Chapter 5: Applications of integrals: Areas and Volumes

Definite Integral:

Definition.

If f is continuous on the interval $[a,b]$, and $f(x) = F'(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 1: Evaluate $\int_0^2 x^2 + 1\, dx$

Sol.

$$\int_0^2 x^2 + 1\, dx = \left(\frac{1}{3}x^3 + x - \frac{18}{31}\right)\Big|_0^2 = \frac{1}{3}(2)^3 + 2 - \frac{18}{31} - \left(\frac{1}{3}(0)^3 + 0 - \frac{18}{31}\right)$$

$$= \frac{14}{3} - \frac{18}{31} + \frac{18}{31} = \frac{14}{3}$$

Example 2:

Evaluate $\int_0^2 x(x^2-1)^7\,dx$.

Sol.

Begin by evaluating the indefinite integral $\int x(x^2-1)^7\,dx$.

Let $u = x^2 - 1$; $du = 2x\,dx$ or $\frac{du}{2} = x\,dx$.

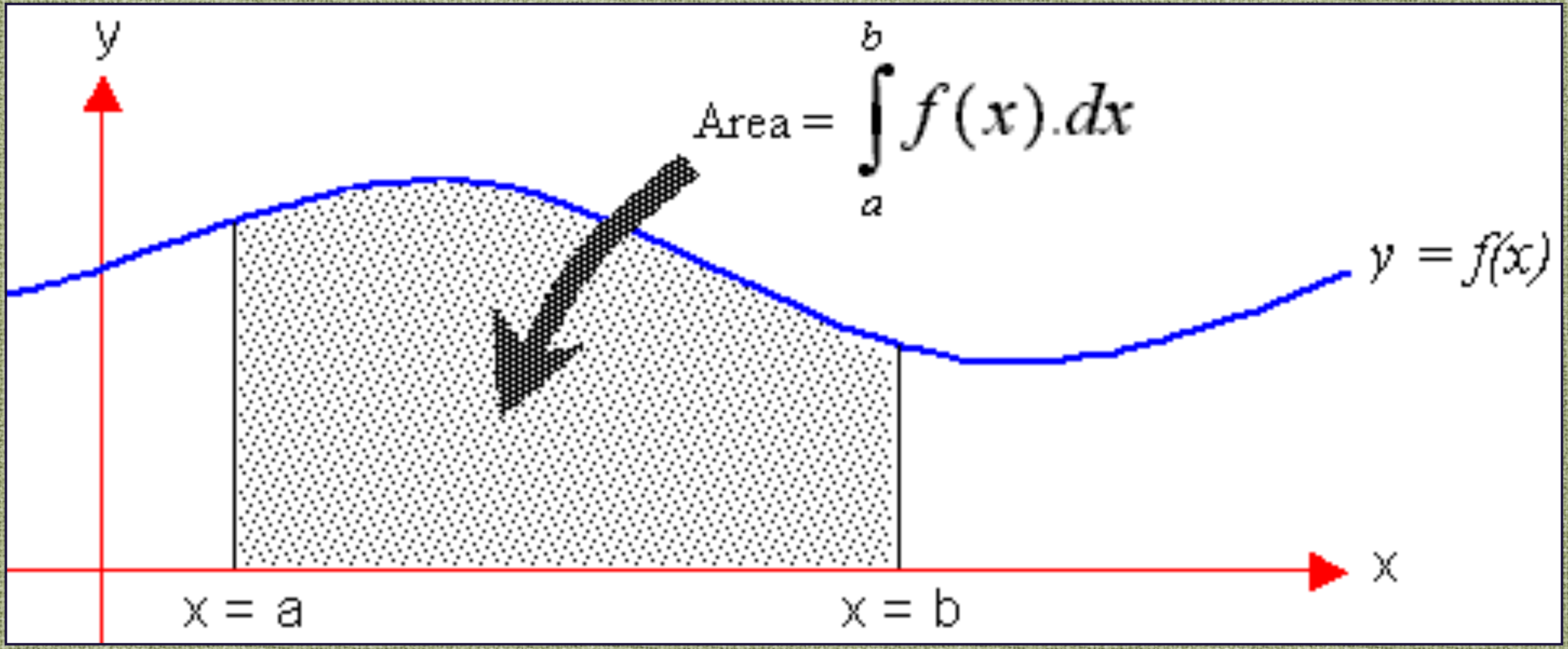
Rewrite: $\int \frac{u^7\,du}{2} = \frac{1}{2}\int u^7\,du = \frac{1}{2}\left(\frac{u^8}{8}\right) + C = \frac{u^8}{16} + C = \frac{(x^2-1)^8}{16} + C.$

Thus the definite integral

$$\int_0^2 x(x^2-1)^7\,dx = \frac{(x^2-1)^8}{16}\Bigg]_0^2 = \frac{(2^2-1)^8}{16} - \frac{(0^2-1)^8}{16} = \frac{3^8}{16} - \frac{(-1)^8}{16} = \frac{3^8-1}{16} = 410.$$

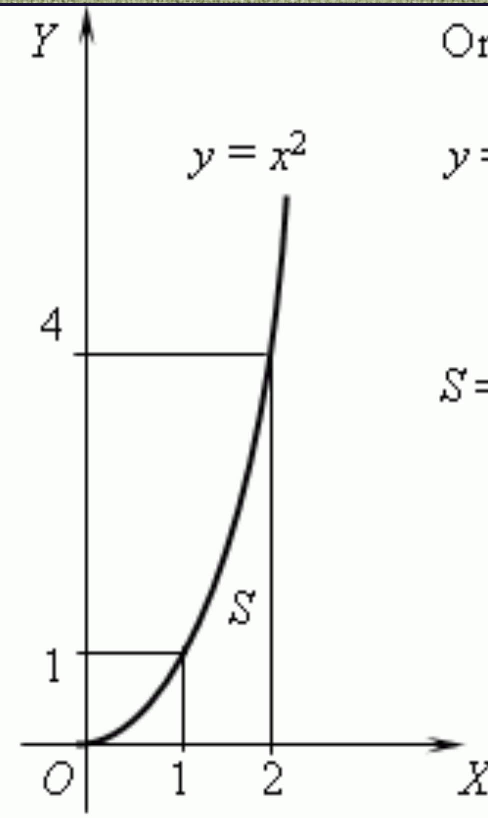
Areas:

Over x axis:



Example 1: Find the area of the region bounded by the graphs:

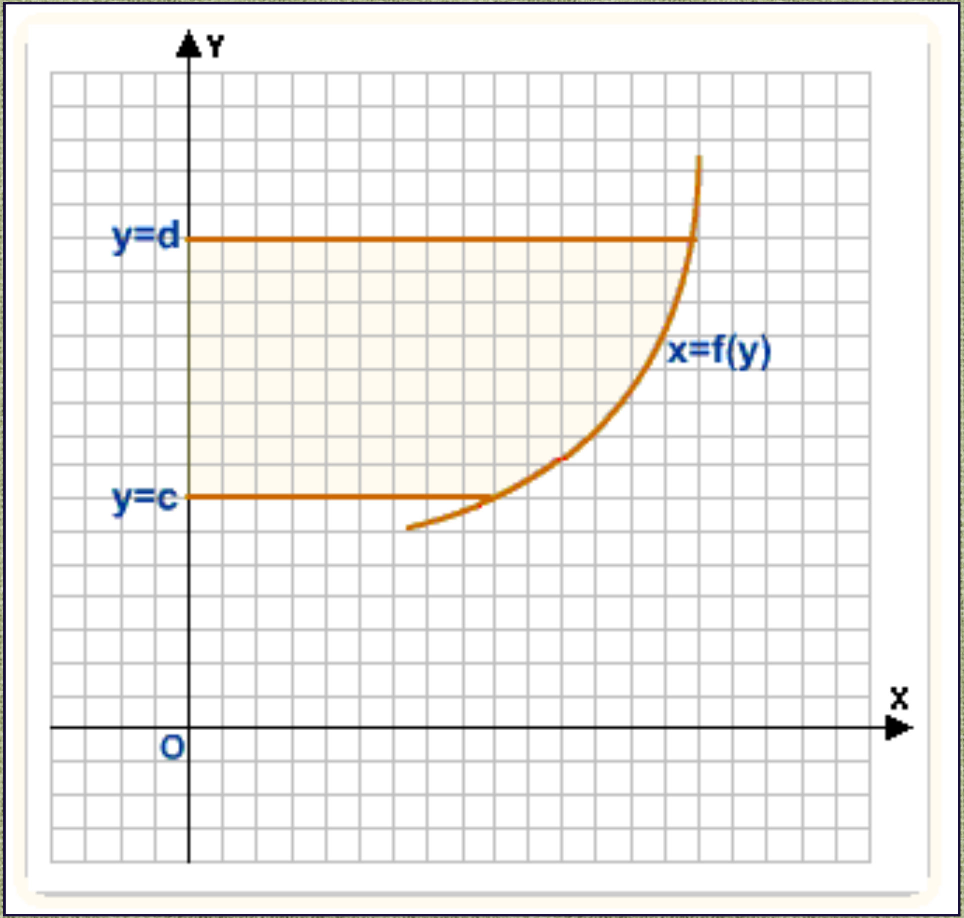
$x = 1,\ x = 2,\ y = 0,\ y = x^2.$



One of primitives for the function $y = x^2$ is $F(x) = x^3/3$. Then

$$S = F(2) - F(1) = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}.$$

Over y axis:



$$Area = \int_c^d f(y)dy$$

Example 2: Find the area of the region bounded by the graphs:

$$y = 1,\ y = 4,\ y = x^2.$$

Solution:

$$Area = \int_1^4 f(y)dy = \int_1^4 y^{\frac{1}{2}} dy = \left[y^{\frac{3}{2}} \right]_1^4 = \left[4^{\frac{3}{2}} - 1 \right] = 8 - 1 = 7\,u.a.$$

Example 2: Find the area of the following regions bounded by the graphs:

1) $x + y = 2, y = 2, y = 2x - 4;$

2) $x + y = 2, x = 0, y = 2x - 4;$

3) $y = 0, x = 1, x = 2, y = 4x^2;$

4) $x = 2, y = 0, \ y = -x^2;$

5) $y = 0, y = 2x + 2, y = -x + 1;$

Solution:

$$1)\ Area = \int_0^2 f(y) - g(y)dy = \int_1^4 (\frac{y}{2} + 2) - (-y + 2)dy$$

$$= \int_1^4 \frac{3y}{2}dy = \frac{3}{2}\left[\frac{y^2}{2}\right]_1^4 = \frac{3}{4}\left[4^2 - 1\right] = \frac{45}{4}u.a.$$

$$2)\ Area = \int_0^2 f(x) - g(x)dx = \int_0^2 (-x+2) - (2x-4)dx$$

$$= \int_0^2 (-3x+6)dx = \left[-\frac{3x^2}{2} + 6x \right]_0^2 = \left[-6+12 \right] = 6u.a.$$

$$3)\ Area = \int_1^2 f(x) - g(x)dx = \int_0^2 (4x^2) - (0)dx = [\frac{4}{3}x^3]_0^2 = \frac{4}{3}2^3 = \frac{32}{3}u.a.$$

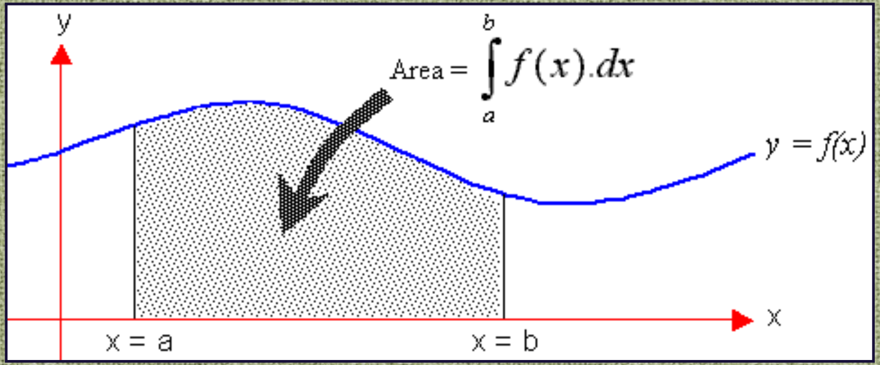
$$4)\ Area = \int_0^2 f(x) - g(x)dx = \int_0^2 (0) - (-x^2)dx$$

$$= \int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \left[\frac{2^3}{3} - \frac{0^3}{3} \right] = \frac{8}{3}u.a.$$

$$5)\ Area = \int_0^2 f(y) - g(y)dy = \int_0^2 (\frac{y}{2} - 1) - (-y-1)dy$$

$$= \int_0^2 (\frac{y}{2} - 1) - (-y-1)dy = \int_0^2 \frac{3}{2} ydx = \left[\frac{3y^2}{4} \right]_0^2 = \left[\frac{12}{4} - \frac{0^3}{4} \right] = 3u.a.$$

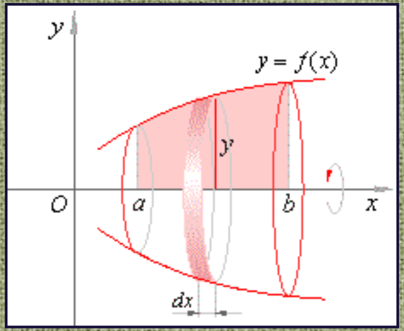
Volumes:

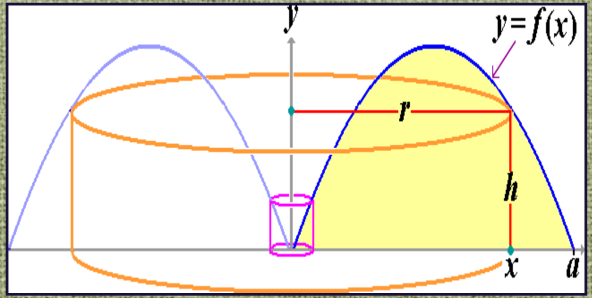


Region 1

about x

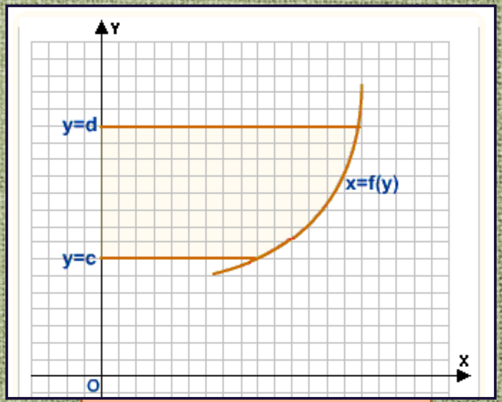
about y





Formula 1

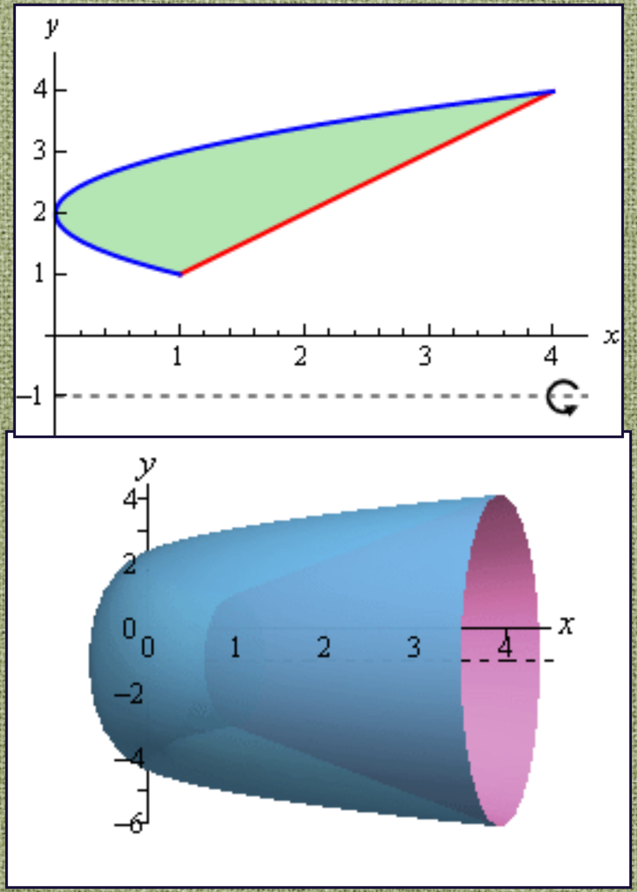
Formula 2

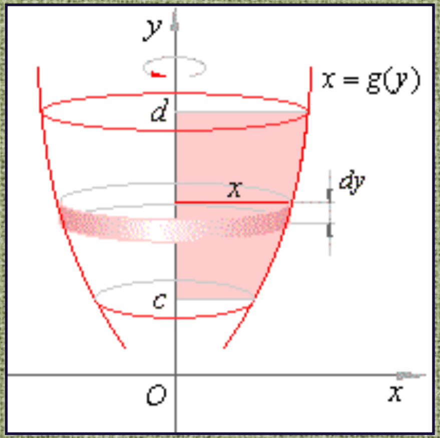


Region 2

about x

about y

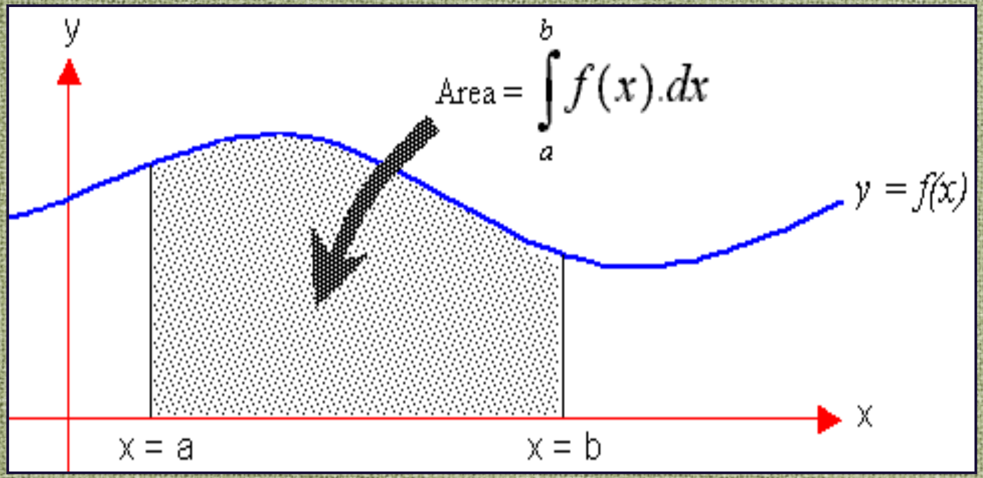




Formula 3

Formula 4

Volumes:



Region 1

about x

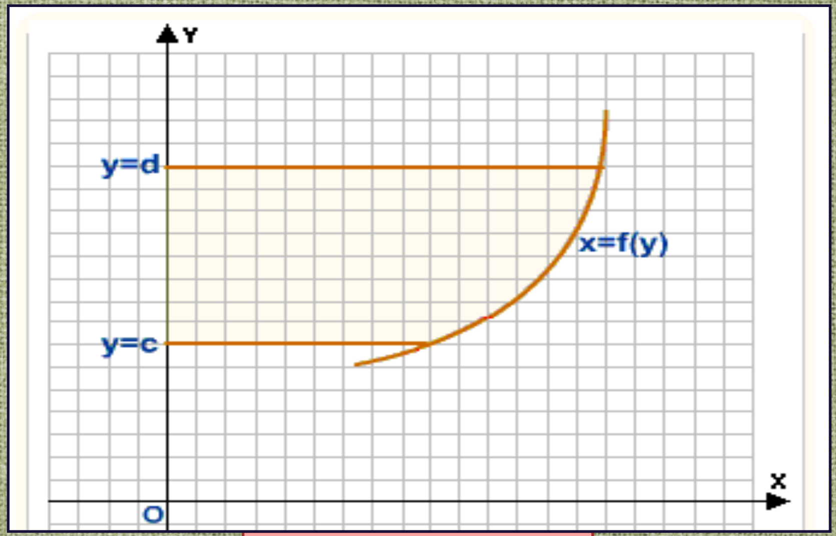
about y

$$V = \int_a^b 2\pi x\, [f(x) - g(x)]dx$$

Formula 2

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]dx$$

Formula 1



Region 2

about x

about y

$$V = \int_c^d 2\pi y\, [f(y) - g(y)]dy$$

Formula 3

$$V = \int_c^d \pi[(f(y))^2 - (g(y))^2]dy$$

Formula 4

Conclusion:

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx$$ → about x — R1

$$V = \int_a^b 2\pi x\,[f(x) - g(x)]\,dx$$ → about y — R1

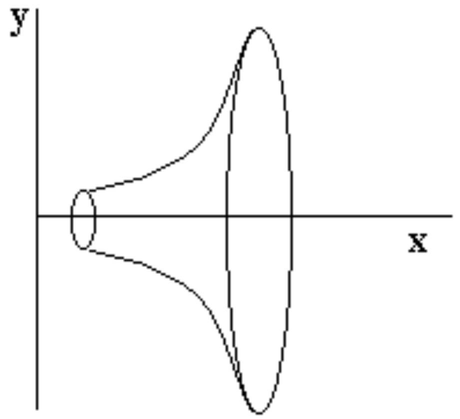
$$V = \int_c^d 2\pi y\,[f(y) - g(y)]\,dy$$ → about x — R2

$$V = \int_c^d \pi[(f(y))^2 - (g(y))^2]\,dy$$ → about y — R2

Example 1:

The graph of $y = x^2$ between $x = 1$ and $x = 3$ is rotated completely around the x-axis. Find the volume generated.

Solution:



$$\text{volume} = \int_1^3 \pi x^4 \, dx$$

$$= \left[\frac{\pi x^5}{5} \right]_1^3$$

$$= \frac{243\pi}{5} - \frac{\pi}{5}$$

$$= 48.4\pi$$

Exercises

1. Find the volume of the solid of revolution generated when the area described is rotated about the x-axis.

   (a) The area between the curve $y = x$ and the ordinates $x = 0$ and $x = 4$.

   (b) The area between the curve $y = x^{3/2}$ and the ordinates $x = 1$ and $x = 3$.

2. The area between the curve $y = 1/x$, the y-axis and the lines $y = 1$ and $y = 2$ is rotated about the y-axis. Find the volume of the solid of revolution formed.

3. The area between the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 2$ is rotated about the y-axis. Find the volume of the solid of revolution formed.

4. The area cut off by the x-axis and the curve $y = x^2 - 3x$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

Answers

1. (a) $21\frac{1}{3}\pi$ (b) 20π
2. $\frac{1}{2}\pi$
3. 2π
4. $\frac{81}{10}\pi$

Exercises:

1. Find the areas of the regions bounded by the curves given in the list of exercises on page 8-9.
2. Find the volume of the solid of revolution generated by rotation about one of the coordinate axis of the region R limited by the curves given in the list of exercises on page 9-10. :