



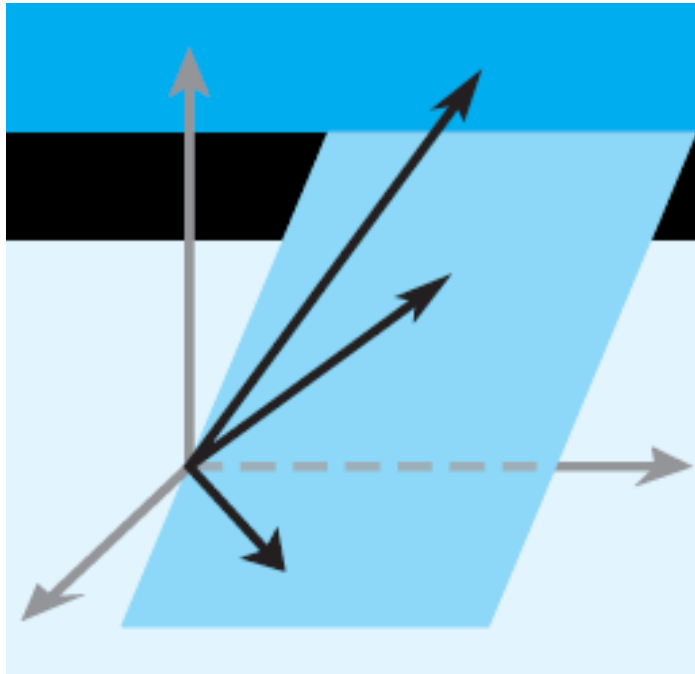
Faculty of Engineering
Mechanical Engineering Department

Linear Algebra and Vector Analysis

MATH 1120

Lecture 16

Elementary Linear Algebra



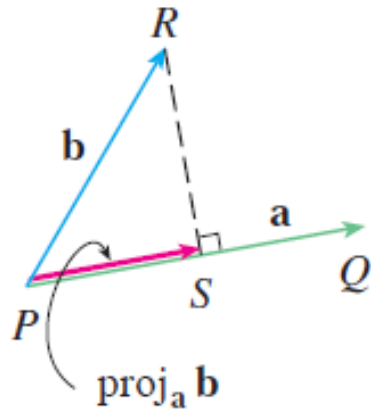
Chapter 3

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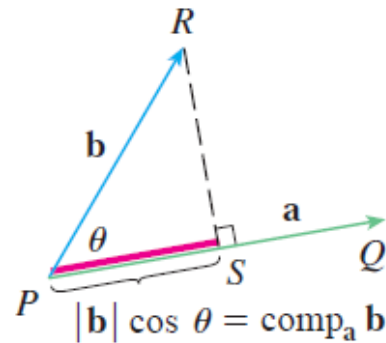
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Projections



Vector projections



Scalar projection

Scalar projection of \mathbf{b} onto \mathbf{a} : $\text{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of \mathbf{b} onto \mathbf{a} : $\text{proj}_a \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

V EXAMPLE 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

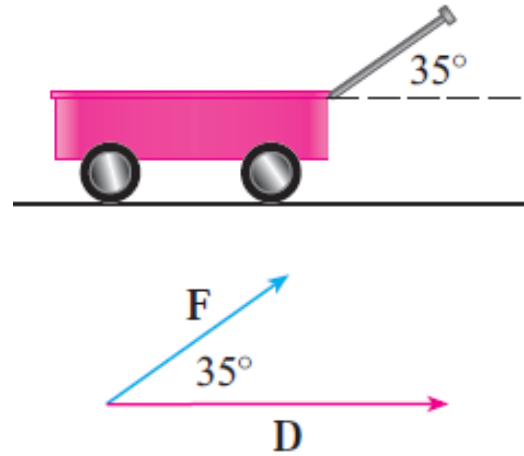
SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of \mathbf{b} onto \mathbf{a} is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of \mathbf{a} :

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14} \mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

EXAMPLE A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.




SOLUTION If \mathbf{F} and \mathbf{D} are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos 35^\circ \\ &= (70)(100) \cos 35^\circ \approx 5734 \text{ N}\cdot\text{m} = 5734 \text{ J} \end{aligned}$$

EXAMPLE A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2 \rangle$, so by Equation 12, the work done is

$$\begin{aligned}W &= \mathbf{F} \cdot \mathbf{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 = 36\end{aligned}$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J. 

The Cross Product

- Given two nonzero vectors $\underline{\mathbf{a}}$ (a_1, a_2, a_3) and $\underline{\mathbf{b}}$ (b_1, b_2, b_3), it is very useful to be able to find a nonzero vector $\underline{\mathbf{c}}$ that is perpendicular to both and

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\boxed{5} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\boxed{6} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\boxed{7} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}\end{aligned}$$

V **EXAMPLE 2** Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector \mathbf{a} in V_3 .

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned}\mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} \\ &= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\end{aligned}$$

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} , we compute their dot product as follows:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\ &= 0\end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} . ■

9 Theorem If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

10 Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to both \vec{PQ} and \vec{PR} and is therefore perpendicular to the plane through P , Q , and R . We know from (12.2.1) that

$$\vec{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\vec{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\ &= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}\end{aligned}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane.

EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 3 we computed that $\vec{PQ} \times \vec{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$|\vec{PQ} \times \vec{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■