  $\int sech \ x \tanh x \ dx = -sech \ x + c$
- $\int \frac{1}{sech \ x \sqrt{4 - \sinh^2 x}} dx = \int \frac{\cosh x}{\sqrt{(2)^2 - (\sinh x)^2}} dx = \sin^{-1}(\frac{\sinh x}{2}) + c$

Definitions

- The inverse hyperbolic sine function is denoted by $\sinh^{-1}$ and it is defined as $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$
- The inverse hyperbolic cosine function is denoted by $\cosh^{-1}$ and it is defined as $y = \cosh^{-1} x \Leftrightarrow x = \cosh y$, where $x \in [1, \infty)$ and $y \in [0, \infty)$
- The inverse hyperbolic tangent function is denoted by $\tanh^{-1}$ and it is defined as $y = \tanh^{-1} x \Leftrightarrow x = \tanh y$, where $x \in [-1, 1]$ and $y \in \mathbb{R}$

Definitions

- The inverse hyperbolic cotangent function is denoted by $\coth^{-1}$ and it is defined as $y = \coth^{-1} x \Leftrightarrow x = \coth y$, where $|x| > 1$ and $y \in \mathbb{R}$.
- The inverse hyperbolic secant function is denoted by $sech^{-1}$ and it is defined as $y = sech^{-1}x \Leftrightarrow x = sech\ y$, where $x \in [0, 1]$ and $y \in [0, \infty)$
- The inverse hyperbolic cosecant function is denoted by $csch^{-1}$ and it is defined as $y = csch^{-1}x \Leftrightarrow x = csch\ y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R} - \{0\}$

Derivatives of the inverse hyperbolic functions

1. $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$, $\frac{d}{dx}\sinh^{-1}f(x) = \frac{f'(x)}{\sqrt{1+f((x))^2}}$.
2. $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$, where $x > 1$
   $\frac{d}{dx}\cosh^{-1}f(x) = \frac{f'(x)}{\sqrt{(f(x))^2-1}}$, where $|f(x)| > 1$
3. $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$, where $|x| > 1$
   $\frac{d}{dx}\tanh^{-1}f(x) = \frac{f'(x)}{1-(f(x))^2}$, where$|f(x)| > 1$
4. $\frac{d}{dx}coth^{-1}x = \frac{-1}{1-x^2}$ where $|x| > 1$
   $\frac{d}{dx}coth^{-1}f(x) = \frac{-f'(x)}{1-(f(x))^2}$ where $|f(x)| > 1$
5. $\frac{d}{dx}sech^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$ where $0 < x < 1$
   $\frac{d}{dx}sech^{-1}f(x) = \frac{-f'(x)}{f(x)\sqrt{1-(f(x))^2}}$ where $0 < f(x) < 1$
6. $\frac{d}{dx}csch^{-1}x = \frac{-1}{|x|\sqrt{1+x^2}}$, where $x \neq 0$
   $\frac{d}{dx}csch^{-1}f(x) = \frac{-f'(x)}{\ldots}$, where $f(x) \neq 0$

Derivatives of the inverse hyperbolic functions (Examples)

1. Find $f'(x)$ if $f(x) = \tanh^{-1} 3x$?
   $f'(x) = \frac{3}{1-(3x)^2} = \frac{3}{1-9x^2}$
2. Find $f'(x)$ if $f(x) = \sinh^{-1} \sqrt{x}$?
   $$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}}$$
3. Find $f'(x)$ if $f(x) = sech^{-1}(\cos 2x)$?
   $$f'(x) = \frac{-(-2\sin 2x)}{\cos 2x\sqrt{1-(\cos 2x)^2}} = \frac{2\sin 2x}{\cos 2x\sqrt{1-\cos^2 2x}}$$

Integration of the inverse hyperbolic functions

1. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}(\frac{x}{a}) + c$

   $\int \frac{f'(x)}{\sqrt{a^2+(f(x))^2}} dx = \sinh^{-1}(\frac{f(x)}{a}) + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}} dx = cosh^{-1}(\frac{x}{a}) + c, (x > a)$

   $\int \frac{f'(x)}{\sqrt{(f(x))^2-a^2}} dx = cosh^{-1}(\frac{f(x)}{a}) + c, \ (f(x) > a)$
3. $\int \frac{1}{a^2-x^2} dx = \frac{1}{a} tanh^{-1}(\frac{x}{a}) + c(|x| < a)$

   $\int \frac{f'(x)}{a^2-(f(x))^2} dx = \frac{1}{a} tanh^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| < a)$
4. $\int \frac{1}{x\sqrt{a^2-x^2}} dx = -\frac{1}{a} sech^{-1}(\frac{x}{a}) + c, \ (0 < x < a)$

   $\int \frac{f'(x)}{f(x)\sqrt{a^2-(f(x))^2}} dx = -\frac{1}{a} sech^{-1}(\frac{f(x)}{a}) + c, \ (0 < f(x) < a)$
5. $\int \frac{1}{x\sqrt{x^2+a^2}} dx = -\frac{1}{a} csch^{-1}(\frac{x}{a}) + c, \ (x \neq 0)$

   $\int \frac{f'(x)}{x\sqrt{(f(x))^2+a^2}} dx = -\frac{1}{a} csch^{-1}(\frac{f(x)}{a}) + c, \ (f(x) \neq 0)$

Integration of the inverse hyperbolic functions

1. $\int \frac{e^x}{1-e^{2x}} dx = \int \frac{e^x}{(1)^2-(e^x)^2} dx = \tanh^{-1}(e^x) + c$
2. $\int \frac{1}{\sqrt{x}\sqrt{4+x}} dx = 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(2)^2+(\sqrt{x})^2}} dx = 2\sinh^{-1}(\frac{\sqrt{x}}{2}) + c$
3. $\int \frac{1}{\sqrt{1+e^{2x}}} dx = \int \frac{e^x}{e^x\sqrt{1+e^{2x}}} dx = -csch^{-1}(e^x) + c$

Indeterminate Forms

Theorem (L'Hopital's Rule)

Suppose that f and g are differentiable on the interval (a, b), except possibly at a point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b), except possibly at c. Suppose further that $\lim_{x \to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L(\text{or } \pm\infty)$. Then, $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Remark

The conclusion of the theorem also holds if $\lim_{x \to c} \frac{f(x)}{g(x)}$ is replaced with $\lim_{x \to c^-} \frac{f(x)}{g(x)}$, $\lim_{x \to c^+} \frac{f(x)}{g(x)}$, $\lim_{x \to -\infty} \frac{f(x)}{g(x)}$ or $\lim_{x \to +\infty} \frac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustment of the hypothesis.)

Types of indeterminate forms:

1. $\frac{0}{0}$ or $\frac{\infty}{\infty}$
2. $\infty - \infty$ or $-\infty + \infty$
3. $0.\infty$ or $0(-\infty)$
4. $0^0, 1^\infty, 1^{-\infty}$ or ∞^0

Example 5.1

$$\lim_{x \to 1} \frac{\sqrt{x}}{\ln x} = \frac{0}{0}$$

Apply L'Hopital's rule

$$\lim_{x \to 1} \frac{\sqrt{x}}{\ln x} = \lim_{x \to 1} \frac{(\frac{1}{2\sqrt{x}})}{(\frac{1}{x})} = \lim_{x \to 1} \frac{x}{2\sqrt{x}} = \frac{1}{2}$$

Example 5.2

1. $\lim_{x \to (\frac{\pi}{2})^-} \frac{2-\sec x}{3\tan x} = \frac{-\infty}{\infty}$

   Apply L'Hopital's rule

   $\lim_{x \to (\frac{\pi}{2})^-} \frac{2-\sec x}{3\tan x} = \lim_{x \to (\frac{\pi}{2})^-} \frac{-\sec x \tan x}{3\sec^2 x} = \lim_{x \to (\frac{\pi}{2})^-} \frac{-\tan x}{3\sec x} =$

   $\lim_{x \to (\frac{\pi}{2})^-} \frac{-\sin x}{3} = -\frac{1}{3}$

2. $\lim_{x \to 1^+} (\frac{3}{\ln x} - \frac{2}{x-1}) = (\infty - \infty)$

   $\lim_{x \to 1^+} \frac{3(x-1)-2\ln x}{(x-1)\ln x} = \frac{0}{0}$

   Apply L'Hopital's rule

   $\lim_{x \to 1^+} \frac{3(x-1)-2\ln x}{(x-1)\ln x} = \lim_{x \to 1^+} \frac{3-\frac{2}{x}}{\ln x+(x-1)\frac{1}{x}} = \lim_{x \to 1^+} \frac{3-\frac{2}{x}}{\ln x+1-\frac{1}{x}} = \infty$

Integration By Parts

It is used to solve integration of a product of two functions using the formula:

$$\int u\ dv = uv - \int v\ du$$

1. $\int xe^x dx$
   $u = x \quad dv = e^x\ dx$
   $du = dx \quad v = e^x$
   $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$
2. $\int_0^\pi x \sin x\ dx$
   $u = x \quad dv = \sin x\ dx$
   $du = dx \quad v = -\cos x$
   $\int_0^\pi x \sin x\ dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x dx = [-x \cos x]_0^\pi + [\sin x]_0^\pi$
   $[(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = \pi$

Notes:

1. $\int xe^x\ dx = (x-1)e^x + c$
   $\int x^2e^x\ dx = (x^2-2x+2)e^x + c$
   $\int x^3e^x\ dx = (x^3-3x^2+6x-6)e^x + c$
2. $\int x\cos x\ dx = x\sin x + \cos x + c$
   $\int x^2\cos x\ dx = (x^2-2)\sin x + 2x\cos x + c$
   $\int x^3\cos x\ dx = (x^3-6x)\sin x + (3x^2-6)\cos x + c$
   $\int x^4\cos x\ dx = (x^4-12x^2+24)\sin x + (4x^3-24x)\cos x + c$
3. $\int x\sin x\ dx = -x\cos x + \sin x + c$
   $\int x^2\sin x\ dx = (-x^2+2)\cos x + 2x\sin x + c$
   $\int x^3\sin x\ dx = (-x^3+6x)\cos x + (3x^2-6)\sin x + c$
   $\int x^4\sin x\ dx = (-x^4+12x^2-24)\cos x + (4x^3-24x)\sin x + c$

Integrals Involving Trigonometric Functions

First : Integrals of the forms

$$\int \sin ax \cos bx \ dx, \quad \int \sin ax \sin bx \ dx, \quad \int \cos ax \cos bx \ dx$$

Wher $a, b \in \mathbb{Z}$

1. The integral $\int \sin ax \cos bx \ dx$ can be solved using the formula $\sin ax \cos bx = \frac{1}{2}[\sin(ax+bx) + \sin(ax-bx)]$
2. The integral $\int \sin ax \sin bx \ dx$ can be solved using the formula $\sin ax \sin bx = \frac{1}{2}[\cos(ax-bx) - \cos(ax+bx)]$
3. The integral $\int \cos ax \cos bx \ dx$ can be solved using the formula $\cos ax \cos bx = \frac{1}{2}[\cos(ax+bx) + \cos(ax-bx)]$

Integrals Involving Trigonometric Functions (Examples)

1. $\int \sin 3x \cos 2x \; dx = \frac{1}{2}\int [\sin 5x + \sin x] dx =$
   $\frac{1}{2}\int \sin 5x \; dx + \frac{1}{2}\int \sin x \; dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$
2. $\int \sin x \sin 3x \; dx = \frac{1}{2}\int [\cos 2x - \cos 4x] dx =$
   $\frac{1}{2}\int \cos 2x \; dx - \frac{1}{2}\int \cos 4x \; dx = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c$
3. $\int \cos 5x \cos 2x \; dx = \frac{1}{2}\int [\cos 7x + \cos 3x] dx =$
   $\frac{1}{2}\int \cos 7x \; dx + \int \cos 3x \; dx = \frac{1}{4}\sin 7x + \frac{1}{6}\sin 3x + c$

Integrals Involving Trigonometric Functions

Second : Integrals of the forms

$$\int \sin^n x \cos^m x \ dx, \quad \int \sinh^n x \cosh^m x \ dx, \qquad Where\ n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd.

1. If n is odd: The substitution $u = \cos x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \cosh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$
2. If m is odd: The substitution $u = \sin x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \sinh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$

Integrals Involving Trigonometric Functions (Examples)

1. $\int \sin^5 x \cos^4 x dx$ to solve this integral put
$u = \cos x \Rightarrow -du = \sin x \ dx$
$\int \sin^5 x \cos^4 x dx = \int (\sin^2 x)^2 \cos^4 x \sin x dx =$
$\int (1 - \cos^2 x) \cos^4 x \sin x \ dx = -\int (1 - u^2)^2 u^4 du = -\int u^4 -$
$2u^6 + u^8 du = -[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}] + c = -[\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9}] + c$
2. $\int \sin^7 \cos^3 x \ dx$ to solve this integral put
$u = \sin x \Rightarrow du = \cos x \ dx$
$\int \sin^7 \cos^3 x \ dx = \int \sin^7 x(1 - \sin^2 x) \cos x \ dx =$
$\int u^7(1 - u^2) \ du = \int u^7 - u^9 \ du = \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8 x}{8} - \frac{sin^{10}}{10} + c$

Integrals Involving Trigonometric Functions (Examples)

- $\int \sinh^3 x \cosh^2 x \; dx$ to solve this integral put
  $u = \cosh x \Rightarrow du = \sinh x$
  $\int \sinh^3 x \cosh^2 x \; dx = \int (\cosh^2 x - 1) \cosh^2 x \sinh x \; dx =$
  $\int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c =$
  $\frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$

Special cases :

1. $\int \sin^2 x \; dx = \frac{1}{2} \int [1 - \cos 2x] \; dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + c$
2. $\int \cos^2 x \; dx = \frac{1}{2} \int [1 + \cos 2x] \; dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + c$

Integrals Involving Trigonometric Functions

Third : Integrals of the forms

$$\int \sec^n x \tan^m x \; dx, \quad \int csc^n x \cot^m x \; dx,$$

$$\int sech^n x \tanh^m x \; dx, \quad \int csch^n x \coth^m x \; dx$$

The above four integrals can be solved by substitution if n is even or m is odd.

Integrals Involving Trigonometric Functions

1. If n is even:
   The substitution $u = \tan x$ can be used to solve $\int \sec^n x \tan^m x \ dx$.
   The substitution $u = \cot x$, $u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.
2. If m is odd:
   The substitution $u = \sec x$ can be used to solve $\int \sec^n x \tan^m x \ dx$.
   The substitutions $u = csc\ x, u = sech\ x$ and $u = csch\ x$ can be used to solve the other theree integrals respictively.

Integrals Involving Trigonometric Functions (Examples)

1. $\int \tan^3 x \sec^3 x \; dx$ to solve this integral put
   $u = \sec x \Rightarrow du = \sec x \tan x \; dx$
   $\int \tan^3 x \sec^3 x \; dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \; dx =$
   $(u^2 - 1)u^2 \; du = \int u^4 - u^2 \; du = \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5}{5} - \frac{\sec^3}{3} + c$
2. $\int tanh^3 x \; sech \; x \; dx$ to solve this integral put
   $u = sech \; x \Rightarrow -du = sech \; x \tanh x \; dx$
   $\int tanh^3 x \; sech \; x \; dx = \int (1 - sech^2 x) sech \; x \; tanh \; x \; dx =$
   $-\int (1 - u^2) du = -u + \frac{u^3}{3} + c = -sech \; x + \frac{sech^3 x}{3} + c$

Trigonometric Substitutions

If the integrand contains a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$, then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
2. An integral involving $\sqrt{a^2 + x^2}$ use the substitution $x = a\tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
3. An integral involving $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec\theta$ where $0 \le \theta < \frac{\pi}{2}$ to solve the integral.

Trigonometric Substitutions (Examples)

Example 5.3

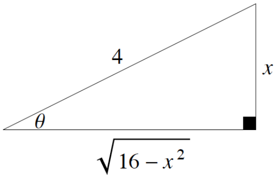
Solve the following integral: $\int \frac{1}{x^2\sqrt{16-x^2}}dx$

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{x^2\sqrt{(4)^2-x^2}}dx,\ Put\ x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$$

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16-16\sin\theta}}d\theta = \int \frac{4\cos\theta}{16\sin^2\theta\ \ 4\cos\theta}d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta = \frac{1}{16}\cot\theta + c$$

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + c$$



Trigonometric Substitutions (Examples)

Example 5.4

Solve the following integral: $\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx$

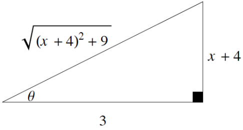
$$\int \frac{1}{[(x^2+8x+16)+9]^{\frac{3}{2}}}\ dx = \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}\ dx.$$

Put $x+4 = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\ d\theta$

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\ d\theta = \int \frac{3\sec^{\theta}}{(9\sec^2\theta)^{\frac{3}{2}}}\ d\theta$$

$$= \int \frac{3\sec^2\theta}{27\sec^3\theta} d\theta = \frac{1}{9}\int \frac{1}{\sec\theta} d\theta = \frac{1}{9}\sin\theta + c$$

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \frac{1}{9}\frac{x+4}{\sqrt{x^2+8x+25}} + c$$



Trigonometric Substitutions (Examples)

Example 5.5

Solve the following integral: $\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

Put $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\;d\theta$

$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \int \frac{\sqrt{4\sec^2\theta-4}\;2\sec\theta\tan\theta}{4\sec^2\theta}d\theta = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{4\sec^2\theta}d\theta$

$= \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta-1}{\sec\theta}d\theta = \int \frac{\sec^2\theta}{\sec\theta}d\theta - \int \frac{1}{\sec\theta}d\theta$

$= \int \sec\theta d\theta - \int \cos\theta d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + c$

$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c$