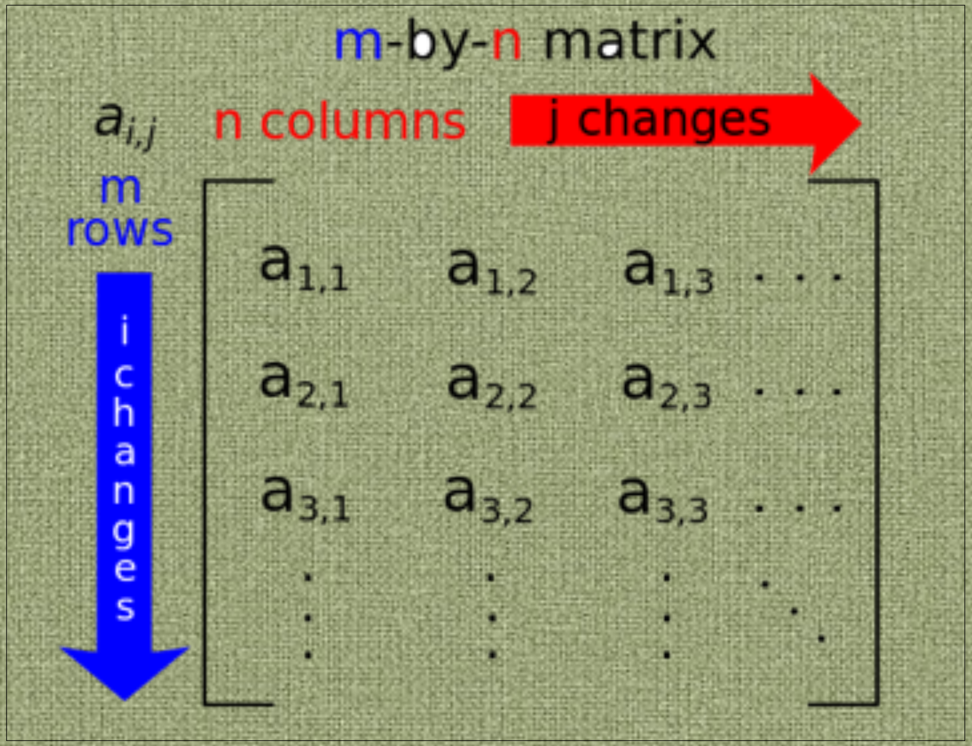
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Chapter 2: Matrices and Determinants

Section 2.1: Matrices

Def. A matrix is a set of numbers arranged in rows and columns in a rectangular table.

$A=(a_{ij})=$



$=(a_{ij})_{m \times n}$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

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Department of Mathematics
King Saud University

Each element (or entry) of a matrix is denoted by a variable with two subscripts. For instance, $a_{2,1}$ represents the element at the second row and first column of a matrix A.

The *order* (or *size*) of a matrix is given by the number of rows and the number of columns. In the following example

$$\begin{bmatrix} 4 & -7 & 5 & 0 \\ -2 & 0 & 11 & 8 \\ 19 & 1 & -3 & 12 \end{bmatrix}$$

the order of A is 3x4 and the order in the first matrix is mxn.

If $m=n$, then the matrix is called a square matrix.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Properties:

- A matrix with just one row is called a row vector.
- A matrix with just one column is called a column vector.
- A scalar in matrix algebra is a 1×1 matrix.
- Types of Matrices: Null (zero) Matrix: If all elements of a matrix is zero the matrix is called null or zero matrix.
- Diagonal Matrix: A square matrix which has at least one nonzero element on its main diagonal and zeros elsewhere is a diagonal matrix.
- Identity (unit) Matrix: A diagonal matrix whose all elements on the main diagonal are equal to one is called identity or unit matrix. A unit matrix is usually denoted by letter I.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

- **Triangular Matrices:** If all elements above the main diagonal of a square matrix are zero the matrix is called "lower triangular matrix".

- Alternatively, if all elements under the main diagonal of a square matrix are zero the matrix is called "upper triangular matrix".

- **Equality in matrices:** Two matrices ***A*** and ***B*** are equal if they have the same order and their corresponding elements are equal.

$$A = B \quad \Leftrightarrow \quad order(A) = order(B) \quad and \;\; a_{ij} = b_{ij}, \forall i,j$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Matrix Operations:

- Scalar Multiplication: If k is a scalar then $k.\,\boldsymbol{A} = k.\,(a_{ij})_{m\times n}$.

  This means that all elements of the matrix are multiplied by the scalar k.
- Matrix Addition & Subtraction: Addition and subtraction are defined for the matrices of the same order. It is not possible to add or subtract matrices from different orders. In both cases the corresponding elements are added or subtracted:

$$\boldsymbol{A}_{m\times n} \pm \boldsymbol{B}_{m\times n} = (a_{ij})_{m\times n} \pm (b_{ij})_{m\times n} = (a_{ij} \pm b_{ij})_{m\times n}$$

Properties of Addition & Subtraction:

- $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ Commutative law
- $\boldsymbol{A} \pm \boldsymbol{B} \pm \boldsymbol{C} = (\boldsymbol{A} \pm \boldsymbol{B}) \pm \boldsymbol{C} = \boldsymbol{A} \pm (\boldsymbol{B} \pm \boldsymbol{C})$ Associative law
- $\boldsymbol{k}.\,(\boldsymbol{A} \pm \boldsymbol{B}) = \boldsymbol{k}\boldsymbol{A} \pm \boldsymbol{k}\boldsymbol{B}$ ($\boldsymbol{k}$ is a scalar)

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Matrix Operations (conti.):

- Matrix Multiplication: Multiplication of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, in the form of $\boldsymbol{A} \cdot \boldsymbol{B}$ or $\boldsymbol{AB}$, is possible if the number of columns in $\boldsymbol{A}$ is equal to the number of rows in $\boldsymbol{B}$. The result of this multiplication is another matrix $\boldsymbol{C}$ where the number of its rows is equal to the number of rows in $\boldsymbol{A}$ and number of its columns is equal to the number of columns in $\boldsymbol{B}$; that is: $\boldsymbol{A}_{m\times n} \times \boldsymbol{B}_{n\times p} = \boldsymbol{C}_{m\times p}$.

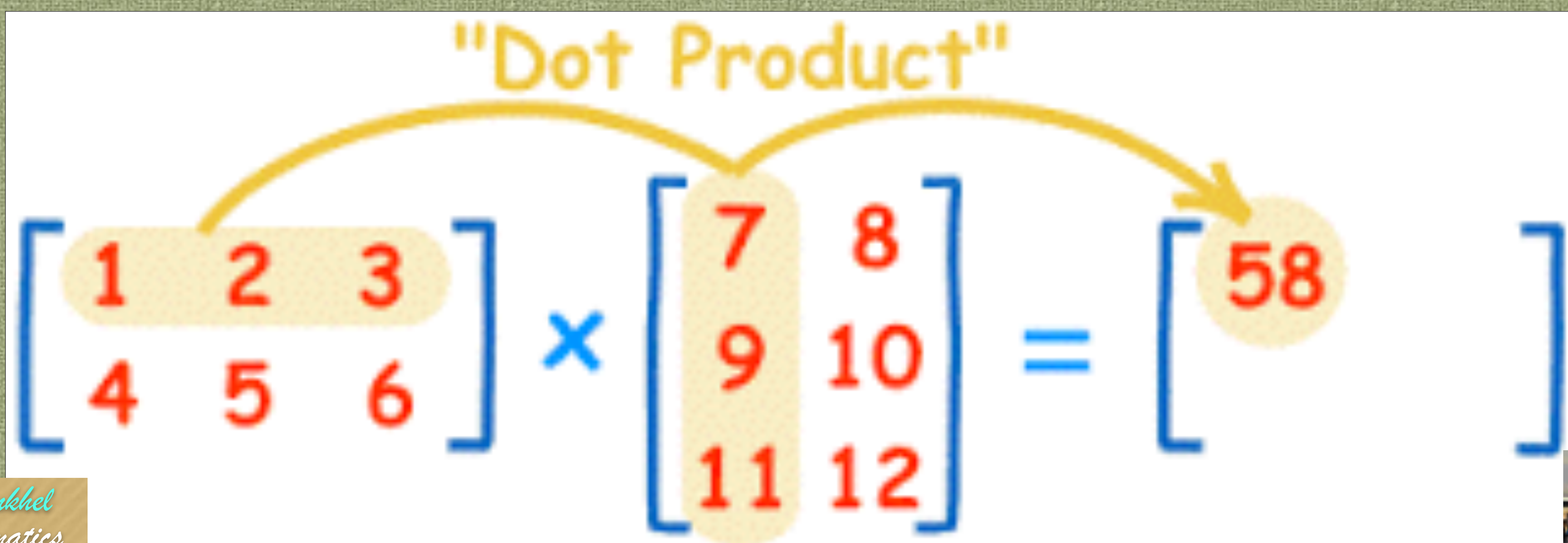
- Elements of $\boldsymbol{C}$ can be calculated by adding some multiplications; multiplications of the elements in the i-th row of $\boldsymbol{A}$ by the corresponding elements in the j-th column of $\boldsymbol{B}$. The element $c_{i\times j}$ is the result of the multiplication of the i^{th} row in $\boldsymbol{A}$ by the j^{th} column in $\boldsymbol{B}$.
- For example, matrix $\boldsymbol{A}_{3\times 3}$ cannot be multiplied by a horizontal vector $\boldsymbol{B}_{1\times 3}$ but it can be multiplied by a vertical vector $\boldsymbol{B}_{3\times 1}$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Matrix Operations (conti.):

- Properties of Matrix Multiplication:
- In general, $AB \neq BA$ if both exist, but there are special cases that this property is not true. If I is an identity matrix, then $IB = BI = B$.
- $A(B+C) = AB + AC$ and $(B + C)A = BA + CA$
- $ABC = A(B\,C) = (AB)C$.
- From $AB = \mathbf{0}$ we cannot conclude necessarily that $A = \mathbf{0}$ *or* $B = \mathbf{0}$.
- From $AB = AC$ we cannot conclude necessarily that $B = C$.



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

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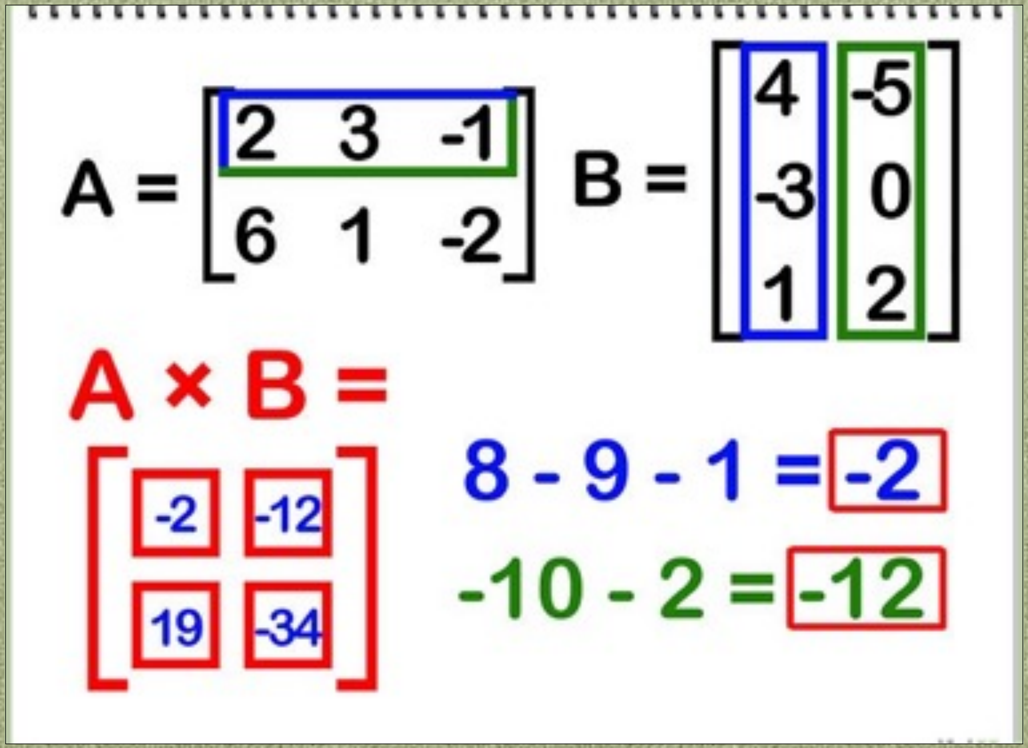
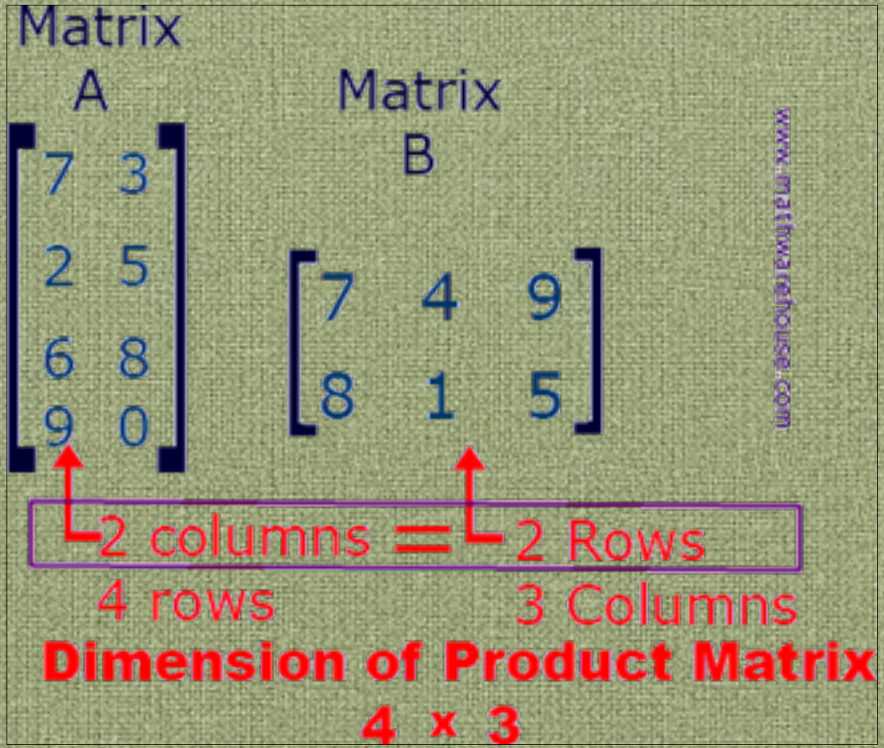
Department of Mathematics
King Saud University





Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Section 2.2: Determinants

Definition. The determinant of a square matrix A is a real number denoted by det(A) or |A|.

- The determinant of 2-by-2 matrices is given by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- Example:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\text{x}4 - 2\text{x}3 = -2$$

Prof. Messaoud Boumkhel
Department of Mathematics
King Saud University

- The determinant of 3-by-3 matrices is given by

$$\det(A) = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

- Example:

$$\begin{matrix} & & & 0 & -105 & 0 \\ -7 & 6 & -5 & -7 & 6 & \\ 0 & -1 & -5 & 0 & -1 & \\ 0 & -3 & 8 & 0 & -3 & \\ & & & 56 & 0 & 0 \end{matrix}$$

$$\begin{vmatrix} -7 & 6 & -5 \\ 0 & -1 & -5 \\ 0 & -3 & 8 \end{vmatrix} = 56 + 0 + 0 - (0 + -105 + 0) = 161$$

Prof. Messaoud Boumkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

- The determinant of 3-by-3 matrices (2nd method):

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

- Example:

$$|A| = \begin{vmatrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 6 & -2 \\ 2 & -3 \end{vmatrix} - (2) \cdot \begin{vmatrix} 5 & -2 \\ 5 & -3 \end{vmatrix} + (-4) \cdot \begin{vmatrix} 5 & 6 \\ 5 & 2 \end{vmatrix}$$

$$= 6\,[-18 - (-4)] - 2\,[-15 - (-10)] - 4\,[10 - 30]$$

$$= 6\,(-14) - 2\,(-5) - 4\,(-20)$$

$$= -84 + 10 + 80$$

$$= 6$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

- The determinant of 4-by-4 matrices:

  It needs a recursive process. We reduce it to the sum of four 3x3 determinants as we show in the following example:

Example: Find the determinant of the matrix:

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}$$

Prof. Messaoud Bounkhel
Department of Mathematics
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Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix} - 4\begin{vmatrix} 0 & 4 & 4 \\ -1 & 1 & 0 \\ 2 & 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 & 4 \\ -1 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 & 4 \\ -1 & 0 & 1 \\ 2 & 0 & 4 \end{vmatrix}$$

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