Math104
General Mathematics 2

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Textbook:

أساسيات الرياضيات لطلبة الكليات العلمية

2014 ,مطابع جامعة الملك سعود .مسعود بونخل.د .مساعد العبداللطيف و أ.د :تأليف ,

Grading:

MidTerm1: 25 + MidTerm2: 25 + Final Exam: 40 + Tutorial: 10 =100


أساسيات الرياضيات
لطلبة الكليات العلمية
تأليف
د. مساعد العبداللطيف
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جامعة الملك سعود
النشر العلمي والمطابع

Topics:

- Chapter1: Conic sections (parabola, ellipse and hyperbola). Second degree equation.

- Chapter2: Matrices (addition and multiplication of matrices), Determinants of square matrices (4 by 4).

- Chapter3: Systems of linear equations (finding the solution by Crammer's rule, Gauss elimination method and Gauss-Jordan elimination method.

- Chapter4: Methods of integration (by substitution, by parts, integration by partial fractions

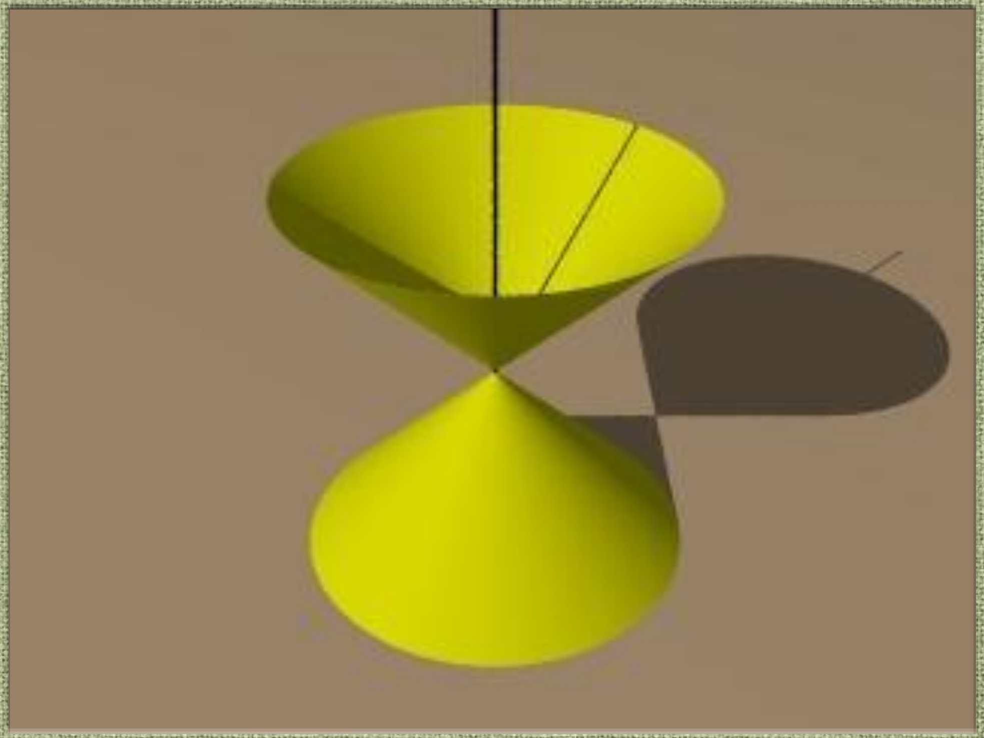
- Chapter5: Applications of Integrals (areas and volumes of revolution (by washer method and by cylindrical shells).

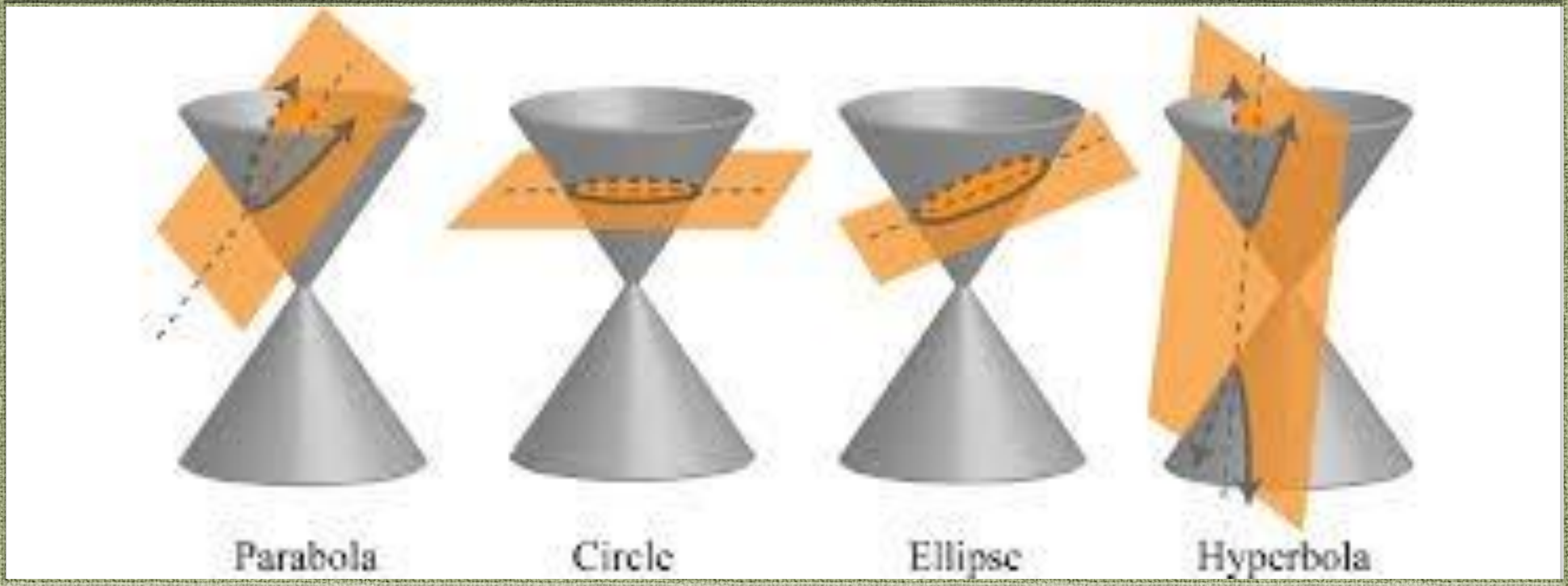
- Chapter6: Polar coordinates and applications.

- Chapter7: Functions of several variables, partial differentiation, chain rules, and implicit differentiation.

- Chapter8: Ordinary differential equations, general and particular solution of differential equations, separable differential equations and first order linear differential equations.

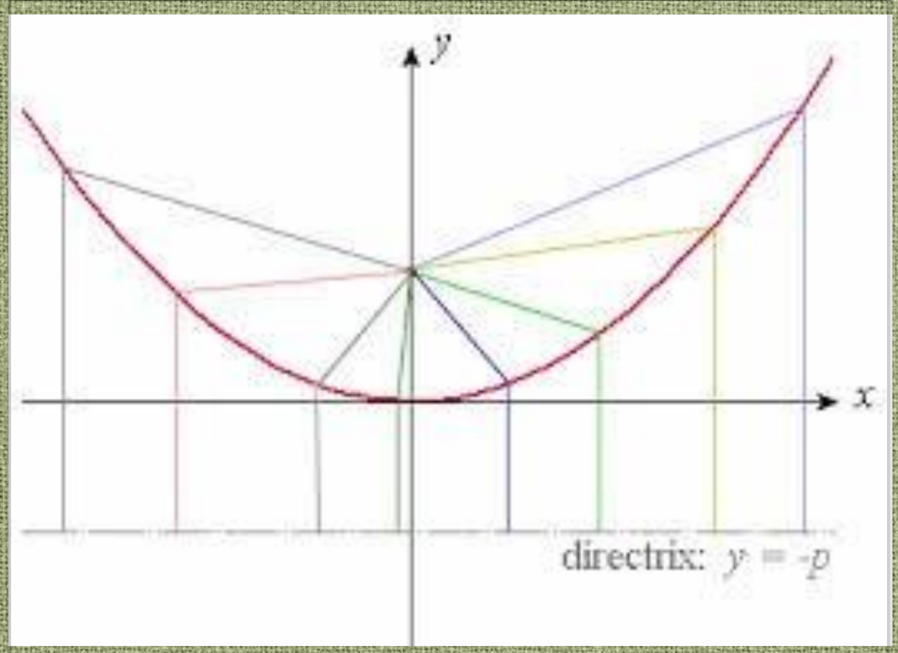
Chapter 1: Conic Sections

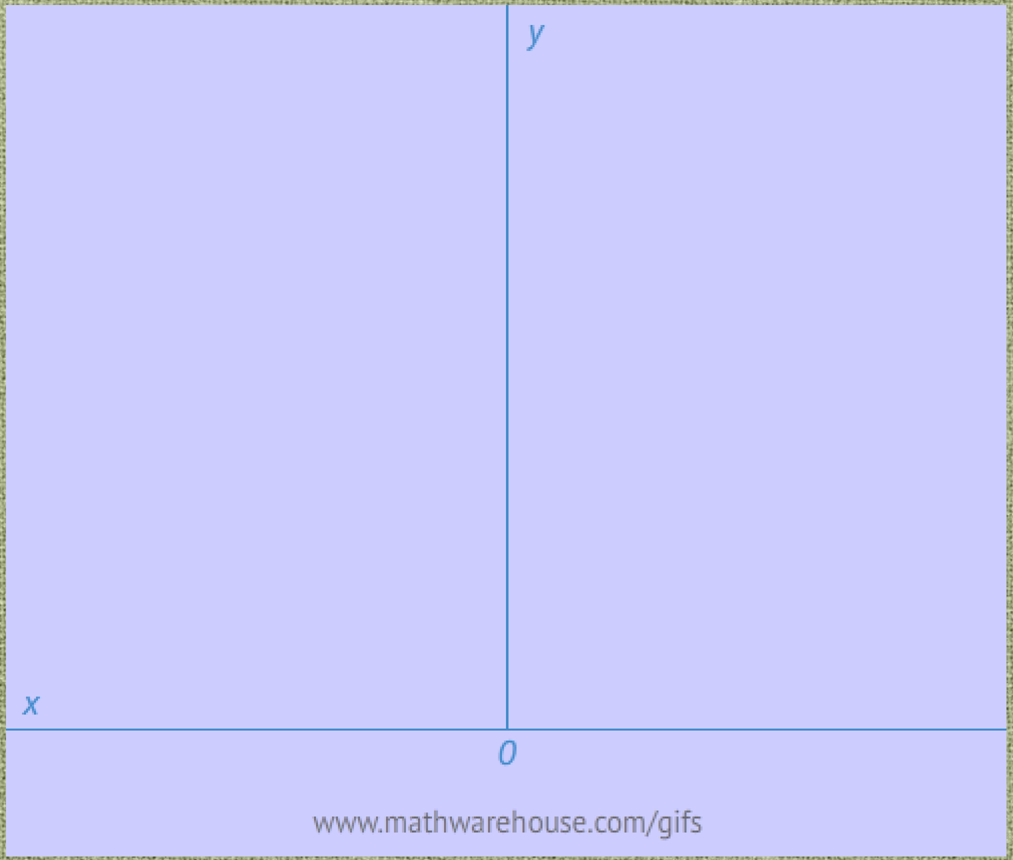


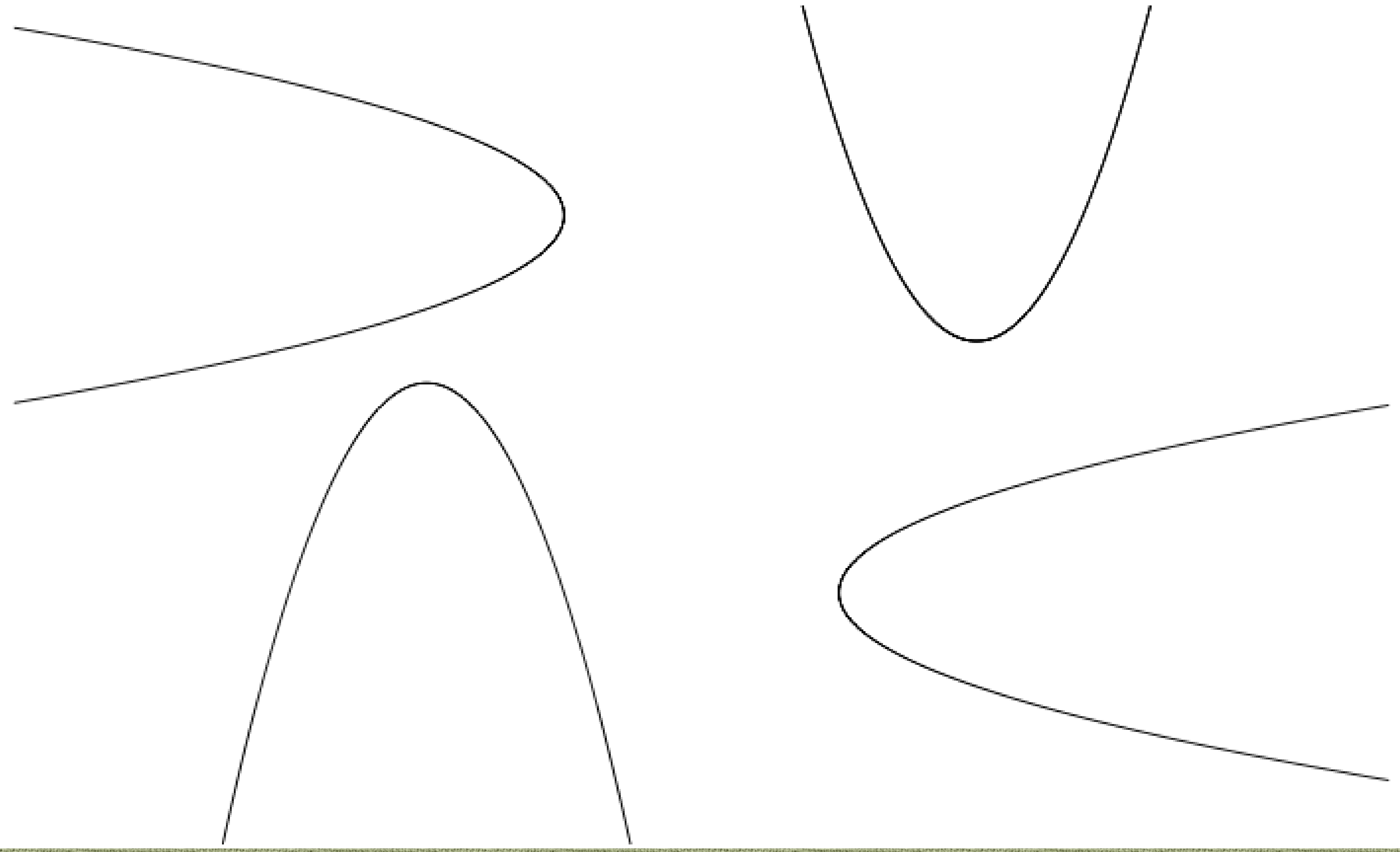


Section 1.1. Parabola.

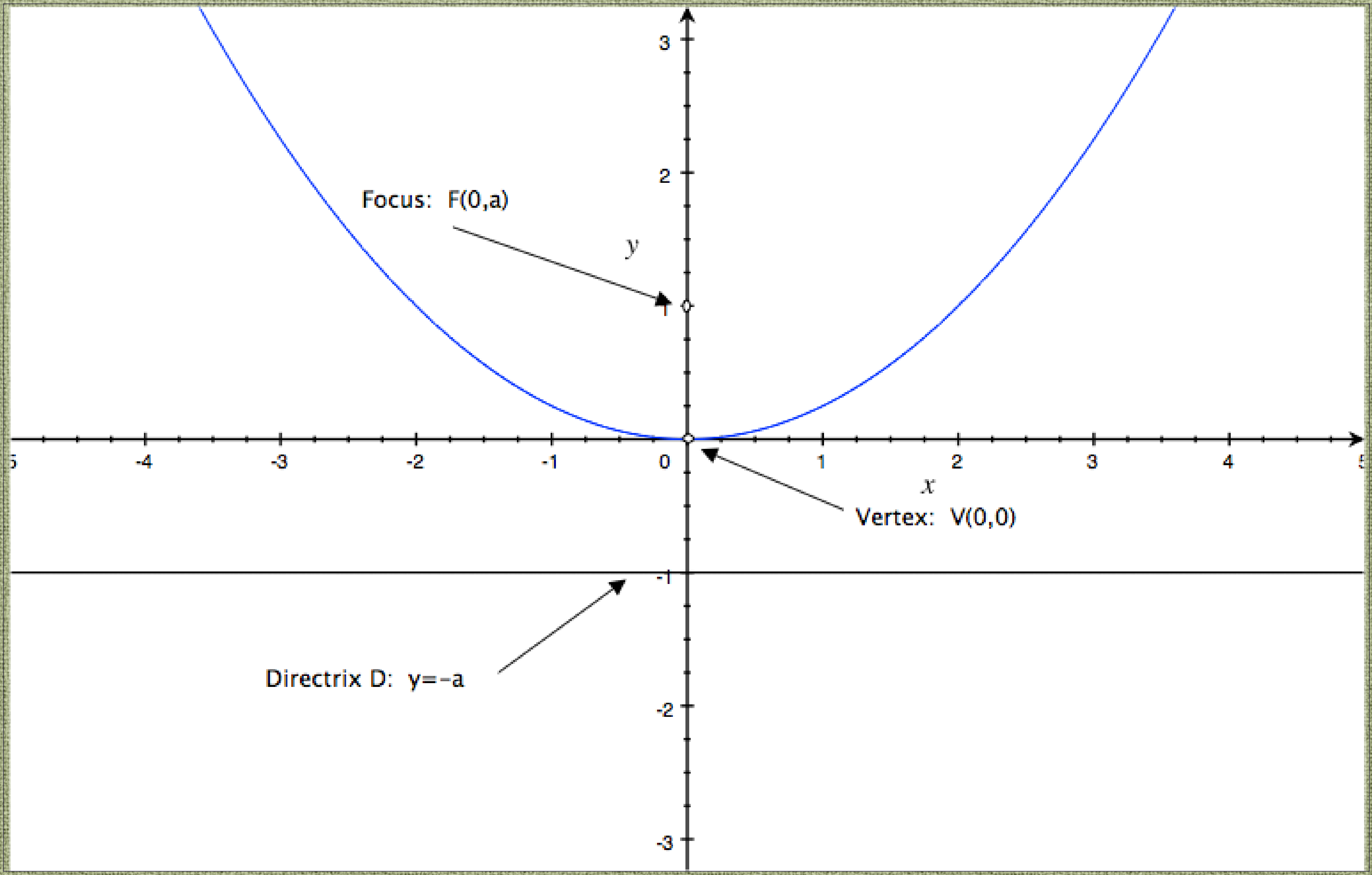
Definition. A parabola is the set of all points on the plane having the same distance between a fixed point F and a fixed straight line D.







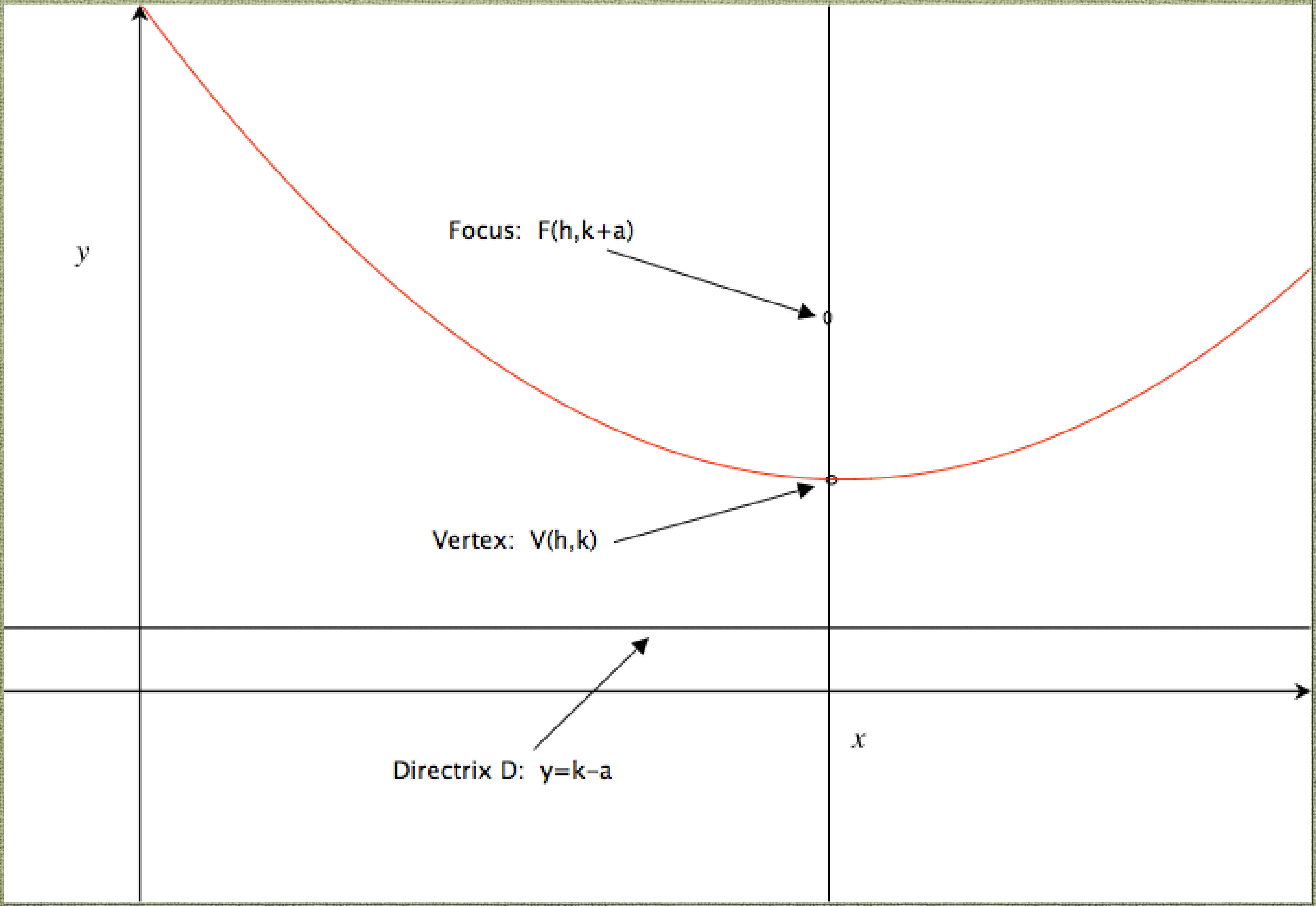
Elements of parabolas:

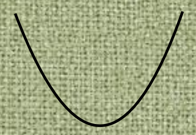
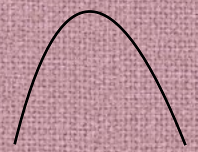
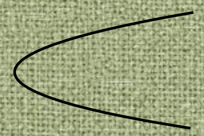
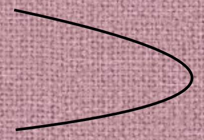


- The point F is called Focus
- The line D is called Directrix
- The point V is called Vertex
- a = distance between F and V
- = distance between D and V

Standard Equation:

Case: $V(h,k)$



Standard Equation	Focus	Directrix	Position
$(x-h)^2 = 4a(y-k)$	$F(h, k+a)$	$y = k - a$	
$(x-h)^2 = -4a(y-k)$	$F(h, k-a)$	$y = k + a$	
$(y-k)^2 = 4a(x-h)$	$F(h+a, k)$	$x = h - a$	
$(y-k)^2 = -4a(x-h)$	$F(h-a, k)$	$x = h + a$	

Example 1. Find the elements (Focus, Vertex, and Directrix) of the parabola $(y-1)^2 = 8(x+1)$ and sketch its graph.

Sol. Compare the given equation with the corresponding standard equation, that is,

$$(y-1)^2 = 8(x+1) \quad \longleftrightarrow \quad (y-k)^2 = 4a(x-h)$$

$$k = 1$$
$$h = -1$$
$$4a = 8 \Rightarrow a = 2.$$

Then the vertex is $V=(h,k) = (-1,1)$.

Since the parabola is open to the right then

$F(-1+a,1) = (-1+2,1) = (1,1)$, that is, $F(1,1)$

and the equation of the directrix D is given by

$x = -1- a = -1-2 = -3$, that is, $x = -3$.

Example 2. Find the elements (Focus, Vertex, and Directrix) of the parabola $2y^2 - 8y - 8x - 16 = 0.$

Sol. Completing the square:

$$2y^2 - 8y - 8x - 16 = 0$$

$$2y^2 - 8y = 8x + 16 \qquad \text{divide by 2}$$

$$y^2 - 4y = 4x + 8$$

$$y^2 - 4y + 4 = 4x + 8 + 4$$

$$y^2 - 4y + 4 = 4x + 12$$

$$(y-2)^2 = 4x + 12$$

$$(y-2)^2 = 4(x+3) \quad \longleftrightarrow \quad (y-k)^2 = 4a(x-h)$$

$$h = -3, \quad k = 2, \quad a = 1.$$

- Then the vertex is $V = (h,k) = (-3,2)$.
- Since the parabola is open to the right then
- $F(-3+a,2) = (-3+1,2) = (-2,2)$, that is, $F(-2,2)$

and the equation of the directrix D is given by

- $x = -3 - a = -3-1 = -4$, that is, $x = -4$.

Example 3. Find the equation of the parabola with Focus $F(3,8)$ and Vertex $V(3,4)$.

Sol. The corresponding equation has the form:

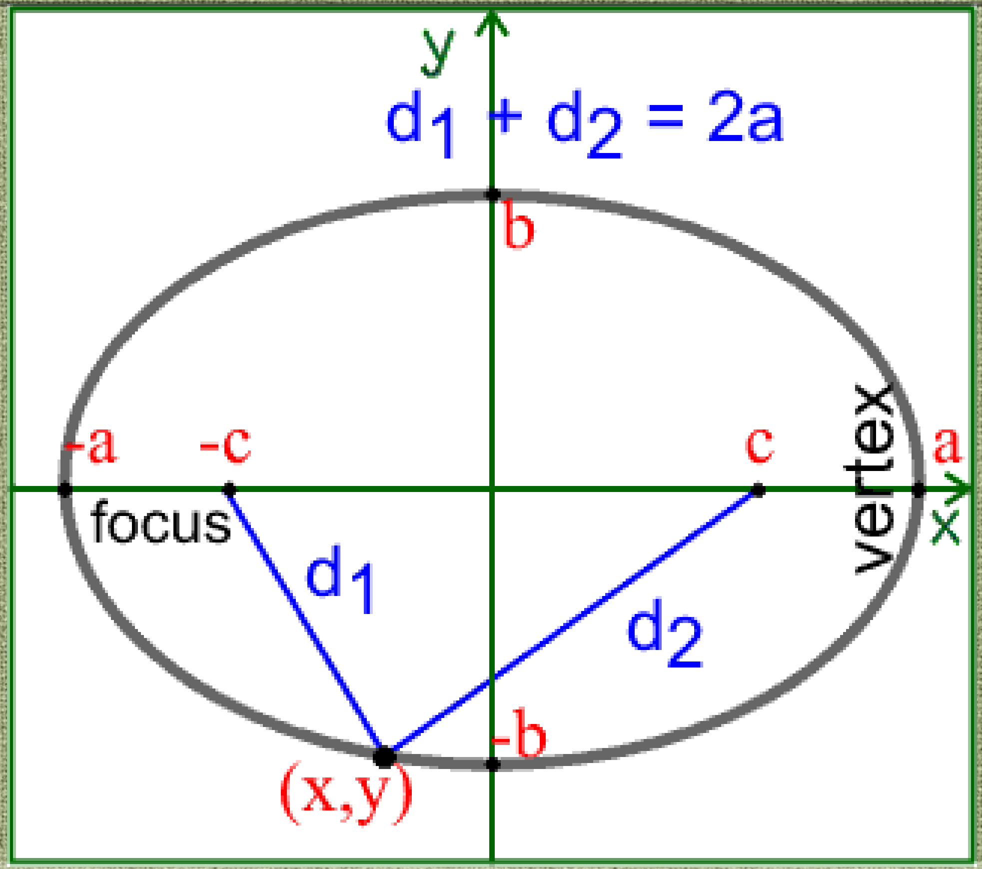
$$(x-h)^2 = 4a(y-k)$$

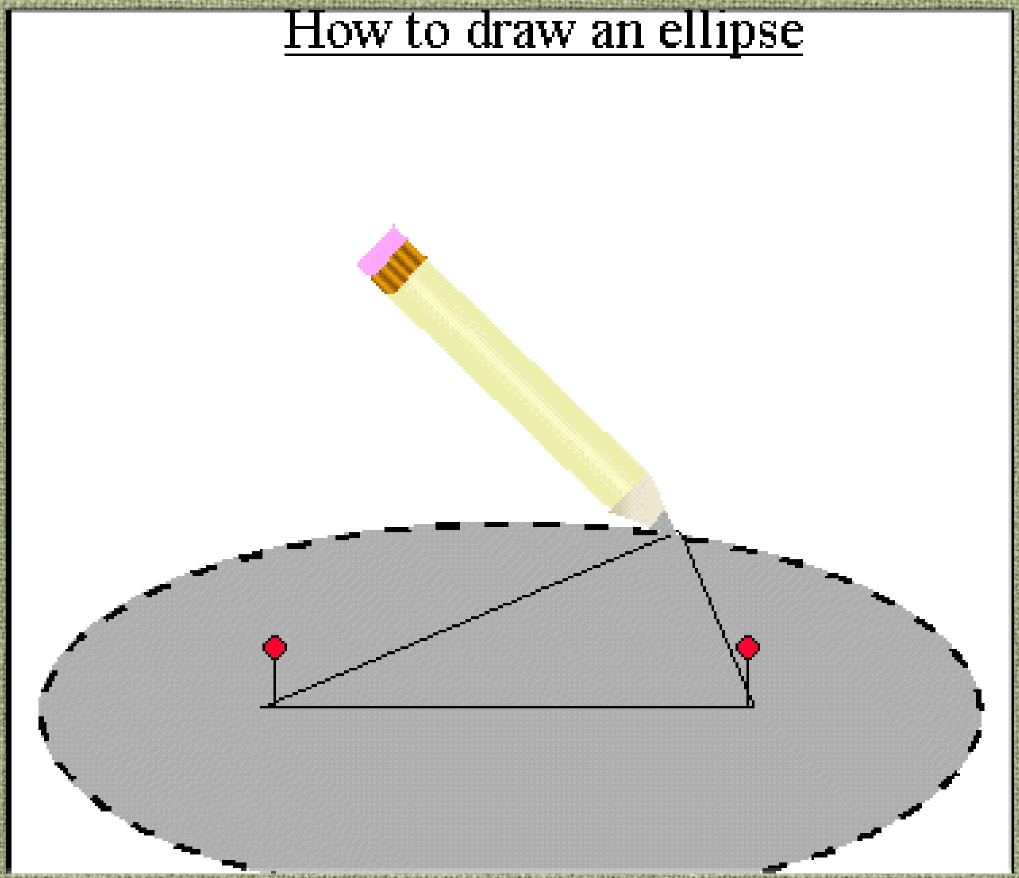
$$h = ?, \quad k = ?, \quad a = ?.$$

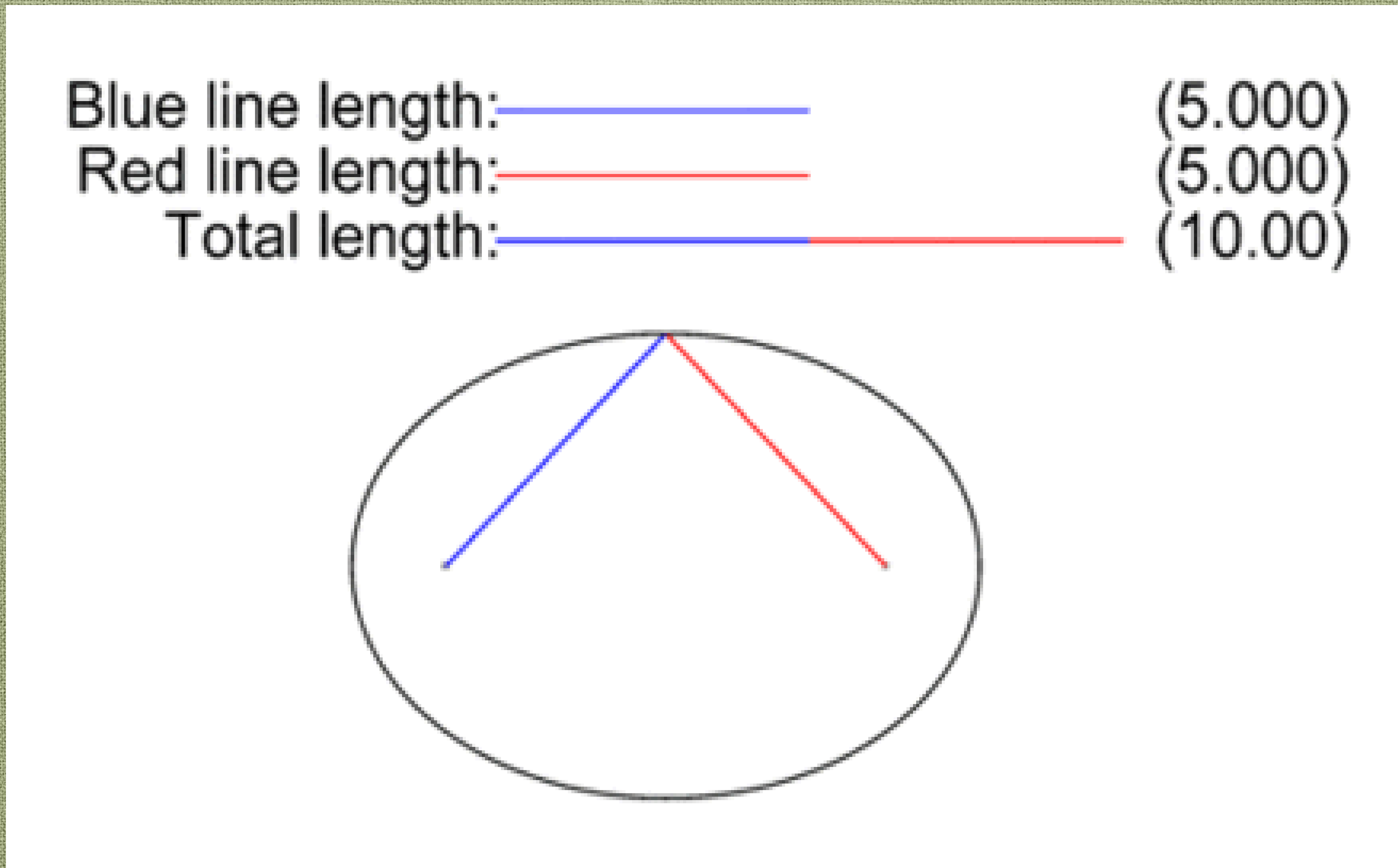
- The vertex is $V(3,4) = (h,k) \Rightarrow h = 3 \ \ and \ \ k = 4$
- Since $a = FV$ then $a = 4.$
- Thus the desired equation is $(x-3)^2 = 16(y-4).$

Section 1.2. Ellipse.

Definition. An ellipse is the set of all points on the plane for which the sum of the distances from any point on the curve to two fixed points F_1 and F_2 is equal to a fixed number $2a$.



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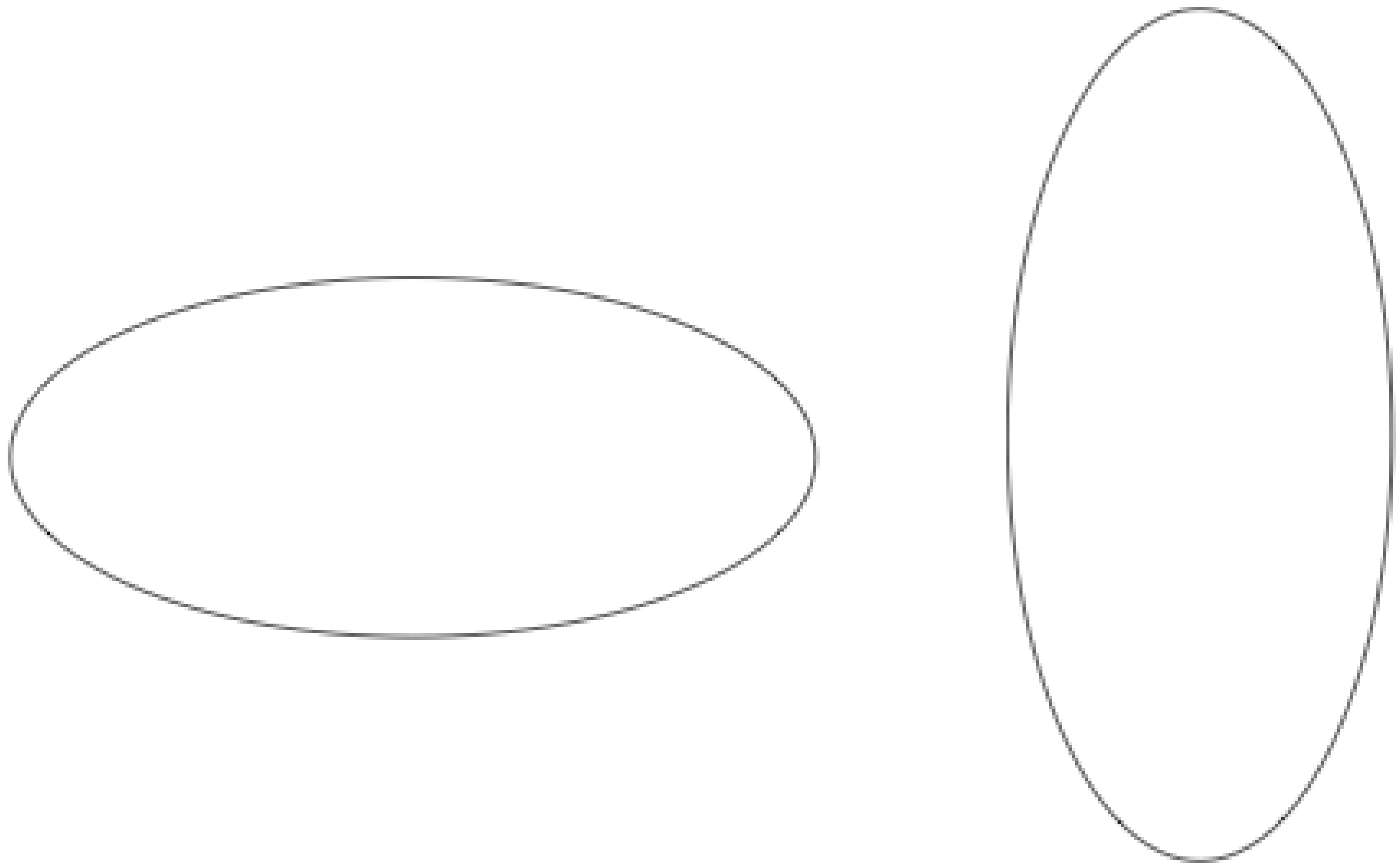
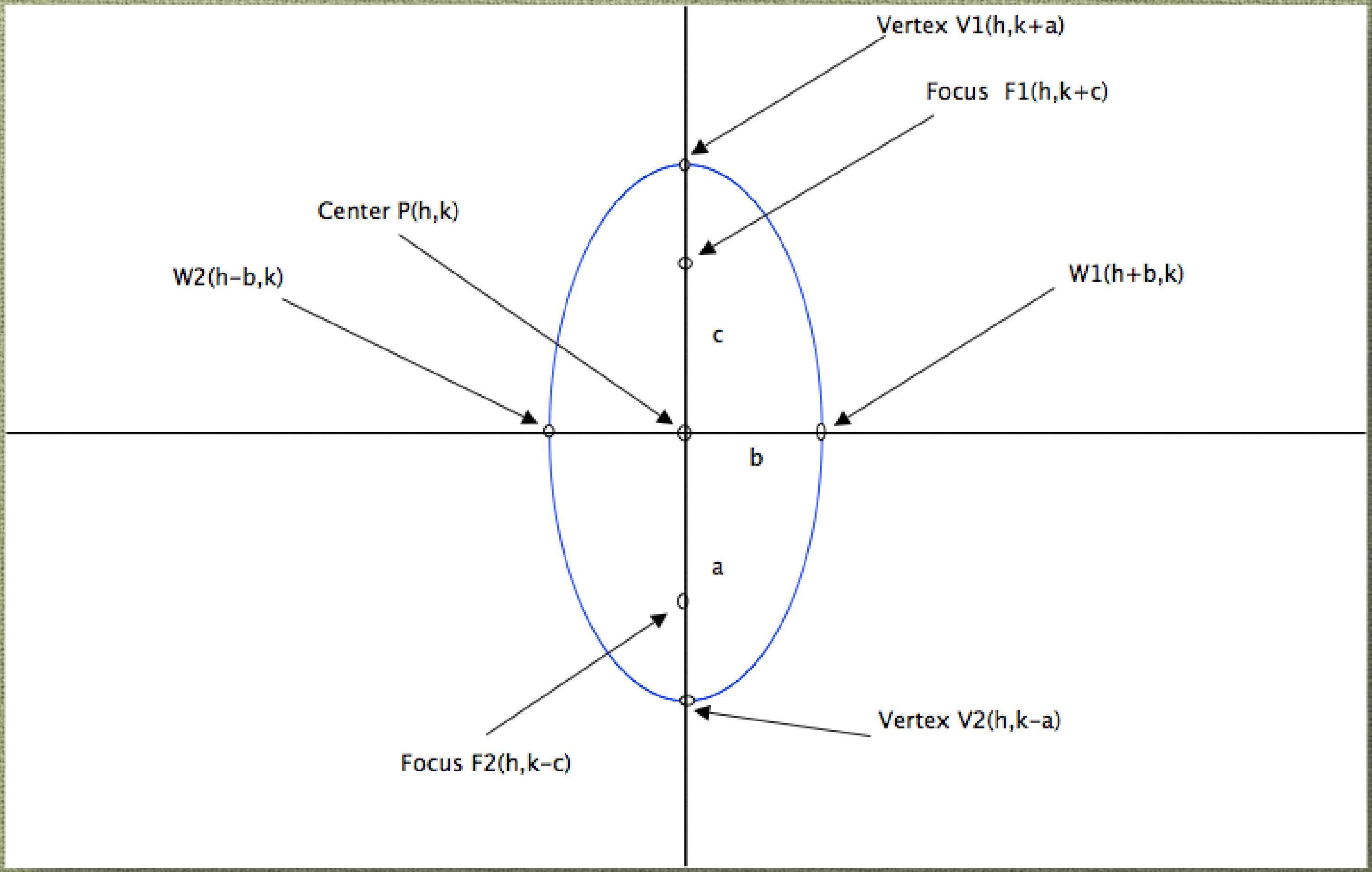
Red line length: (5.000)
Total length: (10.00)



Elements of ellipses:



- The points F_1 and F_2 are called Foci
- The point P is called Center
- The points V_1 and V_2 are called Vertices
- The points W_1 and W_2 are called co-Vertices
- c = distance between F_i $(i=1,2)$ and P
- a = distance between V_i $(i=1,2)$ and P
- b = distance between W_i $(i=1,2)$ and P

$$c^2=a^2-b^2$$

Each type of ellipse has these main properties:

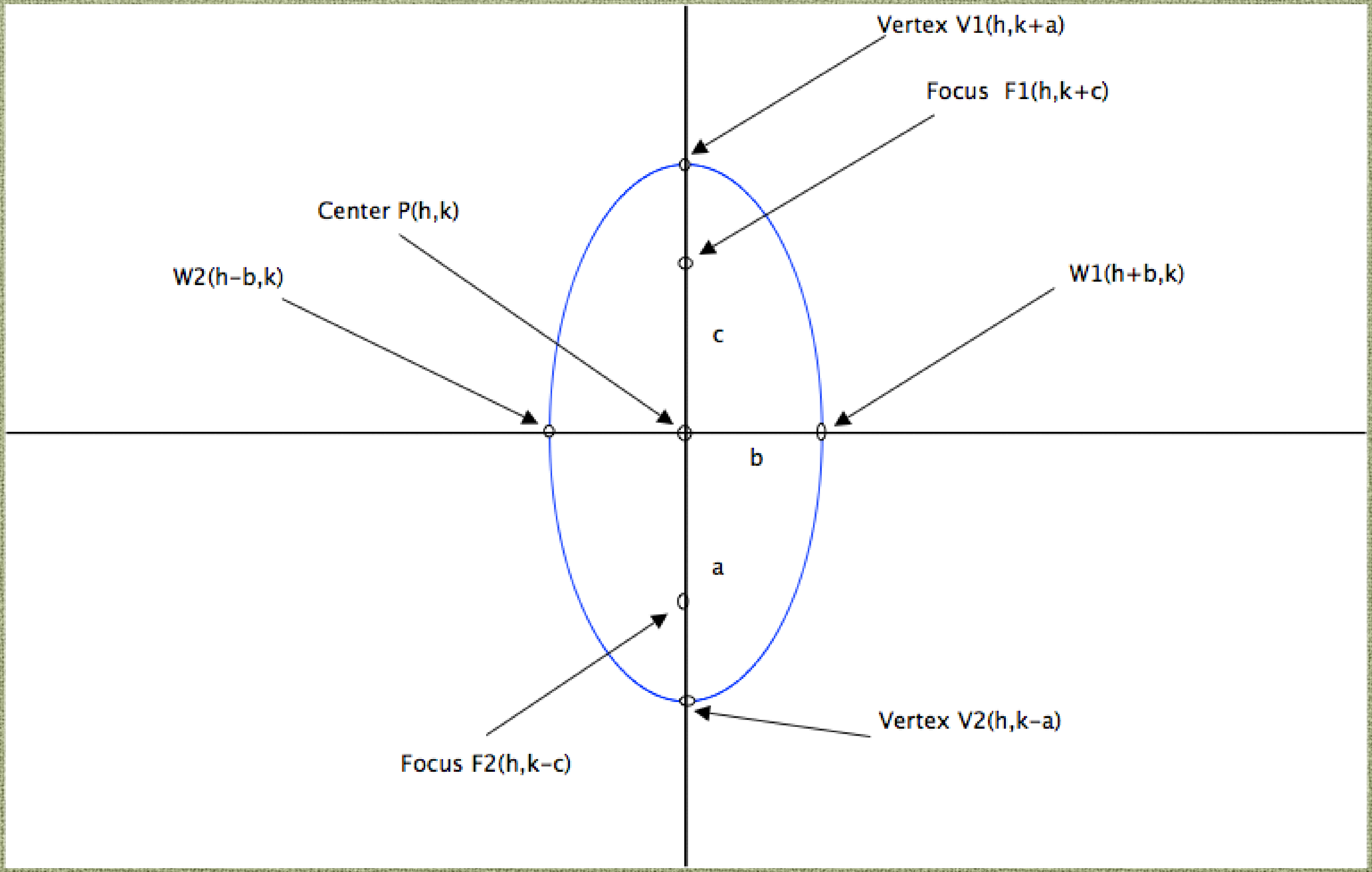
- The major axis is the longest axis of the ellipse.
- The endpoints of the major axis are called vertices V_1 and V_2.
- The variable a is the letter used to name the distance from the center P to each vertex.
- The major axis is the axis containing the foci F_1, F_2, the vertices V_1 , V_2 , and the center P.

- The minor axis is the perpendicular axis to the major axis and it is the shortest one.
- The endpoints on the minor axis are called co-vertices W_1 and W_2.
- The variable b is the letter used to name the distance from the center P to each co-vertex.
- Because the major axis is always longer than the minor one, $a > b$.

- Notice that the length of the major axis is $2a$, and the length of the minor axis is $2b$.
- When the bigger number a is under x, the ellipse is horizontal;
- When the bigger number a is under y, it's vertical.

Standard Equation:

Case: $P(h,k)$



Standard Equation	Foci	Vertices	Co-Vertices	Position
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$F_1(h-c,k)$, $F_2(h+c,k)$	$V_1(h-a,k)$, $V_2(h+a,k)$	$W_1(h,k-b)$, $W_2(h,k+b)$	horizontal
$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	$F_1(h,k-c)$, $F_2(h,k+c)$	$V_1(h,k-a)$, $V_2(h,k+a)$	$W_1(h-b,k)$, $W_2(h+b,k)$	vertical

Example 1. Find the elements (Foci, Vertices, and Co-vertices) of the ellipse : $4(y-1)^2+9(x+1)^2=36$ and sketch its graph.

Sol. Rewrite the given equation in the standard form:

$$4(y-1)^2+9(x+1)^2=36 \iff \frac{(x+1)^2}{4}+\frac{(y-1)^2}{9}=1$$

Compare this equation with the corresponding standard equation:

$$\frac{(x+1)^2}{4}+\frac{(y-1)^2}{9}=1 \iff \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

$$k = 1, h = -1,$$
$$a^2 = 9, b^2 = 4 \Rightarrow a = 3, b = 2$$
$$c^2 = a^2 - b^2 \Rightarrow c = \sqrt{9-4} \Rightarrow c = \sqrt{5}$$

Then the center is $P(-1,1)$. Since the position of the ellipse is vertical the foci and vertices are as follows:

- $F_1(-1,1+c)=(-1,1+\sqrt{5})=(-1,1+\sqrt{5})$,
- $F_2(-1,1-c)=(-1,1-\sqrt{5})=(-1,1-\sqrt{5})$ and
- $V_1(-1,1+a)=(-1,1+3)=(-1,4)$,
- $V_2(-1,1-a)=(-1,1-4)=(-1,-3)$.
- The co-vertices are
- $W_1(-1+b,1)=(-1+2,1)=(1,1)$,
- $W_2(-1-b,1)=(-1-2,-1)=(-3,-1)$.

Example 2. Find the elements (Foci, Vertices, and Co-vertices) of the ellipse : $x^2 + 5y^2 - 40y + 6x + 84 = 0$ and sketch its graph.

Sol. Completing the square:

$$x^2 + 5y^2 - 40y + 6x + 84 = 0$$

$$(x^2+6x) + (5y^2 - 40y) = -84$$

$$(x^2+6x) + 5(y^2 - 8y) = -84$$

$$\left(x^2+6x + \left(\frac{6}{2}\right)^2\right) + 5\left(y^2 - 8y + \left(\frac{8}{2}\right)^2\right) = -84 + 9 + 5 * 16$$

$$(x^2+6x + 9) + 5(y^2 - 8y + 16) = 5$$

$(x+3)^2 + 5(y-4)^2 = 5$ divide by 5 we obtain $\frac{(x+3)^2}{5} + \frac{(y-4)^2}{1} = 1$

Compare this equation with the corresponding standard equation:

$$\frac{(x+3)^2}{5} + \frac{(y-4)^2}{1} = 1 \longleftrightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$k = 4, h = -3,$$

$$a^2 = 5, b^2 = 1 \Rightarrow a = \sqrt{5}, b = 1$$

$$c^2 = a^2 - b^2 \Rightarrow c = \sqrt{5-1} = \sqrt{4} \Rightarrow c = 2$$

Then the center is $P(-3,4)$.

Since the position of the ellipse is horizontal the foci and vertices are as follows:

$$F_1(-3+c,4)=(-3+2,4)=(-1,4);$$
$$F_2(-3-c,4)=(-3-2,4)=(-5,4);$$
$$V_1(-3+a,4)=\left(-3+\sqrt{5},4\right)=\left(-3+\sqrt{5},4\right);$$
$$V_2(-3-a,4)=\left(-3-\sqrt{5},4\right)=\left(-3-\sqrt{5},4\right).$$

The co-vertices are

$$W_1(-3,4+b)=(-3,,4+1)=(-3,5);$$

$$W_2(-3,4-b)=(-3,,4-1)=(-3,3);$$

Example 3. Find the equation of the ellipse with foci $F_1(3,-6)$, $F_2(3,2)$ and the length of its minor axis is 6.

Sol. The corresponding equation has the form:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

$$h=?,\qquad k=?,\qquad a=?,\qquad b=?$$

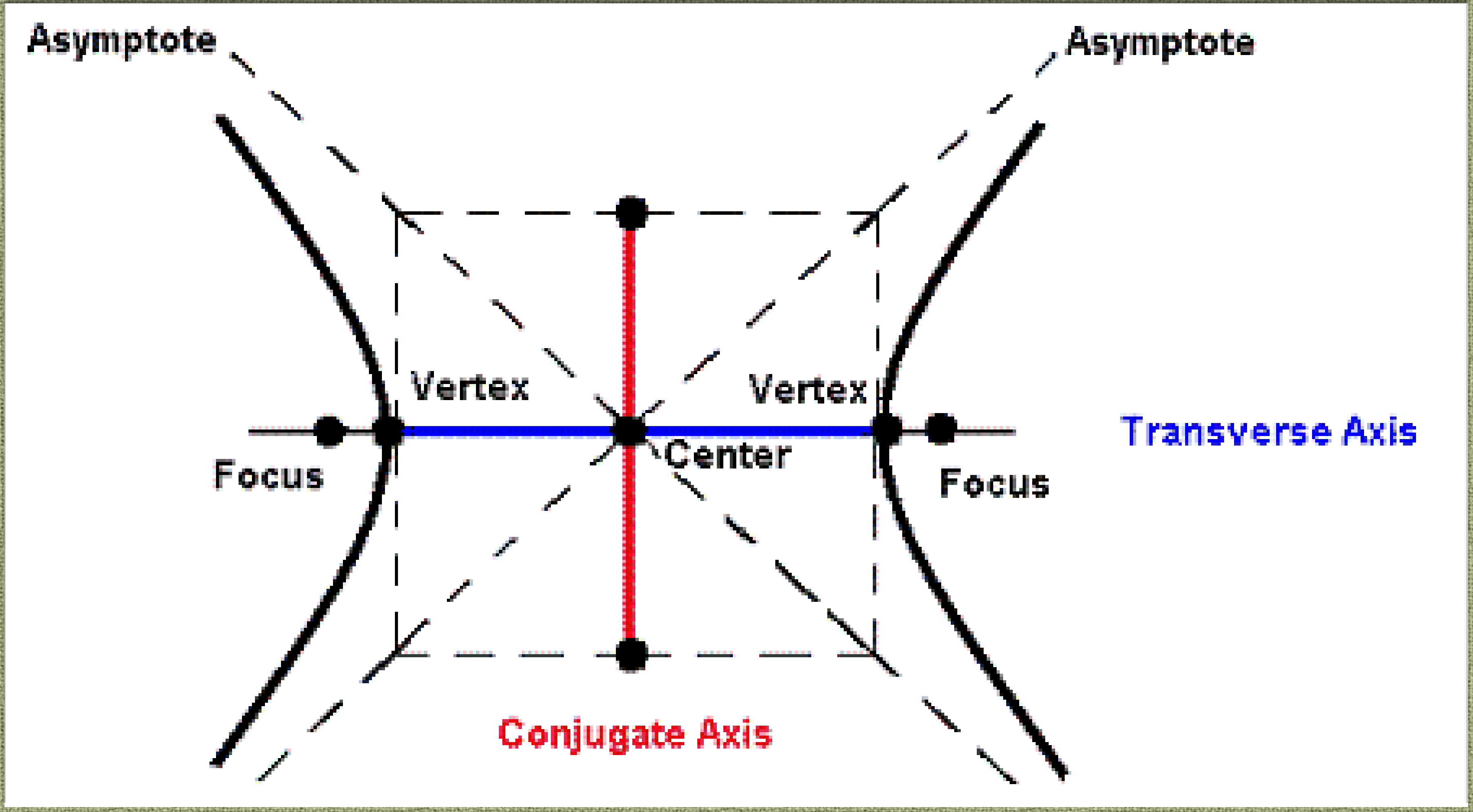
- The center P is the midpoint of the foci, that is, $P=(3,-2)=(h,k)$ and so $h=3$ and $k=-2$.
- The length of the minor axis is $2b=6$, that is, $b=3$.
- In order to find a we find first the value of c.
- $c =$ the distance $PF_1=PF_2=4$. So, $c = 4$ and so

$$c^2=a^2-b^2 \Leftrightarrow 16=a^2-9 \Leftrightarrow a^2=25 \Leftrightarrow a=5.$$

- Thus the desired equation is :

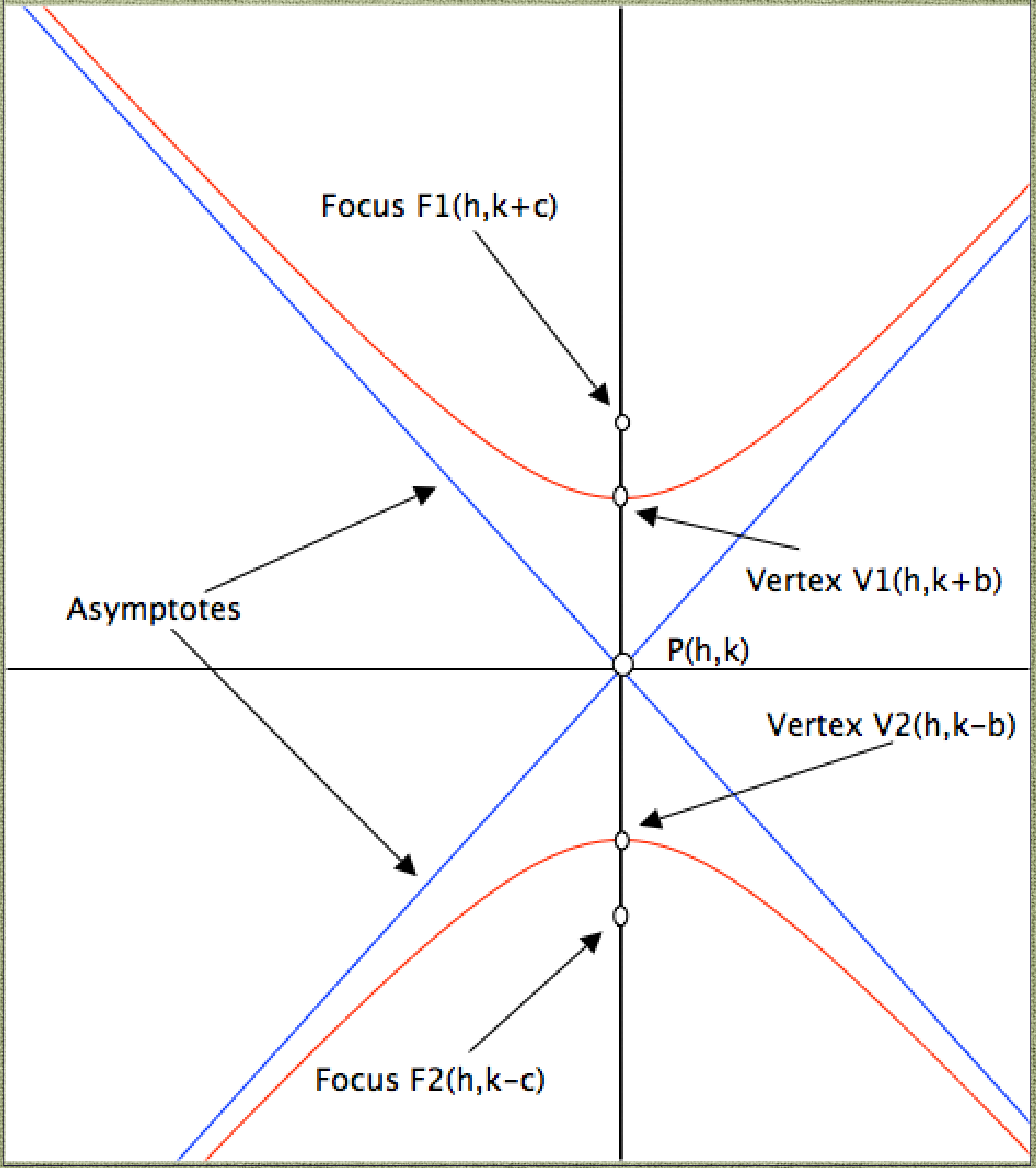
$$\frac{(x-3)^2}{9}+\frac{(y+2)^2}{25}=1$$

Definition. A hyperbola is the set of all points on the plane for which the absolute value of the difference of the distances from any point on the curve to two fixed points F_1 and F_2 is equal to a fixed number $2a$.



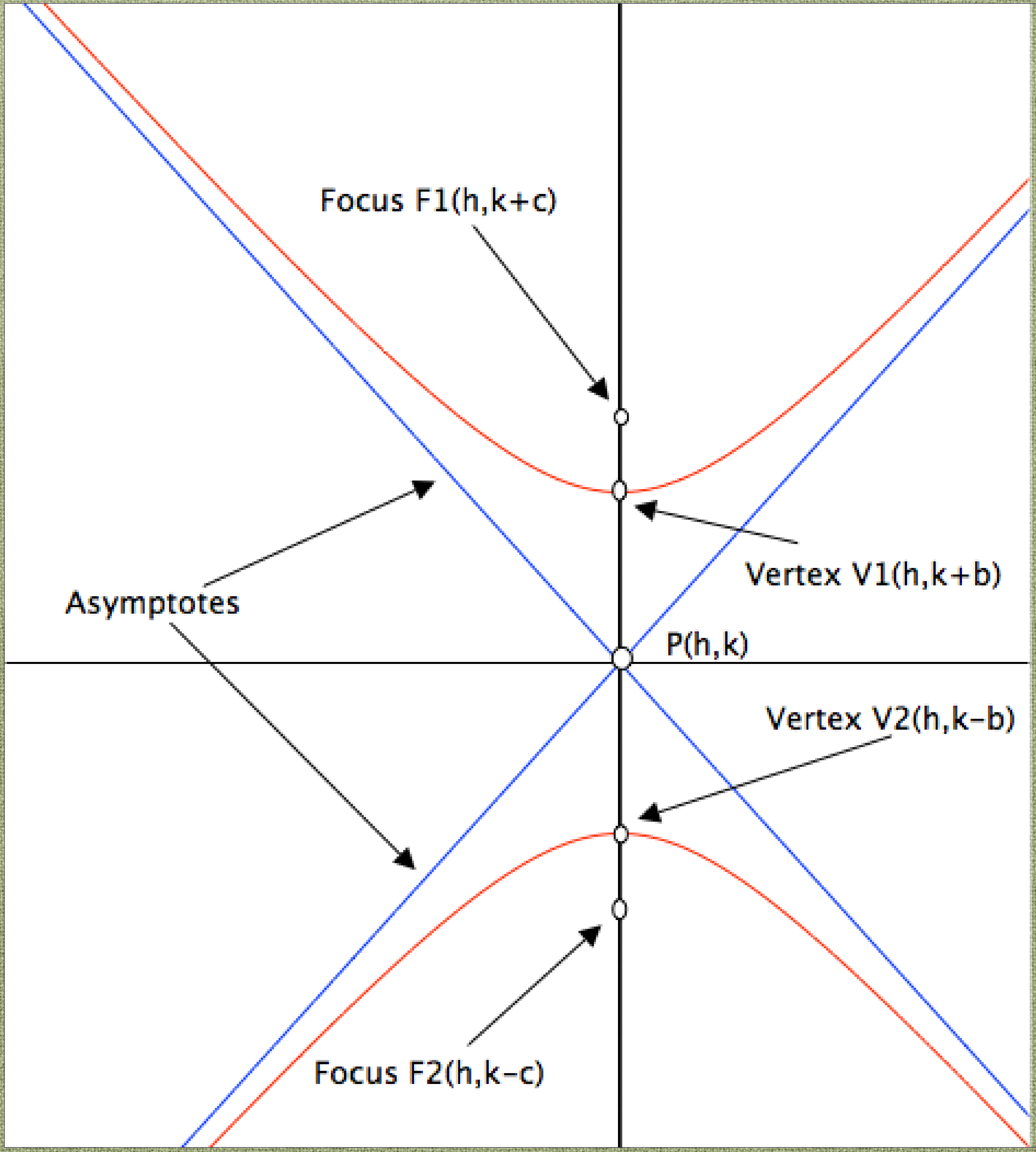
Elements of hyperbolas:

- The points F_1 and F_2 are called Foci
- The point P is called Center
- The points V_1 and V_2 are called Vertices
- c = distance between F_i (i=1,2) and P
- a = distance between V_i (i=1,2) and P when horizontal position
- b = distance between W_i (i=1,2) and P when vertical position
- $c^2=a^2+b^2$



Standard Equation:

Case: $P(h,k)$



Standard Equation	Foci	Vertices	Position
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$F_1(h-c,k)$, $F_2(h+c,k)$,	$V_1(h-a,k)$, $V_2(h+a,k)$,	horizontal
$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$	$F_1(h,k-c)$, $F_2(h,k+c)$,	$V_1(h,k-a)$, $V_2(h,k+a)$,	vertical

$Center \quad P(h,k)$

$$Asymptotes: \quad (y-k) = \pm\frac{b}{a}(x-h)$$

Each type of hyperbola has these main properties:

- The variable *a* is the letter used to name the distance from the center *P* to each vertex when the position is horizontal.
- The variable *b* is the letter used to name the distance from the center *P* to each vertex when the position is vertical.
- The transverse axis is the axis containing the foci F_1, F_2, the vertices V_1, V_2, and the center *P*.

- When the positive term in the equation is the *x*-term, the hyperbola is horizontal;
- When the positive term in the equation is the *y*-term, the hyperbola is vertical.

Example 1. Find the elements (Center, Foci, Vertices, and Asymptotes) of the hyperbola:

$$4(y-1)^2 - 9(x+1)^2 = 36$$

and sketch its graph.

Sol. Rewrite the given equation in the standard form:

$$4(y-1)^2 - 9(x+1)^2 = 36 \iff \frac{(y-1)^2}{9} - \frac{(x+1)^2}{4} = 1$$

Compare this equation with the corresponding standard equation:

$$\frac{(y-1)^2}{9} - \frac{(x+1)^2}{4} = 1 \iff \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

$$h = -1, \;\; k = 1, \;\; a^2 = 4, \;\; b^2 = 9, \;\; c^2 = a^2 + b^2 \Rightarrow$$

$$h = -1, \;\; k = 1, \;\; a = 2, \;\; b = 3, \;\; c = \sqrt{13}$$

Then the center is P(-1,1). Since the position of the ellipse is vertical the foci and vertices are as follows:

- F1(-1,1+c)=$(-1,1+\sqrt{13})$,
- F2(-1,1-c)=$(-1,1-\sqrt{13})$ and
- V1(-1,1+b)=(-1,1+3)=(-1,4),
- V2(-1,1-b)=(-1,1-3)=(-1,-2).

The Asymptotes are

$$(y-1) = \frac{3}{2}(x+1) \quad \text{and} \quad (y-1) = -\frac{3}{2}(x+1)$$

Example 2. Find the elements (Center, Foci, Vertices, and Asymptotes) of the hyperbola : $y^2-5x^2+6y+40x-76=0$ and sketch its graph.

Sol. Completing the square:

$$y^2-5x^2+6y+40x-76=0$$

$$(y^2+6y)+(-5x^2+40x)=76$$

$$(y^2+6y)-5(x^2-8x)=76$$

$$(y^2+6y+9) - 5(x^2-8x+16) = 76 + 9 - 5*16$$

$$(y+3)^2-5(x-4)^2=5 \qquad \text{divide by 5}$$

$$\frac{(y+3)^2}{5}-\frac{(x-4)^2}{1}=1,$$

Compare this equation with the corresponding standard equation:

$$\frac{(y+3)^2}{5}-\frac{(x-4)^2}{1}=1, \quad \longleftrightarrow \quad \frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1,$$

$$k=-3, h=4, a^2=1, b^2=5, c^2=a^2+b^2 \Rightarrow k=-3, h=4, a=1, b=\sqrt{5}, c=\sqrt{6}.$$

Then the center is $P(4,-3)$. Since the position of the hyperbola is vertical the foci and vertices are as follows:

$$F_1(4,-3+c)=(4,-3+\sqrt{6}) \quad \text{and} \quad F_2(4,-3-c)=(4,-3-\sqrt{6})$$

$$V_1(4,-3+b)=(4,-3+\sqrt{5}) \quad \text{and} \quad V_2(4,-3-b)=(4,-3-\sqrt{5})$$

The Asymptotes are

$$(y+3)=\pm\sqrt{5}(x-4)$$

Example 3. Find the equation of the hyperbola with foci $F_1(10,-2), \quad F_2(4,-2),$ and one of the vertices is $V_1(8,-2).$

Sol. The corresponding equation has the form:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

$$h = ?, \ k = ?, \ a = ?, \ b = ?$$

- The center P is the midpoint of the foci, that is, $P(h,k) = (7,-2)$ and so $h = 7$ and $k = -2$.
- Since the position of the hyperbola is horizontal the distance between V_1 and P is the number a, that is, $a=1$.
- In order to find b we find first the value of $c = ?$.
- c = the distance $PF_1 = PF_2 = 3$. So, $c=3$ and so
- $c^2 = a^2 + b^2 \Leftrightarrow 9 = 1 + b^2 \Leftrightarrow b^2 = 8 \Rightarrow b = \sqrt{8}.$
- Thus the desired equation is :

$$\frac{(x-7)^2}{1} - \frac{(y+2)^2}{8} = 1.$$