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Linear operators and Linear functionals on normed spaces

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Submitted in partial fulfillment of the requirements for the degree of Bachelor of Science in mathematics

Jan 2009

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Normed Spaces, Banach Spaces

1.1 *Normed and Banach Space*

Definition (1.1.1)

A *normed space* X is a vector space with a norm defined on it A norm on a vector X is a real valued function $\|.\|: \mathrm{X} \to \Re$ value at an $x \in X$ is denoted by $\|x\|$ and which has the properties:

1- $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0.$

2- $\|\alpha x\| = |\alpha|\|x\|.$

3- $\|x + y\| \leq \|x\| + \|y\|.$

where x, y are arbitrary vector in X and α is any scalar. A normed space is a pair $(X, \|.\|)$ simply by X.

Remark (1.1.2)

Let $\|.\|: \mathrm{X} \to \Re$ be a norm on X , then the norm is continuous on X .

Proof:

Let x_o be an arbitrary point of X , and let $\varepsilon > 0$ be given

Take $\delta = \varepsilon$

$x \in X$ such that $\|x - x_o\| < \delta = \varepsilon$

$$\|x\| = \|x + x_o - x_o\| \leq \|x - x_o\| + \|x_o\| \to \|x\| - \|x_o\| \leq \|x - x_o\| \qquad (1)$$

$$\|x_o\| = \|x_o + x - x\| \leq \|x_o - x\| + \|x\| \to \|x_o\| - \|x\| \leq \|x - x_o\|$$

$$\to \|x\| - \|x_o\| \geq -\|x - x_o\| \qquad (2)$$

from (1) and (2) we have:

$$-\|x - x_o\| \leq \|x\| - \|x_o\| \leq \|x - x_o\|$$

$$\to \big|\|x\| - \|x_o\|\big| \leq \|x - x_o\| < \delta = \varepsilon$$

then $\|.\|: X \to \Re$ is continuous at x_o, since x_o is arbitrary point of X ,then $\|.\|$ is continuous on X .

Remark *(Minkowski inequality)* (1.1.3)

Given two sequences $(\xi_i)_{i=1}^{\infty}, (\eta_i)_{i=1}^{\infty}$ s.t. $\sum_{i=1}^{\infty} |\xi_i|^p < \infty, \sum_{i=1}^{\infty} |\eta_i|^p < \infty, p > 1$

Then $(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p)^{1/p} \leq (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} + (\sum_{i=1}^{\infty} |\eta_i|^p)^{1/p}$.

Examples of normed spaces:

Example (1):

Define $\|.\| : \Re^n \to \Re$ by $\|x\| = (\sum_{i=1}^{n} \xi_i^2)^{1/2}, x = (\xi_1, \xi_2, ..., \xi_n)$

Clearly $\|.\|$ is well defined.

Now, Let $x, y \in \Re^n$ and α is any scalar:

1- $\|x\| = (\sum_{i=1}^{n} \xi_i^2)^{1/2} \geq 0,$

and $\|x\| = 0 \Leftrightarrow (\sum_{i=0}^{n} \xi_i^2)^{1/2} = 0 \Leftrightarrow \xi_i^2 = 0 \forall i \Leftrightarrow \xi_i = 0 \forall i \Leftrightarrow x = 0.$

2- $\|\alpha x\| = (\sum_{i=1}^{n} (\alpha \xi_i)^2)^{1/2} = (\sum_{i=1}^{n} \alpha^2 \xi_i^2)^{1/2} = (\alpha^2 \sum_{i=1}^{n} \xi_i^2)^{1/2}$

$= (\alpha^2)^{1/2} (\sum_{i=1}^{n} \xi_i^2)^{1/2} = |\alpha| \|x\|.$

3- $\|x + y\| = (\sum_{i=1}^{n} (\xi_i + \eta_i)^2)^{1/2}$

$\leq (\sum_{i=1}^{n} \xi_i^2)^{1/2} + (\sum_{i=1}^{n} \eta_i^2)^{1/2} = \|x\| + \|y\|.$ (by Minkowski inequality)

Hence, from 1, 2, and 3 $(\Re^n, \|.\|)$ is norm space.

Example (2):

Let $\Re^2 = \{x = (\xi_1, \xi_2) : \xi_1, \xi_2 \in \Re\}$, Let $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2)$ are any elements in $\Re^2$, α is any scalar, then the following equations are norms on $\Re^2$:

(a) $\|x\|_1 = |\xi_1| + |\xi_2|$

1- $\|x\|_1 = |\xi_1| + |\xi_2| \geq 0,$

and $\|x\|_1 = 0 \Leftrightarrow |\xi_1| + |\xi_2| = 0 \Leftrightarrow \xi_1 = 0, \xi_2 = 0 \Leftrightarrow x = 0.$

2- $\|\alpha x\|_1 = \|(\alpha\xi_1, \alpha\xi_2)\|_1 = |\alpha\xi_1| + |\alpha\xi_2| = |\alpha|(|\xi_1| + |\xi_2|) = |\alpha|\|x\|_1.$

3- $\|x + y\|_1 = \|(\xi_1 + \eta_1, \xi_2 + \eta_2)\|_1 = |\xi_1 + \eta_1| + |\xi_2 + \eta_2|$

$\leq |\xi_1| + |\eta_1| + |\xi_2| + |\eta_2| = (|\xi_1| + |\xi_2|) + (|\eta_1| + |\eta_2|) = \|x\|_1 + \|y\|_1.$

Hence, from 1, 2, and 3 $(\Re^2, \|.\|_1)$ is norm space.

(b) $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2}$

1- $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2} \geq 0,$

and $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2} = 0 \Leftrightarrow \xi_1^2 + \xi_2^2 = 0 \Leftrightarrow \xi_1^2 = 0, \xi_2^2 = 0$

$\Leftrightarrow \xi_1 = 0, \xi_2 = 0 \Leftrightarrow x = 0.$

2- $\|\alpha x\|_2 = ((\alpha\xi_1)^2 + (\alpha\xi_2)^2)^{1/2} = (\alpha^2(\xi_1^2, \xi_2^2))^{1/2}$

$= |\alpha|(\xi_1^2 + \xi_2^2)^{1/2} = |\alpha|\|x\|_2.$

3- $\|x + y\|_2 = \|(\xi_1 + \eta_1, \xi_2 + \eta_2)\|_2 = ((\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2)^{1/2}$

$\leq (\xi_1^2 + \xi_2^2)^{1/2} + (\eta_1^2 + \eta_2^2)^{1/2}$ (by Minkowski inequality)

$= \|x\|_2 + \|y\|_2$

Hence, from 1, 2, and 3 $(\Re^2, \|.\|_2)$ is norm space.

(c) $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\}$

1- $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\} \geq 0,$

and $\|x\|_\infty = 0 \Leftrightarrow \max\{|\xi_1|, |\xi_2|\} = 0 \Leftrightarrow \xi_1 = 0, \xi_2 = 0 \Leftrightarrow x = 0.$

2- $\|\alpha x\|_\infty = \max\{|\alpha\xi_1|, |\alpha\xi_2|\} = |\alpha| \max\{|\xi_1|, |\xi_2|\} = |\alpha|\|x\|_\infty.$

3- $\|x + y\|_\infty = \max\{|\xi_1 + \eta_1|, |\xi_2 + \eta_2|\} \leq \max\{|\xi_1| + |\eta_1|, |\xi_2| + |\eta_2|\}$

$= \max\{|\xi_1|, |\xi_2|\} + \max\{|\eta_1|, |\eta_2|\} = \|x\|_\infty + \|y\|_\infty.$

Hence, from 1, 2, and 3 $(\Re^2, \|.\|_\infty)$ is norm space.

Example (3):

There are several norms of practical importance on the vector space of ordered n-tuples of numbers, notably those defined by

$(a)\|x\|_1 = |\xi_1| + |\xi_2| + ... + |\xi_n|$

$(b)\|x\|_p = (|\xi_1|^p + |\xi_2|^p + ... + |\xi_n|^p)^{1/p}$ $1 < p < +\infty$

$(c)\|x\|_\infty = \max\{|\xi_1|, |\xi_2|, ..., |\xi_n|\}.$

Now, $x = (\xi_1, \xi_2, ..., \xi_n)$, $y = (\eta_1, \eta_2, ..., \eta_n)$ and α is any scalar:

(a) $\|x\|_1 = |\xi_1| + |\xi_2| + ... + |\xi_n|$

1- $\|x\|_1 = |\xi_1| + |\xi_2| + ... + |\xi_n| \geq 0$,

and $\|x\|_1 = 0 \Leftrightarrow |\xi_1| + ... + |\xi_n| = 0 \Leftrightarrow \xi_i = 0 \forall 1 \leq i \leq n \Leftrightarrow x = 0$.

2- $\|\alpha x\|_1 = |\alpha\xi_1| + ... + |\alpha\xi_n| = |\alpha|(|\xi_1| + ... + |\xi_n|) = |\alpha|\|x\|_1$.

3- $\|x + y\|_1 = |\xi_1 + \eta_1| + ... + |\xi_n + \eta_n|$

$\leq |\xi_1| + |\eta_1| + ... + |\xi_n| + |\eta_n| = (|\xi_1| + ... + |\xi_n|) + (|\eta_1| + ... + |\eta_n|)$

$= \|x\|_1 + \|y\|_1$.

(b) $\|x\|_p = (|\xi_1|^p + ... + |\xi_n|^p)^{1/p}$ $\quad 1 < p < +\infty$

1- $\|x\|_p = (|\xi_1|^p + ... + |\xi_n|^p)^{1/p} \geq 0$,

and

$\|x\|_p = 0 \Leftrightarrow (|\xi_1|^p + ... + |\xi_n|^p)^{1/p} = 0 \Leftrightarrow \xi_i = 0 \forall 1 \leq i \leq n \Leftrightarrow x = 0$.

2- $\|\alpha x\|_p = (|\alpha\xi_1|^p + ... + |\alpha\xi_n|^p)^{1/p} = (|\alpha|^p (|\xi_1|^p + ... + |\xi_n|^p)^{1/p}$

$= |\alpha|(|\xi_1|^p + ... + |\xi_n|^p)^{1/p} = |\alpha|\|x\|_p$.

3- $\|x + y\|_p = (\sum_{i=1}^{n} |\xi_i + \eta_i|^p)^{1/p}$

$\leq (\sum_{i=1}^{n} |\xi_i|^p)^{1/p} + (\sum_{i=1}^{n} |\eta_i|^p)^{1/p}$ (by Minkowski inequality)

$= \|x\|_p + \|y\|_p$.

(c) $\|x\|_\infty = \max \{|\xi_1|, ..., |\xi_n|\}$

1- $\|x\|_\infty = \max \{|\xi_1|, ..., |\xi_n|\} \geq 0$, since $|\xi_i| \geq 0 \quad \forall 1 \leq i \leq n$,

$\|x\|_\infty = 0 \Leftrightarrow \max \{|\xi_1|, ..., |\xi_n|\} = 0 \Leftrightarrow \xi_i = 0 \forall 1 \leq i \leq n \Leftrightarrow x = 0$.

2- $\|\alpha x\|_\infty = \max \{|\alpha\xi_1|, ..., |\alpha\xi_n|\} = |\alpha| \max \{|\xi_1|, ..., |\xi_n|\} = |\alpha|\|x\|_\infty$.

3- $\|x + y\|_\infty = \max \{|\xi_1 + \eta_1|, ..., |\xi_n + \eta_n|\} \leq \max \{|\xi_1| + |\eta_1|, ..., |\xi_n| + |\eta_n|\}$

$= \max \{|\xi_1|, ..., |\xi_n|\} + \max \{|\eta_1|, ...|\eta_n|\} = \|x\|_\infty + \|y\|_\infty$.

Example (4):

(Unit sphere), the sphere $S(0;1) = \{x \in X : \|x\| = 1\}$ in a normed space X is called the unit sphere; we want to show that for the following norms:

(a) $\|x\|_1 = |\xi_1| + |\xi_2|$

$S(0;1) = \{x \in \Re^2 : \|x\|_1 = 1\}$

$\|x\|_1 = |\xi_1| + |\xi_2| = 1 \qquad \Rightarrow |\xi_2| = 1 - |\xi_1|$

In 1st quarter $\xi_1 \geq 0, \xi_2 \geq 0$,
hence we get $L_1 : \xi_2 = 1 - \xi_1$,
which is straight line of slope -1,
and cutting the y-axis at (0,1),
and the x-axis at (1,0),
In 2sd quarter $\xi_1 \leq 0, \xi_2 \geq 0$,
hence we get $L_2 : \xi_2 = 1 + \xi_1$,
which is straight line of slope 1,
and cutting the y-axis at (0,1),
and the x-axis at (-1,0),
In 3rd quarter $\xi_1 \leq 0, \xi_2 \leq 0$, hence we get $L_3 : \xi_2 = -1 - \xi_1$,
which is straight line of slope -1, and cutting the y-axis at (0,-1),
and the x-axis at (-1,0),
In 4th quarter $\xi_1 \geq 0, \xi_2 \leq 0$, hence we get $L_4 : \xi_2 = -1 + \xi_1$,
which is straight line of slope 1, and cutting the y-axis at (0,-1),
and the x-axis at (1,0),
Then we have figure (1)

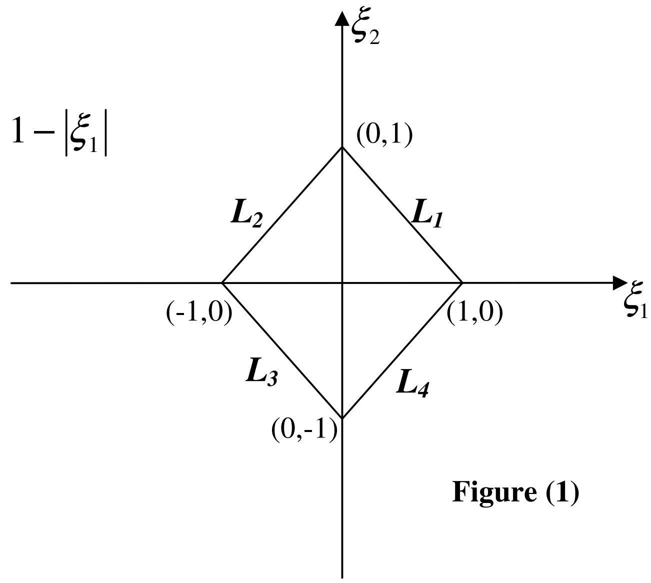

Figure (1)

(b) $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2}$

$S(0;1) = \{x \in \Re^2 : \|x\|_2 = 1\}$

$\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2} = 1 \qquad \Rightarrow \xi_1^2 + \xi_2^2 = 1,$

which is equation of circle
with center (0,0) and radius1,
then we have figure (2)

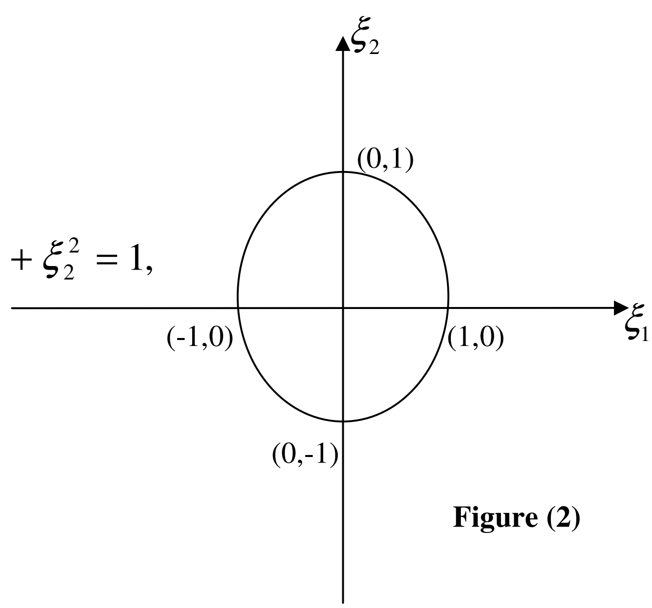

Figure (2)

c) $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\}$ $\quad S(0;1) = \{x \in \Re^2 : \|x\|_\infty = 1\}$

$\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\} = 1 \quad \Rightarrow |\xi_1| = 1 \quad or \quad |\xi_2| = 1,$

if $|\xi_1| = 1 \Rightarrow \xi_1 = 1 \quad or \quad \xi_1 = -1 \quad and \quad \xi_2 = 0,$

hence we get $L_1 : \xi_1 = 1 \quad and \quad L_2 : \xi_1 = -1,$

if $|\xi_2| = 1 \Rightarrow \xi_2 = 1 \quad or \quad \xi_2 = -1 \quad and \quad \xi_1 = 0,$

hence we get $L_3 : \xi_2 = 1 \quad and \quad L_4 : \xi_2 = -1,$

then we have figure (3)

hence the sphere

$S(0;1) = \{x \in \Re^2 : \|x\|_\infty = 1\}$

is the square as given in figure(3)

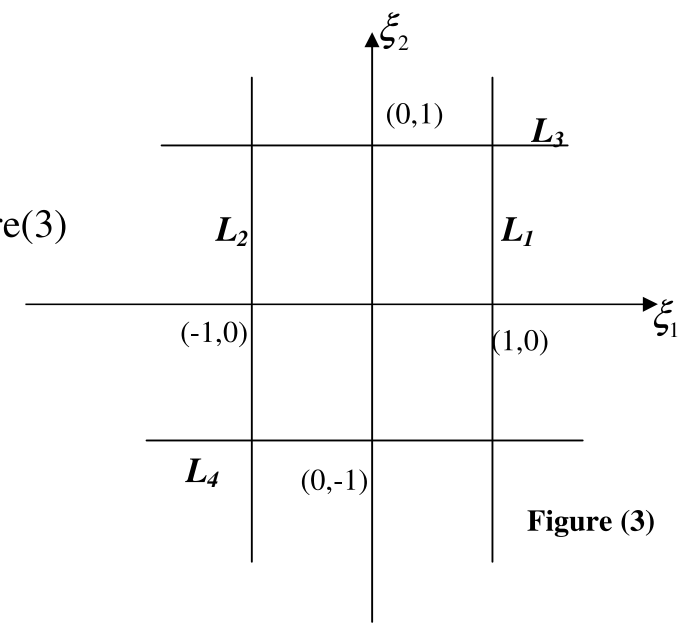

Figure (3)

(d) $\|x\|_4 = (\xi_1^4 + \xi_2^4)^{1/4}$

$S(0;1) = \{x \in \Re^2 : \|x\|_4 = 1\}$

$\|x\|_4 = (\xi_1^4 + \xi_2^4)^{1/4} = 1 \quad \Rightarrow \xi_1^4 + \xi_2^4 = 1$

Then we have the figure (4)

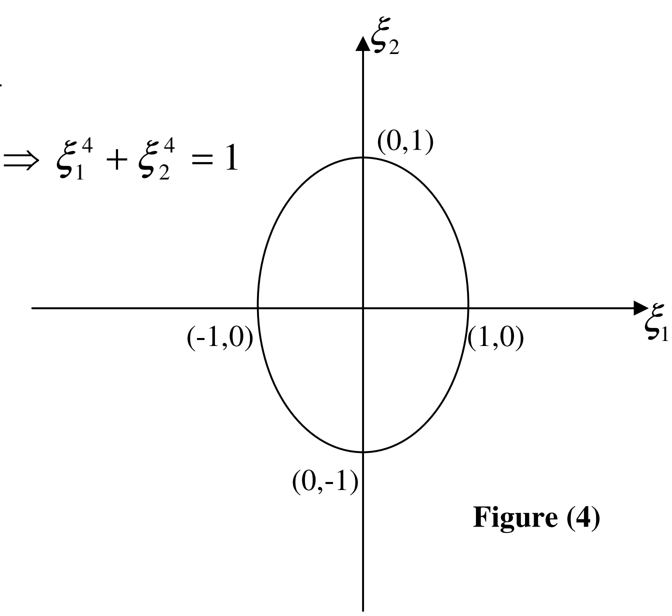

Figure (4)

<u>Definition</u> (1.1.4)

A norm on a vector space X *a* <u>*metric*</u> *d* on $X \times X$ which is given by

$$d(x, y) = \|x - y\| \qquad x, y \in X$$

d is well defined, since the norm is a well defined function

1- $d(x, y) = \|x - y\| \geq 0.$

2- $d(x, y) = 0 \Leftrightarrow x = y,$

$d(x, y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y.$

3- $d(x, y) = d(y, x),$

$d(x, y) = \|x - y\| = \|y - x\| = d(y, x).$

4- $d(x, y) \leq d(x, z) + d(z, y),$

$$d(x, y) = \|x - y\| = \|x - y + z - z\| \leq \|x - z\| + \|z - y\|$$
$$= d(x, z) + d(z, y).$$

Thus true, every normed space is a metric space.
The converse is not true,

<u>*Counterexample*</u>:

Let $d : S \times S \to \Re^{+}$,where S is set of all sequences, *d* defined by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$$

Let $x = (\xi_i), y = (\eta_i), z = (\alpha_i), \qquad x, y, z \in S$

1- $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} \geq 0.$

2- $d(x, y) = 0 \Leftrightarrow \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} = 0 \Leftrightarrow |\xi_i - \eta_i| = 0$

$$\Leftrightarrow \xi_i = \eta_i \forall i \Leftrightarrow x = y.$$

3- $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|\eta_i - \xi_i|}{1 + |\eta_i - \xi_i|} = d(y, x).$

4- $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i + \alpha_i - \alpha_i|}{1 + |\xi_i - \eta_i + \alpha_i - \alpha_i|}$

$$\le \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \alpha_i| + |\alpha_i - \eta_i|}{1 + |\xi_i - \alpha_i| + |\alpha_i - \eta_i|}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{|\xi_i - \alpha_i|}{1 + |\xi_i - \alpha_i| + |\alpha_i - \eta_i|} + \frac{|\alpha_i - \eta_i|}{1 + |\xi_i - \alpha_i| + |\alpha_i - \eta_i|} \right)$$

$$\le \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \alpha_i|}{1 + |\xi_i - \alpha_i|} + \sum_{i=1}^{n} \frac{1}{2^i} \frac{|\alpha_i - \eta_i|}{1 + |\alpha_i - \eta_i|} = d(x, z) + d(z, y).$$

then (S, d) is metric space.

On the other hand,

Let $x = (1,1,0,0,.....)$, $y = (1,0,0,0,.....)$, $\alpha = 3$

$\rightarrow \alpha x = (3,3,0,0,.....)$, $\alpha y = (3,0,0,0,.....)$

Now, $|\alpha| d(x, y) = \alpha \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} = 3 \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$

$$= 3 \left[\frac{1}{2^1} \frac{|1-1|}{1+|1-1|} + \frac{1}{2^2} \frac{|1-0|}{1+|1-0|} + 0 + ... \right] = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}.$$

and $d(\alpha x, \alpha y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\alpha\xi_i - \alpha\eta_i|}{1 + |\alpha\xi_i - \alpha\eta_i|}$

$$= \left[\frac{1}{2^1} \frac{|3-3|}{1+|3-3|} + \frac{1}{2^2} \frac{|3-0|}{1+|3-0|} + 0 + ... \right] = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}.$$

that means $|\alpha| d(x, y) \ne d(\alpha x, \alpha y)$,

hence d is not obtained from a norm, this may immediately be seen from the following lemma which states two basic properties of a metric d obtained from a norm.

Lemma (1.1.5)

A metric d induced by a norm on a norm space X satisfies:

(a) $d(x + a, y + a) = d(x, y)$.

(b) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.

for all $x, y, a \in X$ and every scalar α.

Proof:

$d(x + a, y + a) = \|x + a - (y + a)\| = \|x - y\| = d(x, y)$,

and $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha| \|x - y\| = |\alpha| d(x, y)$.

Definition (1.1.6)

Let $(x_n)_{n=1}^{\infty}$ be a sequence in a normed vector space $(X, \|.\|)$, we say (x_n) *converges* to x_o, and denoted by $x_n \to x_o$ if for any $\varepsilon > 0$ $\exists k_\varepsilon \in N$ such that $\forall n > k_\varepsilon \Rightarrow \|x_n - x_o\| < \varepsilon$.

Definition (1.1.7)

Let $(x_n)_{n=1}^{\infty}$ be a sequence in a normed vector space $(X, \|.\|)$, we say (x_n) is a *Cauchy sequence* if $\forall \varepsilon > 0$ $\exists k_\varepsilon \in N$
such that $\|x_n - x_m\| < \varepsilon$ $\forall n, m > k_\varepsilon$.

Definition (1.1.8)

Let $(X, \|.\|)$ be a normed vector space, we say X is *complete* or *Banach* if every Cauchy sequence in $(X, \|.\|)$ is convergent.

Examples of complete normed spaces:

Example (1):

Let $C[a,b] = \{x : x : [a,b] \to \Re$ is continuous $\}$ we define a norm $\|.\| : C[a,b] \to \Re$ by $\|x\| = \max_{t \in [a,b]} |x(t)|$ (1),

The norm is well defined, since x is continuous on a closed and bounded interval, that means x attains the maximum value on the interval, then $\max_{t \in [a,b]} |x(t)|$ exists and unique.

Now, we want to show that $(C[a,b], \|.\|)$ is norm space

Let x, y are any elements in $C[a,b]$, α is any scalar:

1- $\|x\| = \max_{t \in [a,b]} |x(t)| \geq 0$, since $|x(t)| \geq 0$ $\forall t \in [a,b]$
and $\|x\| = \max_{t \in [a,b]} |x(t)| = 0 \Leftrightarrow x(t) = 0$ $\forall t \in [a,b] \Leftrightarrow x = 0.$

2- $\|\alpha x\| = \max_{t \in [a,b]} |\alpha x(t)| = \max_{t \in [a,b]}(|\alpha| |x(t)|) = |\alpha| \max_{t \in [a,b]} |x(t)| = |\alpha| \|x\|.$

3- $\|x+y\| = \max_{t\in[a,b]}|(x+y)(t)| = \max_{t\in[a,b]}|x(t)+y(t)|$

$\leq \max_{t\in[a,b]}(|x(t)|+|y(t)|) = \max_{t\in[a,b]}|x(t)| + \max_{t\in[a,b]}|y(t)| = \|x\|+\|y\|.$

Hence, from 1, 2, and 3 $(C[a,b],\|.\|)$ is norm space.

Now, we want to show that $C[a,b]$ is complete,

Let $(x_m)_{m=1}^{\infty}$ is any Cauchy sequence in $C[a,b]$, $x_m : [a,b] \to \Re$ is continuous $\Rightarrow \forall \varepsilon > 0 \quad \exists k_\varepsilon \in N$ such that

$$\|x_m - x_n\| < \varepsilon \qquad \forall n,m > k_\varepsilon$$

from (1)

$\Rightarrow \max_{t\in[a,b]}|x_m(t) - x_n(t)| < \varepsilon$

$\Rightarrow \forall t \in [a,b] \qquad n,m \geq k_\varepsilon$

$\Rightarrow |x_m(t) - x_n(t)| \leq \max_{t\in[a,b]}|x_m(t) - x_n(t)| < \varepsilon \qquad (2)$

$\Rightarrow \forall t \in [a,b] \qquad (x_m(t))_{m=1}^{\infty}$ is a Cauchy sequence of numbers, since $\Re$ is complete,

$\Rightarrow (x_m(t))_{m=1}^{\infty}$ is convergent, i.e. $\lim_{m\to\infty} x_m(t)$ exists $\forall t \in [a,b]$

So, we can define a function $x : [a,b] \to \Re$ by

$$x(t) = \lim_{m\to\infty} x_m(t) \qquad (3),$$

clearly x is well defined, since the limit exists

Now, we using (2), for $t \in [a,b] \qquad n \geq k_\varepsilon$

$|x_n(t) - x(t)| = \left|x_n(t) - \lim_{m\to\infty} x_m(t)\right| \qquad$ *from* (3)

$= \lim_{m\to\infty}|x_n(t) - x_m(t)|$ (since the limit is a continuous function).

$< \varepsilon$

Since the limit depends ε

$\Rightarrow (x_n)$ Converges uniformly to x

$\Rightarrow x$ is continuous

that means $x \in C[a,b]$ and $x_n \to x$

$\Rightarrow C[a,b]$ is complete.

Example (2):

Let $l^p = \left\{ x = (\xi_i) : \xi_i \in C, \sum_{i=1}^{\infty} |\xi_i|^p < \infty \right\}$, we define a norm $\|.\| : l^p \to \Re$ by $\|x\| = (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p}$ (1),

The norm is well defined by definition.

Now, we want to show that $(l^p, \|.\|)$ is norm space,

Let $x = (\xi_i), y = (\eta_i)$ are any elements in l^p, α is any scalar:

1- $\|x\| = (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} \geq 0$

and $\|x\| = 0 \Leftrightarrow (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} = 0 \Leftrightarrow \xi_i = 0 \forall i \Leftrightarrow (\xi_i) = 0 \Leftrightarrow x = 0.$

2- $\|\alpha x\| = \|\alpha (\xi_i)_{i=1}^{\infty}\| = (\sum_{i=1}^{\infty} |\alpha \xi_i|^p)^{1/p} = (\sum_{i=1}^{\infty} |\alpha|^p |\xi_i|^p)^{1/p} = (|\alpha|^p \sum_{i=1}^{\infty} |\xi_i|^p)^{1/p}$

$= (|\alpha|^p)^{1/p} (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} = |\alpha| (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} = |\alpha| \|x\|.$

3- $\|x + y\| = \|(\xi_i) + (\eta_i)\| = (\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p)^{1/p}$

$\leq (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} + (\sum_{i=1}^{\infty} |\eta_i|^p)^{1/p}$ (from Minkowski inequality)

$= \|x\| + \|y\|.$

Hence, from 1, 2 and 3 $(l^p, \|.\|)$ is norm space.

Now, we want to show that l^p is complete,

Let (x_m) be a Cauchy sequence in l^p, where $x_m = (\xi_j^{(m)})_{j=1}^{\infty}$,

and let ε>0 be given, then $\exists k_\varepsilon \in$ N such that

$$\|x_m - x_n\| < \varepsilon \qquad \forall m, n \geq k_\varepsilon$$

from(1)

$$\Rightarrow (\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^p)^{1/p} < \varepsilon \qquad \forall m, n \geq k_\varepsilon$$

$$\Rightarrow \sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^p < \varepsilon^p \qquad \forall m, n \geq k_\varepsilon \qquad (2)$$

$$\Rightarrow \forall j \in \mathrm{N}, \forall m, n \geq k_\varepsilon \quad , |\xi_j^{(m)} - \xi_j^{(n)}| \leq \sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon$$

so, $\forall m,n \geq k_\varepsilon \quad j \in \mathrm{N} \Rightarrow \left|\xi_j^{(m)} - \xi_j^{(n)}\right| < \varepsilon$

$\Rightarrow \forall j \in \mathrm{N}$, $(\xi_j^{(m)})_{m=1}^{\infty}$ is a Cauchy sequence of numbers, since C is complete, $\Rightarrow (\xi_j^{(m)})_{m=1}^{\infty}$ is convergent for each $j \in \mathrm{N}$ say, $(\xi_j^{(m)})_{m=1}^{\infty}$ converges to ξ_j, put $x = (\xi_1, \xi_2, \xi_3, ...) = (\xi_j)_{j=1}^{\infty}$

Claim:

1- $x \in l^p$ i.e. $\sum_{j=1}^{\infty} \left|\xi_j\right|^p < \infty$.

2- $(x_m) \to x$.

Now, from (2) $\forall k \in \mathrm{N}, \quad m \geq k_\varepsilon$

$$\sum_{j=1}^{k} \left|\xi_j^{(m)} - \xi_j\right| = \sum_{j=1}^{k} \left|\xi_j^{(m)} - \lim_{n\to\infty} \xi_j^{(n)}\right|^p = \lim_{n\to\infty} \sum_{j=1}^{k} \left|\xi_j^{(m)} - \xi_j^{(n)}\right|^p$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{\infty} \left|\xi_j^{(m)} - \xi_j^{(n)}\right|^p < \varepsilon^p$$

$$\Rightarrow \left\|x_m - x\right\|^p = \sum_{j=1}^{\infty} \left|\xi_j^{(m)} - \xi_j\right| < \varepsilon^p \qquad (3)$$

$\Rightarrow x_m - x$ belong to l^p

since $x_m \in l^p$, and l^p is a vector space

$\Rightarrow x = x_m - (x_m - x) \in l^p, \qquad (4)$

and from (3) it clear that $\forall m \geq k_\varepsilon$,

$$\left\|x_m - x\right\| < \varepsilon$$

$\Rightarrow (x_m) \to x \qquad (5)$

from (4)and(5) $\Rightarrow l^p$ is complete.

<u>Example</u> (3):

We proved that $(\Re^n, \|.\|)$ is norm space with norm given by

$$\|x\| = (\sum_{j=1}^{n} \xi_j^2)^{1/2}, x \in \Re^n.$$

Now, we want to show that $\Re^n$ is complete,

Let (x_m) be a Cauchy sequence in $\Re^n$, $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, ..., \xi_n^{(m)})$

$\Rightarrow \forall \varepsilon > 0 \quad \exists k_\varepsilon \in \mathrm{N}$ such that

$$\left\|x_m - x_r\right\| < \varepsilon \qquad \forall m, r > k_\varepsilon$$

$\Rightarrow (\sum_{j=1}^{n} (\xi_j^{(m)} - \xi_j^{(r)})^2)^{1/2} < \varepsilon \qquad \forall m, r > k_\varepsilon$

$\Rightarrow \forall j \in \mathrm{N}, \quad \forall m, r > k_\varepsilon, \quad \left|\xi_j^{(m)} - \xi_j^{(n)}\right| < \varepsilon$

since $\Re$ is complete $\Rightarrow \forall j \in \mathrm{N}, \quad (\xi_j^{(m)})_{m=1}^{\infty}$ is convergent $\forall j \in \mathrm{N}$,say $(\xi_j^{(m)})_{m=1}^{\infty}$ converges to ξ_j, put $x = (\xi_1, \xi_2, ..., \xi_n) = (\xi_j)_{j=1}^{n}, x \in \Re^n$ we want to prove that $x_m \to x$,since $(\xi_j^{(m)}) \to \xi_j, \quad \lim_{m \to \infty} \xi_j^{(m)} = \xi_j$

$\Rightarrow \exists k_j \in \mathrm{N}$ such that $m \geq k_j \Rightarrow \left|\xi_j^{(m)} - \xi_j\right| < \frac{\varepsilon}{\sqrt{n}} \qquad \forall j = 1, 2, ..., n$

Take $k = \max\{k_1, k_2, ..., k_n\}$

$\Rightarrow \forall m \geq k \Rightarrow \left|\xi_j^{(m)} - \xi_j\right| < \frac{\varepsilon}{\sqrt{n}}$

$$\Rightarrow \sum_{j=1}^{n} (\xi_j^{(m)} - \xi_j)^2 < \sum_{j=1}^{n} \frac{\varepsilon^2}{n} = n\frac{\varepsilon^2}{n} = \varepsilon^2$$

$$\Rightarrow (\sum_{j=1}^{n} (\xi_j^{(m)} - \xi_j)^2)^{1/2} < \varepsilon$$

$$\Rightarrow \|x_m - x\| < \varepsilon$$

$$\Rightarrow x_m \to x$$

$\Rightarrow \Re^n$ is complete.

Example (4):

Let $l^\infty = \{x = (\xi_j), \quad (\xi_j)$ is bounded sequence $\}$, we define $\|.\| : l^\infty \to \Re$ by $\|x\| = \sup_{j \in \mathrm{N}} |\xi_j|$ (1),

The norm is well defined, since $x = (\xi_j) \in l^\infty$ is bounded sequence $\Rightarrow |\xi_j| \leq c_x \forall j \in \mathrm{N}$ for some $c_x > 0 \Rightarrow \{|\xi_j| : j \in \mathrm{N}\}$ is bounded subset of $\Re$, $\Rightarrow \sup_{j \in \mathrm{N}} |\xi_j|$ exists and unique.

Now, we want to show that $(l^\infty, \|.\|)$ is norm space,

Let $x = (\xi_j), y = (\eta_j)$ are any elements in l^∞ , α is any scalar:

1- $\|x\| = \sup_{j \in \mathrm{N}} |\xi_j| \geq 0$,

and $\|x\| = 0 \Leftrightarrow \sup_{j \in \mathrm{N}} |\xi_j| = 0 \Leftrightarrow \xi_j = 0 \qquad \forall j \in \mathrm{N} \Leftrightarrow x = 0.$

2- $\|\alpha x\| = \sup_{j\in N}(|\alpha\xi_j|) = \sup_{j\in N}(|\alpha||\xi_j|) = |\alpha|\sup_{j\in N}|\xi_j| = |\alpha|\|x\|.$

3- $\|x+y\| = \sup_{j\in N}|\xi_j+\eta_j| \le \sup_{j\in N}(|\xi_j|+|\eta_j|) = \sup_{j\in N}|\xi_j| + \sup_{j\in N}|\eta_j| = \|x\|+\|y\|.$

Hence, from 1, 2 and 3 $(l^\infty, \|.\|)$ is norm space.

Now, we want to show that l^∞ is complete.

Let (x_m) be a Cauchy sequence in l^∞, $x_m = (\xi_j^{(m)})_{j=1}^\infty$

$\Rightarrow \forall \varepsilon > 0 \quad \exists k_\varepsilon \in N$ such that

$$\|x_m - x_n\| < \varepsilon \qquad \forall m,n > k_\varepsilon$$

from (1)

$\Rightarrow \sup_{j\in N}|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon \qquad \forall m,n > k_\varepsilon$

$\Rightarrow \forall j \in N, \quad m,n > k_\varepsilon, \quad |\xi_j^{(m)} - \xi_j^{(n)}| < \sup_{j\in N}|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon$ (2)

$\Rightarrow \forall j \in N$, $(\xi_j^{(m)})_{m=1}^\infty$ is Cauchy sequence of numbers, since C is complete, $\Rightarrow (\xi_j^{(m)})_{m=1}^\infty$ is convergent for each $j \in N$, say $(\xi_j^{(m)})_{m=1}^\infty$ converges to ξ_j, put $x = (\xi_1, \xi_2, ...) = (\xi_j)_{j=1}^\infty$

Claim:

1- $x \in l^\infty$ i.e. $x = (\xi_j^{(m)})_{m=1}^\infty$ is bounded sequence.

2- $(x_m) \to x$.

Now, $\forall j \in N, \quad m \ge k_\varepsilon$

$\Rightarrow |\xi_j^{(m)} - \xi_j| = |\xi_j^{(m)} - \lim_{n\to\infty}\xi_j^{(n)}| = \lim_{n\to\infty}|\xi_j^{(m)} - \xi_j^{(n)}|$ from(2)

$< \varepsilon$ (3)

$\Rightarrow x_m - x$ is bounded sequence $\Rightarrow x_m - x$ belong to l^∞

since $x_m \in l^\infty$,and l^∞ is vector space

$\Rightarrow x = x_m - (x_m - x) \in l^\infty$ (4)

and from (3) it clear that $\forall m \ge k_\varepsilon$,

$$\|x_m - x\| < \varepsilon$$

$\Rightarrow (x_m) \to x$ (5)

from (4)and(5) $\Rightarrow l^\infty$ is complete.

Example of non-complete norm space:

Define $\|.\|:\Re^{+}\to\Re^{+}$ by $\|x\|=|x|$

Clearly, the norm is well defined

Now, let $x, y\in\Re^{+}$ and α is any scalar:

1- $\|x\|=|x|\geq 0$ and $\|x\|=0\Leftrightarrow |x|=0\Leftrightarrow x=0$.

2- $\|\alpha x\|=|\alpha x|=|\alpha||x|=|\alpha|\|x\|$.

3- $\|x+y\|=|x+y|\leq |x|+|y|=\|x\|+\|y\|$.

Hence, from 1, 2, and 3 $(\Re^{+},\|.\|)$ is norm space

Now, let x_n be a sequence in $\Re^{+}$, $x_n=(\frac{1}{n})_{n=1}^{\infty}$ $n\in N$

$\forall\varepsilon>0$ $\exists k_{\varepsilon}\in N$ such that $k_{\varepsilon}>\frac{2}{\varepsilon}$

$m,n>k_{\varepsilon}$ $\|x_n-x_m\|=|x_n-x_m|=\left|\frac{1}{n}-\frac{1}{m}\right|=\left|\frac{1}{n}+(-\frac{1}{m})\right|\leq\left|\frac{1}{n}\right|+\left|\frac{1}{m}\right|$

$=\frac{1}{n}+\frac{1}{m}<\frac{1}{k_{\varepsilon}}+\frac{1}{k_{\varepsilon}}=\frac{2}{k_{\varepsilon}}<\varepsilon$

$\Rightarrow$ x_n is Cauchy sequence

but $x_n=(\frac{1}{n})\to 0$ $,0\notin\Re^{+}$

$\Rightarrow$ $(\Re^{+},\|.\|)$ is not complete.

1.2 ***Linear operators***

Definition (1.2.1)

A *linear operator* T is an operator such that:
(a) The domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field.
(b) for all $x, y \in D(T)$ and scalars α ,

$$T(x+y) = T(x) + T(y).$$
$$T(\alpha x) = \alpha T(x).$$

Definition (1.2.2)

The *Null space* of T is the set of all $x \in D(T)$ such that $T(x) = 0$.

Examples of linear operators:

Example (1)

The Identity operator $I_X : X \to X$ is defined by

$$I_X(x) = x \qquad \forall x \in X$$

this operator is linear, since

$$I(x+y) = x+y = I(x) + I(y) \qquad \forall x, y \in X.$$
$I(\alpha x) = \alpha x = \alpha I(x)$, where α any scalar, $x \in X$

Example (2)
Let be X a vector space of all polynomials on the closed bounded interval $[a,b]$, we define the operator $T : X \to Y$ by:

$$T(x(t)) = x'(t) \qquad \forall x \in X$$

this operator is linear, since $\forall x, y \in X \qquad t \in [a,b]$

$$(T(x+y))(t) = T((x+y)(t)) = (x+y)'(t) = x'(t) + y'(t)$$
$$= T(x(t)) + T(y(t)) = (T(x) + T(y))(t)$$

there for $T(x+y) = T(x) + T(y)$.
and

$$(T(\alpha x))(t) = T((\alpha x)(t)) = (\alpha x)'(t) = \alpha x'(t) = \alpha T(x(t)) = (\alpha T(x))(t).$$

there for $T(\alpha x) = \alpha T(x)$. Hence $T : X \to Y$ is linear operator.

Example (3)

The operator T from $C[a,b]$ into itself $T : C[a,b] \to C[a,b]$ can be defined by

$$T(x(t)) = \int_a^t x(\tau)d\tau \qquad t \in [a,b]$$

this operator is linear, since $\forall x, y \in X \qquad t \in [a,b]$

$$(T(x+y))(t) = \int_a^t (x+y)(\tau)d\tau = \int_a^t (x(\tau)+y(\tau))d\tau$$

$$= \int_a^t x(\tau)d\tau + \int_a^t y(\tau)d\tau = (Tx(t)) + (Ty(t))$$

then $T(x+y) = T(x) + T(y)$.

and $(T(\alpha x)(t)) = \int_a^t (\alpha x)(\tau)d\tau = \alpha \int_a^t x(\tau)d\tau = (\alpha T(x))(t)$

then $T(\alpha x) = \alpha T(x)$. Hence $T : C[a,b] \to C[a,b]$ is linear operator.

Example (4)

The cross product with one factor kept fixed defines a linear operator $T : \Re^3 \to \Re^3$ by $Tx = x \times a = (x_2\alpha_3 - x_3\alpha_2, x_3\alpha_1 - x_1\alpha_3, x_1\alpha_2 - x_2\alpha_1)$

where $a = (\alpha_i) \in \Re^3$ is fixed, $a \neq 0$ say $\alpha_1 \neq 0$

this operator is linear, since $\forall x, y \in \Re^3$, α is any scalar:

1- $T(x+y) = (x+y) \times a = (x \times a) + (y \times a) = Tx + Ty$.

2- $T(\alpha x) = (\alpha x) \times a = \alpha(x \times a) = \alpha Tx$. Hence, T is linear.

The null space of this operator is $N(T) = \{x \in \Re^3 : Tx = (0,0,0)\}$,

$Tx = (0,0,0) \Leftrightarrow (x_2\alpha_3 - x_3\alpha_2, x_3\alpha_1 - x_1\alpha_3, x_1\alpha_2 - x_2\alpha_1) = (0,0,0)$

$\Leftrightarrow$ (1) $x_2\alpha_3 - x_3\alpha_2 = 0$ (2) $x_3\alpha_1 - x_1\alpha_3 = 0$ (3) $x_1\alpha_2 - x_2\alpha_1 = 0$

since $\alpha_1 \neq 0$,then from(2) we get $x_3 = \dfrac{\alpha_3}{\alpha_1}x_1$,and from (3)we get $x_2 = \dfrac{\alpha_2}{\alpha_1}x_1$

$$\Rightarrow x = (x_1, x_2, x_3) = (x_1, \frac{\alpha_2}{\alpha_1}x_1, \frac{\alpha_3}{\alpha_1}x_1) = x_1(1, \frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1})$$

Now, multiplying both said by α_1 we get $\alpha_1 x = x_1(\alpha_1, \alpha_2, \alpha_3)$

$\Rightarrow x = \dfrac{x_1}{\alpha_1} \cdot a \Rightarrow x = \beta \cdot a$, where $\beta = \dfrac{x_1}{\alpha_1}$

Hence the Null space is $N(T) = span\,\{a\}$.

Theorem (Range and null space) (1.2.3)

Let T be a linear operator, then:
(a) The range $R(T)$ is a vector space.
(b) If $\dim D(T) = n < \infty$, then $\dim R(T) \leq n$.
(c) The null space $N(T)$ is a vector space.

Proof:

(a) Let $y_1, y_2 \in R(T)$ we want to show that $\alpha y_1 + \beta y_2 \in R(T)$ for any scalars α, β
Now, we have $y_1 = T(x_1), y_2 = T(x_2)$ for some $x_1, x_2 \in D(T)$
and $\alpha x_1 + \beta x_2 \in D(T)$ because $D(T)$ is a vector space
and since T is linear, we have
$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) = \alpha y_1 + \beta y_2$
hence $\alpha y_1 + \beta y_2 \in R(T)$,since $y_1, y_2 \in R(T)$ were arbitrary and so were the scalars this prove that $R(T)$ is a vector space.

(b) We choos $n+1$ element $y_1, y_2,..., y_{n+1}$ of $R(T)$ in an arbitrary fashion. Then we have $y_1 = T(x_1),..., y_{n+1} = T(x_{n+1})$ for some $x_1, x_2,..., x_{n+1}$ in $D(T)$
Since $\dim D(T) = n$, this set $\{x_1, x_2,..., x_{n+1}\}$ must be linearly dependent. Hence $\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1} = 0_X$
for some scalars $\alpha_1,..., \alpha_{n+1}$ not all zero. Since T is linear and $T 0_X = 0_Y$, application of T on both sides gives

$$T(\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + ... + \alpha_{n+1} y_{n+1} = 0_Y$$

This shows that $\{y_1,..., y_{n+1}\}$is linearly dependent set because the α_j's are not all zero. Remembering that this subset of $R(T)$ was chosen in an arbitrary fashion, we conclude that $R(T)$ has no linearly independent subsets of $n+1$ or more element. By definition this means that $\dim R(T) \leq n$.

(c) Let $x_1, x_2 \in N(T)$, then $T(x_1) = T(x_2) = 0$, α any scalar,
Since T is linear
$T(x_1 + x_2) = T(x_1) + T(x_2) = 0 + 0 = 0$, hence $x_1 + x_2 \in N(T)$ (1)
$T(\alpha x_1) = \alpha T(x_1) = \alpha 0 = 0$, hence $\alpha x_1 \in N(T)$ (2)
Then, from (1), (2) $N(T)$ is a vector space.

<u>Definition</u> (1.2.4)

Let X,Y be a vector spaces, $T : D(T) \to Y$ is said to be *<u>injective</u>* or *<u>one to one</u>*, if for any $x_1, x_2 \in D(T)$

$$x_1 \neq x_2 \Rightarrow T(x_1) \neq T(x_2)$$

equivalently,

$$T(x_1) = T(x_2) \Rightarrow x_1 = x_2.$$

<u>Definition</u> (1.2.5)

Let $T : D(T) \to R(T)$ is one to one,

The mapping $T^{-1} : R(T) \to D(T)$ defined by

$$T^{-1}(y) = x$$

which maps every $y \in R(T)$ onto that $x \in D(T)$ for which $T(x) = y$, the mapping T^{-1} is called the *<u>inverse</u>* of T. we clearly have

$$T^{-1}T(x) = x \qquad \forall x \in D(T)$$

$$TT^{-1}(y) = y \qquad \forall y \in R(T).$$

<u>Theorem (Inverse theorem)</u> (1.2.6)

Let X,Y be a vector spaces, let $T : D(T) \to Y$ be a linear operator with domain $D(T) \subset X$ and range $R(T) \subset Y$, then:

(a) The inverse $T^{-1} : R(T) \to D(T)$ exist if and only if

$$T(x) = 0 \Rightarrow x = 0.$$

(b) If T^{-1} exists, it is a linear operator.

(c) If $\dim D(T) = n < \infty$ and T^{-1} exists, then $\dim R(T) = \dim D(T)$.

<u>Proof:</u>

(a) Suppose that $T^{-1} : R(T) \to D(T)$ exists, then $T : D(T) \to R(T)$ is one to one, suppose $T(x) = 0$, then

$$T(x) = T(0) = 0 \Rightarrow x = 0$$

Conversely

Suppose that $T(x) = 0 \Rightarrow x = 0$, let $T(x_1) = T(x_2)$, since T is linear,

$T(x_1 - x_2) = T(x_1) - T(x_2) = 0$

so that $x_1 - x_2 = 0$
Hence $x_1 = x_2$
Hence T is one to one and so T^{-1} exists.

(b) If $T^{-1} : R(T) \to D(T)$ exists, it is a linear operator. Indeed,
Let $y_1, y_2 \in D(T^{-1}) = R(T)$, then $\exists x_1, x_2 \in X$ such that
$y_1 = T(x_1), y_2 = T(x_2)$, then $x_1 = T^{-1}(y_1), x_2 = T^{-1}(y_2)$
Now,

$$
\begin{aligned}
T^{-1}(\alpha y_1 + \beta y_2) &= T^{-1}(\alpha T(x_1) + \beta T(x_2)) \\
&= T^{-1}(T(\alpha x_1 + \beta x_2)) \\
&= \alpha x_1 + \beta x_2 \\
&= \alpha T^{-1}(y_1) + \beta T^{-1}(y_2).
\end{aligned}
$$

Hence, T^{-1} is a linear operator.

(c) Suppose $\dim D(T) = n < \infty$, and $T^{-1} : R(T) \to X$ exists,
By theorem (1.2.3(b)) we have $\dim R(T) \le \dim D(T) = n$
Now, $n = \dim D(T) = \dim R(T^{-1}) \le \dim D(T^{-1}) = \dim R(T) \le n$
Hence, $\dim D(T) = \dim R(T)$.

***Applications*:**

A- Let $T_1 : \Re^2 \to \Re^2$ be defined by

$$T_1(\xi_1, \xi_2) = (\xi_1, 0)$$

Then T_1 is linear operator.

Proof:

Let $x = (\xi_1, \xi_2) \in \Re^2$, $y = (\eta_1, \eta_2) \in \Re^2$, and α is any scalar

$$
\begin{aligned}
T_1(x + y) &= T_1((\xi_1, \xi_2) + (\eta_1, \eta_2)) = T_1(\xi_1 + \eta_1, \xi_2 + \eta_2) \\
&= (\xi_1 + \eta_1, 0) = (\xi_1, 0) + (\eta_1, 0) = T_1(x) + T_1(y).
\end{aligned}
$$

$$T_1(\alpha x) = T_1(\alpha\xi_1, \alpha\xi_2) = (\alpha\xi_1, 0) = \alpha(\xi_1, 0) = \alpha T_1(x).$$

and $R(T_1) = \{(\xi_1, 0) : \xi_1 \in \Re\} = \Re \times \{0\}$.

$$
\begin{aligned}
N(T_1) &= \{(\xi_1, \xi_2) \in \Re^2 : T_1(\xi_1, \xi_2) = (0,0)\} \\
&= \{(\xi_1, \xi_2) \in \Re^2 : (\xi_1, 0) = (0,0)\} \\
&= \{(\xi_1, \xi_2) \in \Re^2 : \xi_1 = 0\}.
\end{aligned}
$$

B- Let $T_2 : \Re^2 \to \Re^2$ defined by

$$T_2(\xi_1, \xi_2) = (0, \xi_2)$$

Then T_2 is linear operator.

Proof:

Let $x = (\xi_1, \xi_2) \in \Re^2$, $y = (\eta_1, \eta_2) \in \Re^2$, and α is any scalar

$$T_2(x + y) = T_2((\xi_1, \xi_2) + (\eta_1, \eta_2)) = T_2(\xi_1 + \eta_1, \xi_2 + \eta_2)$$
$$= (0, \xi_2 + \eta_2) = (0, \xi_2) + (0, \eta_2) = T_2(x) + T_2(y).$$
$$T_2(\alpha x) = T_2(\alpha\xi_1, \alpha\xi_2) = (0, \alpha\xi_2) = \alpha(0, \xi_2) = \alpha T_2(x).$$

and $R(T_2) = \{(0, \xi_2) : \xi_2 \in \Re\} = \{0\} \times \Re.$

$$N(T_2) = \{(\xi_1, \xi_2) \in \Re^2 : T_2(\xi_1, \xi_2) = (0,0)\}$$
$$= \{(\xi_1, \xi_2) \in \Re^2 : (0, \xi_2) = (0,0)\}$$
$$= \{(\xi_1, \xi_2) \in \Re^2 : \xi_2 = 0\}.$$

C- Let $T_3 : \Re^2 \to \Re^2$ defined by

$$T_3(\xi_1, \xi_2) = (\xi_2, \xi_1)$$

Then T_3 is linear operator.

Proof:

Let $x = (\xi_1, \xi_2) \in \Re^2$, $y = (\eta_1, \eta_2) \in \Re^2$, and α is any scalar

$$T_3(x + y) = T_3((\xi_1, \xi_2) + (\eta_1, \eta_2)) = T_3(\xi_1 + \eta_1, \xi_2 + \eta_2)$$
$$= (\xi_2 + \eta_2, \xi_1 + \eta_1) = (\xi_2, \xi_1) + (\eta_2, \eta_1) = T_3(x) + T_3(y).$$
$$T_3(\alpha x) = T_3(\alpha\xi_1, \alpha\xi_2) = (\alpha\xi_2, \alpha\xi_1) = \alpha(\xi_2, \xi_1) = \alpha T_3(x).$$

and $R(T_3) = \{(\xi_2, \xi_1) : \xi_1, \xi_2 \in \Re\} = \Re^2.$

D- Let $T_4 : \Re^2 \to \Re^2$ defined by

$$T_4(\xi_1, \xi_2) = (\gamma\xi_1, \gamma\xi_2)$$

Then T_4 is linear operator.

Proof:

Let $x = (\xi_1, \xi_2) \in \Re^2$, $y = (\eta_1, \eta_2) \in \Re^2$, and α is any scalar

$$T_4(x+y) = T_4((\xi_1,\xi_2)+(\eta_1,\eta_2)) = T_4(\xi_1+\eta_1,\xi_2+\eta_2)$$
$$= (\gamma\xi_1+\gamma\eta_1,\gamma\xi_2+\gamma\eta_2) = (\gamma\xi_1,\gamma\xi_2)+(\gamma\eta_1,\gamma\eta_2)$$
$$= T_4(x)+T_4(y).$$
$$T_4(\alpha x) = T_4(\alpha\xi_1,\alpha\xi_2) = (\gamma\alpha\xi_1,\gamma\alpha\xi_2) = \alpha(\gamma\xi_1,\gamma\xi_2) = \alpha T_4(x).$$

and $R(T_4) = \{(\gamma\xi_1,\gamma\xi_2) : \xi_1,\xi_2 \in \Re\} = \Re^2$.

$$N(T_4) = \{(\xi_1,\xi_2) \in \Re^2 : T_4(\xi_1,\xi_2) = (0,0)\}$$
$$= \{(\xi_1,\xi_2) \in \Re^2 : (\gamma\xi_1,\gamma\xi_2) = (0,0)\}$$
$$= \{(\xi_1,\xi_2) \in \Re^2 : \xi_1 = 0, \xi_2 = 0\}$$

E- Let $T : D(T) \to Y$ be a linear operator whose inverse exists. If $\{x_1,\dots,x_n\}$ is a linearly independent set in $D(T)$, then the set $\{Tx_1,\dots,Tx_n\}$ is linearly independent.

Proof:

We want to show $\{Tx_1,\dots,Tx_n\}$ is linearly independent.

So, let $\alpha_1,\dots,\alpha_n$ be scalars such that

$$\alpha_1 Tx_1 + \dots + \alpha_n Tx_n = 0_Y$$

we want to prove $\alpha_i = 0, \forall i = 1,\dots,n$

since T is linear, then $T(\alpha_1 x_1 + \dots + \alpha_n x_n) = 0_Y$

and since T^{-1} exists, then

$$T^{-1}(T(\alpha_1 x_1 + \dots + \alpha_n x_n)) = T^{-1}(0_Y)$$

$\Rightarrow \alpha_1 x_1 + \dots + \alpha_n x_n = 0_X$

since $\{x_1,\dots,x_n\}$ linearly independent, then $\alpha_i = 0, \forall i = 1,\dots,n$

Hence $\{Tx_1,\dots,Tx_n\}$ is linearly independent.

F- Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$, then $R(T) = Y$ if and only if T^{-1} exists.

***Proof*:**

Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$, and $R(T) = Y$, we want to show that T^{-1} exists, i.e. T is one to one, i.e. $Tx = 0 \Rightarrow x = 0$,

let $B = \{e_1,\dots,e_n\}$ be a basis for X, and let $y \in Y = R(T)$, then

$y = Tx$ for some $x \in X$, $x = \sum_{i=1}^{n} \alpha_i e_i$

$y = T(\sum_{i=1}^{n} \alpha_i e_i) = \sum_{i=1}^{n} \alpha_i T(e_i)$, then $\{Te_1, ..., Te_n\}$ generates $Y = R(T)$

since $\dim Y = n < \infty$, then $\{Te_1, ..., Te_n\}$ is a basis for Y

Now, let $Tx = 0$

$$\Rightarrow T(\sum_{i=1}^{n} \alpha_i e_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i T(e_i) = 0$$

since $\{Te_1, ..., Te_n\}$ is linearly independent (from ***E***)

$$\Rightarrow \alpha_i = 0, \forall i$$

$$\Rightarrow x = 0$$

That means T is one to one, so T^{-1} exists.

Conversely

Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$, and T^{-1} exists, we want to show that $R(T) = Y$,

Since T is linear operator, $T : X \to R(Y)$

$$\Rightarrow \dim R(T) \le \dim X = n \qquad (1)$$

since T^{-1} exists, $T^{-1} : R(T) \to X$

$$\Rightarrow n = \dim X \le \dim R(T) \qquad (2)$$

from (1)and (2) we get

$$\dim R(T) = n$$

since $R(T)$ subspace of Y, and $\dim Y = n$

Hence $R(T) = Y$.

1.3 ***Bounded and continuous linear operators***

Definition (1.3.1)

Let X and Y be normed space and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$ .The operator T is said to be *bounded* if there is a real number c such that for all $x \in D(T)$

$$\|Tx\| \le c\|x\| \qquad (1)$$

the smallest possible c in (1)

$$\frac{\|Tx\|}{\|x\|} \le c \qquad x \ne 0$$

is that supremum. This quantity is denoted by $\|T\|$; thus

$$\|T\| = \sup_{\substack{x\in D(T)\\ x\ne 0}} \frac{\|Tx\|}{\|x\|} \qquad (2)$$

$\|T\|$ is called the norm of the operator T, if $D(T) = \{0\}$,we define $\|T\| = 0$.

Lemma (1.3.2)

Let T be a bounded linear operator, then:

(a) An alternative formula for the norm of T is: $\|T\| = \sup\limits_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$

(b) The norm defined by (2) satisfies the properties of norm.

Proof:

(a) we write $\|x\| = a > 0$, and set $y = \frac{1}{a}x$, where $x \ne 0$,then

$\|y\| = \left\|\frac{1}{a}x\right\| = \frac{\|x\|}{a} = \frac{a}{a} = 1$, and since T is linear (2) gives

$$\|T\| = \sup_{\substack{x\in D(T)\\ x\ne 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T)\\ x\ne 0}} \frac{1}{a}\|Tx\| = \sup_{\substack{x\in D(T)\\ x\ne 0}} \left\|T\left(\frac{1}{a}x\right)\right\| = \sup_{\substack{y\in D(T)\\ \|y\|=1}} \|Ty\|$$

writing x for y on right, we have $\|T\| = \sup\limits_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$.

(b) $\|T\| = \sup\limits_{\substack{x\in D(T)\\ x\neq 0}} \dfrac{\|Tx\|}{\|x\|}$

1- $\|T\| = \sup\limits_{\substack{x\in D(T)\\ x\neq 0}} \dfrac{\|Tx\|}{\|x\|} \geq 0$

and $\|T\| = 0 \Leftrightarrow Tx = 0 \quad \forall x \in D(T)$, so that $T = 0$.

2- $\|\alpha T\| = \sup\limits_{\|x\|=1} \|\alpha Tx\| = \sup\limits_{\|x\|=1} |\alpha| \|Tx\| = |\alpha| \sup\limits_{\|x\|=1} \|Tx\| = |\alpha| \|T\|$.

3- $\|T_1 + T_2\| = \sup\limits_{\|x\|=1} \|(T_1 + T_2)x\| = \sup\limits_{\|x\|=1} \|T_1 x + T_2 x\| \leq \sup\limits_{\|x\|=1} \|T_1 x\| + \sup\limits_{\|x\|=1} \|T_2 x\|$

$= \|T_1\| + \|T_2\|. \qquad , x \in D(T).$

Examples:

Example (1):

The identity operator $I : X \to X$ on a normed space $X \neq \{0\}$ defined by $Ix = x \quad \forall x \in X$, is bounded and has norm $\|I\| = 1$, since

$$\|Ix\| \leq c\|x\| \qquad c > 0$$

$$\Rightarrow \frac{\|Ix\|}{\|x\|} \leq c$$

$$\Rightarrow \frac{\|x\|}{\|x\|} \leq c \qquad \Rightarrow 1 \leq c \qquad \Rightarrow \|I\| = 1.$$

Example (2):

The zero operator $0 : X \to Y$ on a normed space X defined by $0x = 0 \quad \forall x \in X$, is bounded and has norm $\|0\| = 0$, since

$$\|0x\| \leq c\|x\| \qquad c > 0$$

$$\Rightarrow \frac{\|0x\|}{\|x\|} \leq c$$

$$\Rightarrow \frac{0}{\|x\|} \leq c \qquad \Rightarrow 0 \leq c \qquad \Rightarrow \|0\| = 0.$$

Example (3):

Let X be the normed space of all polynomials on $J = [0,1]$ with norm given $\|x\| = \max |x(t)|, t \in J$ .A differentiation operator T is defined on X by

$$Tx(t) = x'(t)$$

this operator is linear but not bounded, to proof this let $x_n(t) = t^n$, where $n \in N$

$$\|x_n\| = \max_{t\in[0,1]} |x_n(t)| = \max_{t\in[0,1]} |t^n| = 1$$

and

$$Tx_n(t) = x_n' = nt^{n-1}$$

$$\Rightarrow \|Tx_n\| \max_{t\in[0,1]} |Tx_n(t)| = \max_{t\in[0,1]} |nt^{n-1}| = n$$

$$\Rightarrow \frac{\|Tx_n\|}{\|x_n\|} = \frac{n}{1} = n \qquad n \in N$$

Now, $\dfrac{\|Tx_n\|}{\|x_n\|} = n \le c \qquad n \in N$

But no fixed number c such that $\dfrac{\|Tx_n\|}{\|x_n\|} = n \le c$

$\Rightarrow T$ is not bounded.

Example (4):

We defined an integral operator $T : C[0,1] \to C[0,1]$ by

$$y = Tx \text{ , where } y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$$

k is given function, which is called the kernel of T , and is continuous on the closed square $G = J \times J$, $J = [0,1]$, this operator is linear,

$$T(x+y) = \int_0^1 k(t,\tau)(x+y)(\tau)d\tau = \int_0^1 k(t,\tau)(x(\tau)+y(\tau))d\tau$$

$$= \int_0^1 k(t,\tau)x(\tau)d\tau + \int_0^1 k(t,\tau)y(\tau)d\tau = Tx + Ty.$$

$$T(\alpha x) = \int_0^1 k(t,\tau)\alpha x(\tau)d\tau = \alpha\int_0^1 k(t,\tau)x(\tau)d\tau = \alpha Tx.$$

T is bounded , to proof this, we first note that since k is continuous on the closed square $\Rightarrow$ k is bounded

$\Rightarrow \exists \mathrm{M} > 0$ such that $|k(t,\tau)| \leq \mathrm{M} \qquad \forall (t,\tau) \in G \qquad (1)$

and since $\|x\| = \max_{t \in J} |x(t)|$

$$\Rightarrow |x(t)| \leq \max_{t \in J} |x(t)| = \|x\| \qquad (2)$$

Now,

$$\|y\| = \|Tx\| = \max_{t \in J} |Tx(t)| = \max_{t \in J} \left| \int_0^1 k(t,\tau)x(\tau)d\tau \right|$$

$$\leq \max_{t \in J} \int_0^1 |k(t,\tau)||x(\tau)|d\tau \leq \max_{t \in J} \int_0^1 \mathrm{M}\|x\| \qquad \textit{from } (1), (2)$$

$$\leq \mathrm{M}\|x\|$$

$\Rightarrow \|Tx\| \leq \mathrm{M}\|x\| \qquad \mathrm{M} = c$

$\Rightarrow \|Tx\| \leq c\|x\|$

$\Rightarrow T$ is bounded.

Lemma (1.3.3)

Let $\{x_1, \ldots\ldots, x_n\}$ be a linearly independent set of vector in a normed space X (of any dimension), then there is number $c > 0$ such that for every choice of scalars $\alpha_1, \ldots\ldots, \alpha_n$ we have

$$\|\alpha_1 x_1 + \ldots\ldots + \alpha_n x_n\| \geq c(|\alpha_1| + \ldots.. + |\alpha_n|).$$

Theorem (Finite dimension) (1.3.4)

If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof:

Let $\dim X = n$ and $\{e_1, \ldots., e_n\}$ a basis for X , we take any $x = \sum_{i=1}^{n} \xi_i e_i$ and consider any linear operator T on X .

Since T is linear

$$\Rightarrow \|Tx\| = \left\|T(\sum_{i=1}^{n} \xi_i e_i)\right\| = \left\|\sum_{i=1}^{n} \xi_i T(e_i)\right\| \le \max_k \|T(e_k)\| \sum_{i=1}^{n} |\xi_i| \qquad (1)$$

we apply lemma (1.3.4) with $\alpha_i = \xi_i, x_i = e_i$,we get

$$c\sum_{i=1}^{n} |\xi_i| \le \left\|\sum_{i=1}^{n} \xi_i e_i\right\|$$

$$\Rightarrow \sum_{i=1}^{n} |\xi_i| \le \frac{1}{c}\left\|\sum_{i=1}^{n} \xi_i e_i\right\| = \frac{1}{c}\|x\| \qquad (2)$$

from (1) and(2)

$$\Rightarrow \|Tx\| \le \max_k \|T(e_k)\| \sum_{i=1}^{n} |\xi_i| \le \frac{1}{c}\|x\| \max_k \|T(e_k)\|$$

$$\Rightarrow \|Tx\| \le \gamma\|x\| \qquad where \qquad \gamma = \frac{1}{c}\max_k \|T(e_k)\|$$

From this we see that is T bounded.

Definition (1.3.5)

Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$, and X, Y are normed spaces, we say T is *continuous* at x_o if for any $\varepsilon > 0 \quad \exists\delta > 0$ such that if $\|x - x_o\| < \delta$

$$\Rightarrow \|Tx - Tx_o\| < \varepsilon \qquad \forall x \in D(T).$$

Theorem (Continuity and boundedness) (1.3.6)

Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$, and X, Y are normed spaces, then:

(a) T is continuous if and only if T is bounded.
(b) If T is continuous at a single point, it is continuous.

Proof:

(a) Suppose that T is bounded,

$$\Rightarrow \exists c > 0 \text{ such that } \|Tx\| \le c\|x\| \qquad \forall x \in D(T) \qquad (1)$$

We want to prove T is continuous, so let ε>0 be given and let $x_o \in D(T)$ be any point

Let $\delta = \frac{\varepsilon}{c}$, where c given in (1), then if $\|x - x_o\| < \delta$

$$\Rightarrow \|Tx - Tx_o\| = \|T(x - x_o)\| \qquad \text{Since } T \text{ is linear}$$

$$\leq c\|x - x_o\| \qquad \text{Since } T \text{ is bounded}$$

$$< c\delta$$

$$= c\frac{\varepsilon}{c} = \varepsilon$$

$\Rightarrow T$ is continuous at x_o, since x_o is an arbitrary point in $D(T)$,hence T is continuous on X.

Conversely, assume that T is continuous at an arbitrary $x_o \in D(T)$, then given $\varepsilon > 0 \quad \exists \delta > 0$ such that if $\|x - x_o\| < \delta$

$$\Rightarrow \|Tx - Tx_o\| < \varepsilon \qquad \forall x \in D(T) \qquad (2)$$

take any $y \in D(T)$, $y \neq 0$ and set $x = x_o + \frac{\delta}{\|y\|} y \Rightarrow x - x_o = \frac{\delta}{\|y\|} y$

$$\Rightarrow \|x - x_o\| = \left\| \frac{\delta}{\|y\|} y \right\| = \frac{\|\delta\|}{\|y\|} \|y\| = \delta$$

$$\Rightarrow \|Tx - Tx_o\| = \|T(x - x_o)\| \qquad \text{Since } T \text{ is linear}$$

$$= \left\| T\left(\frac{\delta}{\|y\|} y\right) \right\|$$

$$= \frac{\delta}{\|y\|} \|Ty\| \qquad \text{Since } T \text{ is linear}$$

$$\Rightarrow \|Tx - Tx_o\| = \frac{\delta}{\|y\|} \|Ty\| < \varepsilon \qquad \textit{from } (2)$$

$$\Rightarrow \|Ty\| \leq \frac{\varepsilon}{\delta} \|y\|$$

$$\Rightarrow \|Ty\| \leq c\|y\| \qquad \textit{where} \quad c = \frac{\varepsilon}{\delta}$$

$\Rightarrow T$ is bounded.

(b) Continuity of T at a point implies bounded of T by the second part of the proof of (a), which in turn implies continuity of T by (a).

Corollary (Continuity, null space) (1.3.7)

Let T be a bounded linear operator, then:

(a) $x_n \to x$ (where $x_n, x \in D(T)$) implies $Tx_n \to Tx$.
(b) The null space $N(T)$ is closed.

Proof:

(a) $\|Tx_n - Tx\| = \|T(x_n - x)\|$ Since T is linear

$\leq \|T\|\|x_n - x\|$ Since T is bounded

$\|T\|\|x_n - x\| \to 0$ Since $x_n \to x \Rightarrow \|x_n - x\| \to 0$

$\Rightarrow \|Tx_n - Tx\| \to 0$

$\Rightarrow Tx_n \to Tx$.

(b) let $x \in \overline{N(T)}$, then there is a sequence (x_n) in $N(T)$ such that

$x_n \to x \quad \Rightarrow Tx_n \to Tx \quad$ *from* (a)

Since (x_n) in $N(T)$

$\Rightarrow Tx_n = 0$

$\Rightarrow Tx = 0$

$\Rightarrow x \in N(T)$

$\Rightarrow N(T)$ is closed.

Applications:

A- Let X and Y be normed spaces, then a linear operator $T : X \to Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

Proof:

Let $T : X \to Y$ be a bounded linear operator i.e. $\exists c \in \Re$ such that

$$\|Tx\| \leq c\|x\| \qquad \forall x \in X \qquad (1)$$

and let $A \subset X$, A is bounded set $\Rightarrow \exists M > 0$ such that

$$\|x\| \leq M \qquad \forall x \in A \qquad (2)$$

and $T(A) = \{Tx : x \in A\}$

Now, for all $x \in A$

$$\Rightarrow \|Tx\| \le c\|x\| \qquad \textit{from } (1)$$
$$\le cM \qquad \textit{from } (2)$$

$\Rightarrow T(A)$ is bounded.

Conversely, suppose that T is a linear operator such that T maps bounded sets in X into bounded sets in Y, we want to show that T is bounded i.e. $\exists c \in \Re$ such that $\|Tx\| \le c\|x\| \qquad \forall x \in X$,

So let $x \in X, x \ne 0 \Rightarrow \dfrac{x}{\|x\|} \in X$, and let $A = \left\{ \dfrac{x}{\|x\|} : x \in X \setminus \{0\} \right\}$,

then $\|y\| = 1 \qquad \forall y \in A$, A is bounded $\Rightarrow T(A)$ is bounded

i.e. $\Rightarrow \exists M > 0$ such that $\|Ty\| \le M \qquad \forall y \in A$

Then $\forall x \in X$

$$\Rightarrow \left\| T\left(\frac{x}{\|x\|}\right) \right\| \le M$$

$$\Rightarrow \frac{1}{\|x\|}\|Tx\| \le M$$

$$\Rightarrow \|Tx\| \le M\|x\|$$

$$\Rightarrow \|Tx\| \le c\|x\| \qquad \textit{where} \qquad c = M$$

$\Rightarrow T$ is bounded.

B- Let $T : l^\infty \to l^\infty$ be an operator defined by

$$y = (\eta_i) = Tx, \eta_i = \frac{\xi_i}{i}, x = (\xi_i)$$

Then T is linear and bounded, but the range $R(T)$ of T need not be closed.

<u>Proof:</u>

First we want to show that T is linear,

Let $x_1, x_2 \in l^\infty, x_1 = (\xi_i^{(1)}), x_2 = (\xi_i^{(2)})$, and α is nay scalar:

1- $T(x_1 + x_2) = (\dfrac{\xi_i^{(1)} + \xi_i^{(2)}}{i}) = (\dfrac{\xi_i^{(1)}}{i}) + (\dfrac{\xi_i^{(2)}}{i}) = Tx_1 + Tx_2.$

2- $T(\alpha x_1) = (\alpha \frac{\xi_i^{(1)}}{i}) = \alpha(\frac{\xi_i^{(1)}}{i}) = \alpha T x_1$. Hence T is linear.

Now, we want to show that T is bounded,

$\|Tx_1\| = \sup_{i\in N}\left|\frac{\xi_i^{(1)}}{i}\right| \leq \sup_{i\in N}\left|\xi_i^{(1)}\right| = \|x_1\|$, hence T is bounded.

Finally, we want to show that the range $R(T)$ of T need not be closed,

$R(T) = \left\{(\frac{\xi_i}{i}) : x = (\xi_i) \in l^\infty\right\}$ is not closed i.e. $\exists (y_n)$ any sequence in $R(T)$ such that $y_n \to y$ but $y \notin R(T)$,

Now, let $x_n = (1, \sqrt{2}, ..., \sqrt{n}, 0, 0, ...)$, $x_n \in l^\infty$ for all $n \in N$, then

$y_n = T(x_n) = (1, \frac{1}{\sqrt{2}}, ..., \frac{1}{\sqrt{n}}, 0, 0, ...) \Rightarrow y_n \in l^\infty$ for all $n \in N$,

Clearly $y_n \to y = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, ...) = (\frac{1}{\sqrt{i}})$,

Now, suppose that $y = Tx$ for some $x \in l^\infty$

$\Rightarrow (\frac{1}{\sqrt{i}}) = (\frac{\xi_i}{i}) \Rightarrow \frac{1}{\sqrt{i}} = \frac{\xi_i}{i} \quad \forall i \in N \Rightarrow \xi_i = \sqrt{i} \quad \forall i \in N$

$\Rightarrow x = (\xi_i) = (1, \sqrt{2}, \sqrt{3}, ...) \notin l^\infty$

Therefore $y \notin R(T)$, so $R(T)$ not closed.

C- Let T be a bounded linear operator from a normed space X onto normed space Y. If there is a positive b such that

$$\|Tx\| \geq b\|x\| \qquad \forall x \in X$$

Then $T^{-1} : Y \to X$ exists and bounded.

<u>Proof:</u>

Let $Tx = 0 \Rightarrow 0 = \|Tx\| \geq b\|x\| \Rightarrow \|x\| = 0 \Rightarrow x = 0$, then T^{-1} exists.

Now, let $y \in Y \Rightarrow T^{-1}(y) = x$ for some $x \in X$, then

$\|T^{-1}(y)\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|$

$\Rightarrow \|T^{-1}(y)\| \leq \frac{1}{b}\|y\|$

$\Rightarrow T^{-1}$ is bounded.

1.4 ***Linear functionals***

Definition (1.4.1)

A *linear functional* f is a linear operator with domain in a vector space X and range in the scalar field K of X ; thus:
$f : D(T) \to K$
where $K = \Re$ if X is real, and $K = C$ if X is complex.

Definition (1.4.2)

A *bounded* linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $D(f)$ lies.
Thus there exist a real number c such that, for all $x \in D(f)$

$$|f(x)| \le c\|x\| .$$

Furthermore, the norm of f is

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \ne 0}} \frac{|f(x)|}{\|x\|} \text{ or } \|f\| = \sup_{\substack{x \in D(f) \\ \|x\|=1}} |f(x)|$$

$$\Rightarrow |f(x)| \le \|f\| \|x\|.$$

Examples:

Example (1):

The familiar dot product with one factor kept fixed defines a functional $f : \Re^3 \to \Re$ by means of:
$f(x) = x.a = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3$
where $a = (\alpha_1, \alpha_2, \alpha_3) \in \Re^3$ is a fixed, $x = (\xi_1, \xi_2, \xi_3)$
f is linear and bounded,
first we want to prove f is linear,
1- $f(x + y) = (x + y).a = (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3).a$
$= (\xi_1 + \eta_1)\alpha_1 + (\xi_2 + \eta_2)\alpha_2 + (\xi_3 + \eta_3)\alpha_3$
$= (\xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3) + (\eta_1\alpha_1 + \eta_2\alpha_2 + \eta_3\alpha_3)$
$= (\xi_1 + \xi_2 + \xi_3).a + (\eta_1 + \eta_2 + \eta_3).a = x.a + y.a = f(x) + f(y).$

2- $f(\alpha x) = (\alpha x).a = \alpha\xi_1\alpha_1 + \alpha\xi_2\alpha_2 + \alpha\xi_3\alpha_3$
$= \alpha(\xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3) = \alpha(x.a) = \alpha f(x).$
Now, we want to prove f is bounded

$$|f(x)| = \left|\sum_{i=1}^{3} \xi_i\alpha_i\right| \le \sum_{i=1}^{3} |\xi_i\alpha_i| \le (\sum_{i=1}^{3} |\xi_i|^2)^{1/2}(\sum_{i=1}^{3} |\alpha_i|^2)^{1/2} = \|x\|\|a\|.$$

By holder inequality,
There for f is bounded
So, $\forall x \in \Re^3, x \ne 0$

$$\frac{|f(x)|}{\|x\|} \le \|a\| \Rightarrow \sup \frac{|f(x)|}{\|x\|} \le \|a\| \Rightarrow \|f\| \le \|a\| \qquad (1)$$

$$\|f\| = \sup \frac{|f(x)|}{\|x\|} \ge \frac{|f(a)|}{\|a\|} = \frac{|\alpha_1\alpha_1 + \alpha_2\alpha_2 + \alpha_3\alpha_3|}{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2}}$$

$$= (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2} = \|a\| \Rightarrow \|f\| \ge \|a\| \qquad (2)$$

from (1) and (2)we get $\|f\| = \|a\|.$

Example (2):

We can obtain a linear functional f on the Hilbert space l^2 by choosing a fixed $a = (\alpha_i) \in l^2$, and define $f_a : l^2 \to C$ by $f_a(x) = \sum_{i=1}^{\infty} \xi_i\alpha_i$, where $x = (\xi_i) \in l^2$
Now, by holder inequality

$$\sum_{i=1}^{\infty} |\alpha_i\xi_i| \le (\sum_{i=1}^{\infty} |\alpha_i|^2)^{1/2}(\sum_{i=1}^{\infty} |\xi_i|^2)^{1/2} < \infty$$

$\Rightarrow \sum_{i=1}^{\infty} \xi_i\alpha_i$ is absolutely convergent, then is convergent

$\Rightarrow$ for each $x \in l^2$ there corresponds number $\sum_{i=1}^{\infty} \alpha_i\xi_i$

$\Rightarrow$ f_a is well defined.

$$|f(x)| = \left|\sum_{i=1}^{\infty} \alpha_i\xi_i\right| \le \sum_{i=1}^{\infty} |\alpha_i\xi_i| \le (\sum_{i=1}^{\infty} |\alpha_i|^2)^{1/2}(\sum_{i=1}^{\infty} |\xi_i|^2)^{1/2} = \|a\|\|x\|.$$

Theorem (1.4.3)

If $f \neq 0$ be any linear functional on vector space X and x_o any fixed element of $X - N(f)$, where $N(f)$ is the null space of f, then any $x \in X$ has a unique representation $x = ax_o + y$, where $y \in N(f)$.

Proof:

Let $x \in X$, x_o any fixed element of $X - N(f)$, let $\alpha = \dfrac{f(x)}{f(x_o)}$

$$f(x - \frac{f(x)}{f(x_o)} \cdot x_o) = f(x) - \frac{f(x)}{f(x_o)} \cdot f(x_o) = 0$$

Hence $x - \dfrac{f(x)}{f(x_o)} \cdot x_o$ belong to $N(f)$

$$\Rightarrow x - \frac{f(x)}{f(x_o)} \cdot x_o = y \text{ for some } y \in N(f)$$

$$\Rightarrow x = \frac{f(x)}{f(x_o)} \cdot x_o + y$$

Hence, every $x \in X$ can be written of the form $x = \alpha x_o + y \quad y \in N(f)$.
To prove this form is unique
Let $x = \alpha x_o + y = \alpha' x_o + y' \qquad y, y' \in N(f); \alpha, \alpha' \in K; \alpha \neq \alpha'$

$$\Rightarrow \alpha x_o - \alpha' x_o = y' - y$$

$$\Rightarrow x_o(\alpha - \alpha') = y' - y$$

$$\Rightarrow (\alpha - \alpha')x_o \in N(f)$$

$$\Rightarrow x_o \in N(f),$$

a contradiction, hence the representation is unique.

Application:

A- Let $f : X \to K$ be a linear functional, then two elements $x_1, x_2 \in X$ belong to the same element of the quotient space $X/N(f)$ if and only if $f(x_1) = f(x_2)$.

Proof:

Suppose that $x_1, x_2 \in x_o + N(f)$ for some $x_o \in X$, we want to prove that $f(x_1) = f(x_2)$,

Since $x_1, x_2 \in x_o + N(f)$

$\Rightarrow x_1 = x_o + y_1, x_2 = x_o + y_2 \qquad y_1, y_2 \in N(f)$

Now, $f(x_1) = f(x_o + y_1) = f(x_o) + f(y_1) = f(x_o)$

and $f(x_2) = f(x_o + y_2) = f(x_o) + f(y_2) = f(x_o)$

Therefore $f(x_1) = f(x_2)$.

Conversely:

Suppose that for $x_1, x_2 \in X$, $f(x_1) = f(x_2)$

$\Rightarrow f(x_1) - f(x_2) = 0$

$\Rightarrow f(x_1 - x_2) = 0$

$\Rightarrow x_1 - x_2 \in N(f)$

$\Rightarrow (x_1 - x_2) + N(f) = N(f)$

$\Rightarrow x_1 + N(f) = x_2 + N(f)$

$\Rightarrow x_1 = x_1 + 0 \in x_1 + N(f) = x_2 + N(f), x_2 \in x_2 + N(f)$

Hence, $x_1, x_2 \in X$ belong to the same element of the quotient space $X/N(f)$.

B- Let $f : X \rightarrow K$ be a non zero linear functional on X , then $\dim(X/N(f)) = 1$.

Proof:

We want to prove that $X/N(f) = span\{x_o + N(f)\}$ for some $x_o \notin N(f)$

Clearly, $span\{x_o + N(f)\} \subseteq X/N(f) \qquad (1)$

Now, let $y \in X/N(f)$

$y = x + N(f)$ for some $x \in X$, from(1.4.3) $x = \alpha x_o + y_1, y_1 \in N(f)$

$\Rightarrow y = x + N(f) = \alpha x_o + y_1 + N(f) = \alpha x_o + N(f) = \alpha(x_o + N(f))$

$y \in span\{x_o + N(f)\} \Rightarrow X/N(f) \subseteq span\{x_o + N(f)\} \qquad (2)$

Hence, from (1) and (2) we get $X/N(f) = span\{x_o + N(f)\}$,

so $\dim(X/N(f)) = 1$.

C- Let f_1, f_2 be two non-zero linear functional on the same vector space such that $N(f_1) = N(f_2)$, then f_1 and f_2 are proportional.

Proof:

Since $f_1, f_2 \neq 0$, then $\exists x_o \in X$ such that $f_1(x_o) \neq 0$

Since $N(f_1) = N(f_2)$, $f_2(x_o) \neq 0$

from theorem (1.4.3) any $x \in X, x = \alpha x_o + y$ for some scalar α, $y \in N(f_1)$

$$x = \frac{f_1(x)}{f_1(x_o)} x_o + y$$

$y \in N(f_1) = N(f_2) \Rightarrow f_2(y) = 0$

Now,

$$f_2(x) = \frac{f_1(x)}{f_1(x_o)} f_2(x_o) + f_2(y)$$

$$\Rightarrow f_2(x) = \frac{f_2(x_o)}{f_1(x_o)} f_1(x).$$

Remark (1.4.4)

Note that if Y is a subspace of vector space X and f is a linear functional on X such that $f(Y) \neq K$, then $f(y) = 0$ for all $y \in Y$.

Indeed suppose that $\exists y_o \in Y \subseteq X$ such that $f(y_o) = \alpha_o \neq 0$, then for any $\beta \in K \Rightarrow \beta = \frac{\beta}{\alpha_o}\alpha_o = \frac{\beta}{\alpha_o} f(y) = f(\frac{\beta}{\alpha_o} y) \in f(Y)$

$\Rightarrow K = f(Y)$, a contradiction

$$\Rightarrow f(y) = 0 \qquad \forall y \in Y.$$

Fundamental theorem for normed and Banach spaces

2.1 *Zorn's lemma*

Definition (Partially ordered set, Chain) (2.1.1)

A *partially ordered set* is a set M on which there is defined a partial ordering, that is a binary relation which is written $(\leq)$ and satisfies the conditions:

$a \leq a$ for every $a \in M$	(Reflexivity)
If $a \leq b$ and $b \leq a$, then $a = b$	(Antisymmetry)
If $a \leq b$ and $b \leq c$, then $a \leq c$	(Transitivity)

*If neither $a \leq b$ nor $b \leq a$ holds, then a and b called *incomparable elements*, in contrast, two elements a and b are called *comparable elements* if they satisfy $a \leq b$ or $b \leq a$ (or both).

*A *totally ordered set* or *Chain* is partially ordered set such that every elements of the set are comparable.

*An *upper bound* of a subset W of a partially ordered set M is an element $u \in M$ such that

$$x \leq u \qquad \text{for every } x \in W$$

*A *maximal element* of M is an $m \in M$ such that

$$m \leq x \quad \text{implies} \quad m = x$$

Examples:

(a) Let M be the set of all real numbers and let $x \leq y$ have a usual meaning, M is totally ordered, M has no maximal element.
(b) Let $P(X)$ be the power set (set of all subset) of a given set X and let $A \leq B$ mean $A \subset B$, that is A is subset of B, then $P(X)$ is partially ordered, and the only maximal element of $P(X)$ is X .
(c) Let M be the set of all ordered n-tuples $\{x = (\xi_1, ..., \xi_n | \xi_i \in \Re\}$, and $x \leq y$ mean $\xi_i \leq \eta_i$ for every $i = 1, ..., n$, where $\xi_i \leq \eta_i$ has its usual meaning, M is partially ordered, M has no maximal element.
(d) Let $M = N$, the set of all positive integers, let $m \leq n$ mean that m divides n, N is partially ordered.

Zorn's lemma (2.1.2)

Let M be a partially ordered set, suppose that every chain $C \subset M$ has upper bound, than M has at lest one maximal element.

Definition (2.1.3)

A *sublinear functional* is a real-valued functional p on a vector space X which is
**Subaddative*, that is

$$p(x+y) \leq p(x)+p(y) \qquad \forall x, y \in X.$$

**Positive-homogenous*, that is

$$p(\alpha x)=\alpha p(x) \qquad \forall \alpha \in \Re, \alpha \geq 0, x \in X.$$

2.2 ***Hahn-Banach theorem***

Hahn-Banach theorem (Extension of linear functional) (2.2.1)

Let X be a real vector space and p a sublinear functional on X, furthermore, let f be a linear functional which is defined on subspace Z of X and satisfies:

$$f(x) \le p(x) \qquad \forall x \in Z$$

Then f has a linear extension $\tilde{f}$ from Z to X satisfying:

$$\tilde{f}(x) \le p(x) \qquad \forall x \in X$$

That is, $\tilde{f}$ is a linear functional on X, satisfying

$$\tilde{f}(x) \le p(x) \text{ on } X \text{ , and } \tilde{f}(x) = f(x) \qquad \forall x \in Z.$$

Proof:

We shall prove:

(a) The set E of all linear extensions g of f satisfying $g(x) \le p(x)$ on their domain $D(g)$ can be partially ordered and Zorn's lemma yields a maximal element $\tilde{f}$ of E.

(b) $\tilde{f}$ is defined on the entire space X.

(c) An auxiliary relation which was used in (b).

We start with part (a)

Let E be the set of all linear extensions g of f which satisfy the condition:

$$g(x) \le p(x) \qquad \forall x \in D(g)$$

Clearly, $E \ne \phi$ since $f \in E$,

On E we can define a partial ordering by $g \le h$ meaning h is an extension of g,

$\Rightarrow$ By definition, $D(g) \subset D(h)$ and $h(x) = g(x) \qquad \forall x \in D(g)$

Let $C \subset E$ is chain, we define $\hat{g}$ by

$$\hat{g}(x) = g(x) \text{ if } x \in D(g) \qquad (g \in C)$$

$\hat{g}$ is linear functional, the domain being

$$D(\hat{g}) = \bigcup_{g \in C} D(g)$$

which is vector space, since C is a chain,

The definition of $\hat{g}$ is unambiguous, Indeed, for an $x \in D(g_1) \cap D(g_2)$ with $g_1, g_2 \in C$, we have $g_1(x) = g_2(x)$,
and $g_1 \leq g_2$ or $g_2 \leq g_1$ since C is chain
Clearly, $g \leq \hat{g}$ for all $g \in C$ since $D(g) \subset D(\hat{g})$ for all $g \in C$
$\Rightarrow$ $\hat{g}$ is an upper bound of C
Since $C \subset E$ was arbitrary, then by Zone's lemma E has a maximal element $\tilde{f}$, and by the definition of E
$\Rightarrow$ $\tilde{f}$ is linear extension of f which satisfies:

$$\tilde{f}(x) \leq p(x) \qquad \forall x \in D(\tilde{f}).$$

(b) We want to show that $D(\tilde{f})$ is all of X ,
Suppose that this false
$\Rightarrow \exists y_1$ such that $y_1 \in X - D(\tilde{f})$
Consider the subspace Y_1 of X spanned by $D(\tilde{f})$ and y_1
Note that $y_1 \neq 0$, since $0 \in D(\tilde{f})$
Now, any $x \in Y_1$ can be written

$$x = y + \alpha y_1 \qquad y \in D(\tilde{f})$$

This representation is unique, since
Let $x = y + \alpha y_1$ and $x = y' + \beta y_1$ $\qquad y, y' \in D(\tilde{f})$
$\Rightarrow y + \alpha y_1 = y' + \beta y_1 \Rightarrow y - y' = (\beta - \alpha) y_1$
Since $y_1 \notin D(\tilde{f})$, $y - y' \in D(\tilde{f})$, then the only solution is
$y - y' = 0$ and $\beta - \alpha = 0 \Rightarrow y = y'$ and $\beta = \alpha$,
Hence the representation is unique.
Now, a functional g_1 on Y_1 is defined by

$$g_1(y + \alpha y_1) = \tilde{f}(y) + \alpha c \qquad (1)$$ where c any real constant

g_1 is linear, since for $x_1, x_2 \in Y_1 \Rightarrow x_1 = y + \alpha y_1, x_2 = y' + \beta y_1$,

1- $g_1(x_1 + x_2) = g_1((y + \alpha y_1) + (y' + \beta y_1)) = g_1((y + y') + (\alpha + \beta) y_1)$
$= \tilde{f}(y + y') + (\alpha + \beta)c = \tilde{f}(y) + \tilde{f}(y') + \alpha c + \beta c$ since $\tilde{f}$ is linear
$= g_1(x_1) + g_1(x_2)$.

2- $g_1(rx_1) = g_1(r(y + \alpha y_1)) = g_1(ry + r\alpha y_1) = \tilde{f}(ry) + r\alpha c$
$= r\tilde{f}(y) + r\alpha c$ since $\tilde{f}$ is linear
$= r(\tilde{f}(y) + \alpha c) = rg_1(x_1)$. , where r is any scalar.

Now, for $\alpha = 0 \Rightarrow x = y \Rightarrow g_1(y) = \tilde{f}(y)$, then g_1 is proper extension of $\tilde{f}$, since $D(\tilde{f}) \subset D(g_1)$

Now, if we can prove that $g_1 \in E$ by showing that

$$g_1(x) \le p(x) \qquad \forall x \in D(g_1)$$

this will contradict the maximality of $\tilde{f}$, so that $D(\tilde{f}) \ne X$ is false and $D(\tilde{f}) = X$ is true.

(c) We must finally show that g_1 with a suitable c in (1) satisfies:

$$g_1(x) \le p(x) \qquad \forall x \in D(g_1)$$

consider any $y, z \in D(\tilde{f})$

$$\Rightarrow \tilde{f}(y) - \tilde{f}(z) = \tilde{f}(y - z) \le p(y - z) = p(y + y_1 - y_1 - z)$$

$$\le p(y + y_1) + p(-y_1 - z) \qquad \text{since } p \text{ is sublinear}$$

$$\Rightarrow -p(-y_1 - z) - \tilde{f}(z) \le p(y + y_1) - \tilde{f}(y)$$

where y_1 is fixed, since y does not appear on the left and z not on the right, if we take the supremum over $z \in D(\tilde{f})$ on the left (call it m_o) and the infimum over $y \in D(\tilde{f})$ on the right (call it m_1)

then $m_o \le m_1$, and for a c with $m_o \le c \le m_1$

$$\Rightarrow -p(-y_1 - z) - \tilde{f}(z) \le c \qquad \forall z \in D(\tilde{f}) \qquad (2)$$

$$c \le p(y + y_1) - \tilde{f}(y) \qquad \forall y \in D(\tilde{f}) \qquad (3)$$

Now, for $\alpha < 0$ and z replaced by $\alpha^{-1} y$ in(2)

$$\Rightarrow -p(-y_1 - \frac{1}{\alpha} y) - \tilde{f}(\frac{1}{\alpha} y) \le c, \text{ multiplication by} -\alpha > 0$$

$$\Rightarrow \alpha p(-y_1 - \frac{1}{\alpha} y) + \alpha\tilde{f}(\frac{1}{\alpha} y) \le -\alpha c$$

$$\Rightarrow \alpha p(-y_1 - \frac{1}{\alpha} y) + \tilde{f}(y) \le -\alpha c$$

$$\Rightarrow \tilde{f}(y) + \alpha c \le -\alpha p(-y_1 - \frac{1}{\alpha} y)$$

$$\Rightarrow g_1(x) \le p(\alpha y_1 + y)$$

$$\Rightarrow g_1(x) \le p(x).$$

for $\alpha > 0$ and y replaced by $\alpha^{-1} y$ in (3)

$$\Rightarrow c \le p(\frac{1}{\alpha} y + y_1) - \tilde{f}(\frac{1}{\alpha} y), \text{ multiplication by } \alpha > 0$$

$$\Rightarrow \alpha c \le \alpha p(\frac{1}{\alpha} y + y_1) - \alpha \tilde{f}(\frac{1}{\alpha} y)$$

$$\Rightarrow \alpha c \le p(y + \alpha y_1) - \tilde{f}(y)$$

$$\Rightarrow \tilde{f}(y) + \alpha c \le p(y + \alpha y_1)$$

$$\Rightarrow g_1(x) \le p(x).$$

for $\alpha = 0$ we have $x \in D(\tilde{f})$ and nothing to prove.

Applications:

A- A sublinear functional p satisfies $p(0) = 0$ and $p(-x) \ge -p(x)$.

Proof:

Since p is sublinear functional $p : X \to \Re$

$$\Rightarrow p(x + y) \le p(x) + p(y) \qquad \forall x, y \in X$$

and $\quad p(\alpha x) = \alpha p(x) \qquad \forall \alpha \in \Re, \alpha \ge 0, x \in X$

let $\alpha = 0$

$$p(0) = p(0x) = 0\, p(x) = 0$$

and

$$0 = p(0) = p(x - x) \le p(x) + p(-x)$$

$$\Rightarrow p(-x) \ge -p(x).$$

B- If a subadditive functional p on a normed space X is continuous at 0 and $p(0) = 0$, then p is continuous for all $x \in X$.

Proof:

Let x_o be an arbitrary (but fixed) point in X, we want to show that p is continuous at x_o,

so let $\varepsilon > 0$ be given, since p continuous at 0

$\Rightarrow \exists \delta > 0$ such that if $\|y - 0\| < \delta, y \in X$, then $|p(y)| < \varepsilon$

thus, of $y = x - x_o$

$$\|x - x_o\| < \delta \Rightarrow |p(x - x_o)| < \varepsilon \qquad (1)$$

Now,

$$p(x) = p(x - x_o + x_o) \le p(x - x_o) + p(x_o)$$

$$\Rightarrow p(x) - p(x_o) \le p(x - x_o) \qquad (2)$$

and

$$p(x_o) = p(x_o - x + x) \le p(x_o - x) + p(x)$$

$$\Rightarrow p(x) - p(x_o) \ge -p(x - x_o) \qquad (3)$$

then, from(2) and (3)we get

$$-p(x - x_o) \le p(x) - p(x_o) \le p(x - x_o)$$

$$\Rightarrow |p(x) - p(x_o)| < |p(x - x_o)| < \varepsilon \qquad \textit{from } (1)$$

hence p is continuous at x_o, and since x_o an arbitrary, then p is continuous for all $x \in X$.

C- If a subadditive functional defined on a normed space X is nonnegative outside a sphere $\{x \| \|x\| = r\}$, then it is nonnegative for all $x \in X$.

<u>Proof:</u>

Let $p : X \to \Re$, be a subadditive functional defined on a normed space X, and let $p(x) \ge 0$ for x such that $\|x\| > r$ (1)

we want to prove that $p(x) \ge 0$ for $x \in X$

(a) Let $x \in X$ such that $\|x\| = r$

$$\Rightarrow \|2x\| = 2\|x\| = 2r > r \Rightarrow p(2x) \ge 0 \qquad \textit{from } (1)$$

$$\Rightarrow 2p(x) \ge 0 \Rightarrow p(x) \ge 0$$

(b) Let $y \in X$, $y \ne 0$ then, $\left\| \dfrac{ry}{\|y\|} \right\| = r \dfrac{\|y\|}{\|y\|} = r \Rightarrow p(r \dfrac{y}{\|y\|}) \ge 0$ from(a)

$$\Rightarrow \frac{r}{\|y\|} p(y) \ge 0 \Rightarrow p(y) \ge 0 \qquad \text{for } y \in X, y \ne 0$$

if $y = 0 \Rightarrow p(0) = 0$

Then, from (1), (a) and (b) $p(x) \ge 0 \qquad \forall x \in X$.

D- If p is sublinear functional on a real vector space X, then there exists a linear functional $\tilde{f}$ on X such that $-p(-x) \le \tilde{f}(x) \le p(x)$.

<u>Proof:</u>

From theorem (2.2.1) we have $\tilde{f}(x) \le p(x)$ (1)

and $-\tilde{f}(x) = \tilde{f}(-x) \le p(-x)$ since $\tilde{f}$ is linear

$\Rightarrow \tilde{f}(x) \ge -p(-x)$ (2)

from (1) and(2) we get $-p(-x) \le \tilde{f}(x) \le p(x)$.

E- Let p be a sublinear functional on a real vector space X, and let f be defined on $Z = \{x \in X \,|\, x = \alpha x_o, \alpha \in \Re\}$ by $f(x) = \alpha p(x_o)$ with fixed x_o, then f is a functional on Z satisfying $f(x) \le p(x)$.

<u>Proof:</u>

First we want to prove that f is linear functional on Z, $f : Z \to \Re$

Let $x, y \in Z \Rightarrow x = \alpha x_o, y = \beta x_o, \alpha, \beta \in \Re \Rightarrow x + y = (\alpha + \beta) p(x_o)$,

and let r is any scalar

1- $f(x + y) = (\alpha + \beta) p(x_o) = \alpha p(x_o) + \beta p(x_o) = f(x) + f(y)$.

2- $f(rx) = r\alpha p(x_o) = rf(x)$, hence f is linear functional on Z.

Now we want to prove that $f(x) \le p(x)$,

Since $f(x) = \alpha p(x_o), x = \alpha x_o$

if $\alpha \ge 0 \Rightarrow f(x) = \alpha p(x_o) = p(\alpha x_o) = p(x) \Rightarrow f(x) = p(x)$ (1)

if $\alpha < 0 \Rightarrow -\alpha > 0$

$\Rightarrow f(x) = \alpha p(x_o) = -(-\alpha) p(x_o) = -p(-\alpha x_o) = -p(-x) < p(x)$

$\Rightarrow f(x) < p(x)$ (2)

from (1) and (2) we get $f(x) \le p(x)$.

2.3 ***Hahn Banach theorem for complex vector spaces and normed spaces***

Hahn Banach theorem(Generalized) (2.3.1)

Let X be a real or complex vector space and p a real-valued functional on X which is subadditive, that is

$$p(x+y) \le p(x)+p(y) \qquad \forall x, y \in X \qquad (1)$$

and for every scalar α satisfies

$$p(\alpha x) = |\alpha| p(x) \qquad (2)$$

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$|f(x)| \le p(x) \qquad \forall x \in Z \qquad (3)$$

Then, f has a linear extension $\tilde{f}$ from Z to X satisfying

$$|\tilde{f}(x)| \le p(x) \qquad \forall x \in X \qquad (4)$$

Proof:

(a) Real vector space:
If X is real, the situation is simple

$$f(x) \le |f(x)| \le p(x) \qquad \textit{from } (3)$$

$$\Rightarrow f(x) \le p(x) \qquad \forall x \in Z$$

then, by theorem (2.2.1) there is a linear extension $\tilde{f}$ from Z to X such that

$$\tilde{f}(x) \le p(x) \qquad \forall x \in X \qquad (5)$$

Now, $-\tilde{f}(x) = \tilde{f}(-x) \le p(-x) = |-1| p(x) = p(x) \qquad \textit{from } (2)$

$$\Rightarrow -\tilde{f}(x) \le p(x)$$

$$\Rightarrow \tilde{f}(x) \ge -p(x) \qquad (6)$$

Then from (5) and (6)

$$\Rightarrow -p(x) \le \tilde{f}(x) \le p(x)$$

$$\Rightarrow |\tilde{f}(x)| \le p(x).$$

(b) Complex vector space:
Let X be complex, then Z is a complex vector space, too

$\Rightarrow f$ is complex-valued

$\Rightarrow$ we can write $\quad f(x) = f_1(x) + if_2(x) \qquad x \in Z$

where f_1 and f_2 are real-valued

for a moment we regard X and Z as real vector space and denote them by X_r and Z_r respectively, this simply means that we restrict multiplication by scalars to real numbers (instead of complex numbers), since f is linear on Z, and f_1, f_2 are real-valued $\Rightarrow f_1, f_2$ are linear functional on Z, also $f_1(x) \leq |f(x)|$

$$\Rightarrow f_1(x) \leq p(x) \qquad \forall x \in Z_r \qquad \textit{from } (3)$$

$\Rightarrow$ by theorem (2.2.1), there is a linear extension $\tilde{f}_1$ of f_1 from Z_r to X_r, such that

$$\tilde{f}_1(x) \leq p(x) \qquad \forall x \in X_r \qquad (7)$$

this take care of f_1 and we now turn of f_2

Now, returning to Z and using $f = f_1 + if_2$, we have for every $x \in Z$

$$i[f_1(x) + if_2(x)] = if(x) = f(ix) = f_1(ix) + if_2(ix)$$

$$\Rightarrow if_1(x) - f_2(x) = f_1(ix) + if_2(ix)$$

the real parts on both sides must be equal

$$\Rightarrow -f_2(x) = f_1(ix)$$

$$\Rightarrow f_2(x) = -f_1(ix) \qquad \forall x \in Z \qquad (8)$$

$$\Rightarrow f(x) = f_1(x) - if_1(ix)$$

$\Rightarrow$ if for all $x \in X$ we set

$$\tilde{f}(x) = \tilde{f}_1(x) - i\tilde{f}_1(ix) \qquad (9)$$

then from(8) $\quad \tilde{f}(x) = f(x)$ on Z

this shows that $\tilde{f}$ is an extension of f from Z to X, now we want to prove that:

(a) $\tilde{f}$ is linear functional on the complex vector space X.

(b) $\tilde{f}$ satisfies (4) on X.

To prove (a) let $x, y \in X$ and $\alpha \in C, \alpha = a + ib \qquad a, b \in \Re$

$$\begin{aligned}\tilde{f}(x+y) &= \tilde{f}_1(x+y) - i\tilde{f}_1(i(x+y)) \qquad \textit{from } (9)\\ &= \tilde{f}_1(x) + \tilde{f}_1(y) - i(\tilde{f}_1(ix) + \tilde{f}_1(iy))\\ &= \tilde{f}_1(x) - i\tilde{f}_1(ix) + \tilde{f}_1(y) - i\tilde{f}_1(iy)\\ &= \tilde{f}(x) + \tilde{f}(y).\end{aligned}$$

and,

$$\begin{aligned}
\tilde{f}(\alpha x) &= \tilde{f}((a+ib)x) \\
&= \tilde{f}(ax+ibx) \\
&= \tilde{f}_1(ax+ibx) - i\tilde{f}_1(iax-bx) \\
&= a\tilde{f}_1(x) + b\tilde{f}_1(ix) - i\left[a\tilde{f}_1(ix) - b\tilde{f}_1(x)\right] \\
&= a\tilde{f}_1(x) + b\tilde{f}_1(ix) - ia\tilde{f}_1(ix) + ib\tilde{f}_1(x) \\
&= a\left[\tilde{f}_1(x) - i\tilde{f}_1(ix)\right] + b\left[\tilde{f}_1(ix) + i\tilde{f}_1(x)\right] \\
&= a\left[\tilde{f}_1(x) - i\tilde{f}_1(ix)\right] + ib\left[\tilde{f}_1(x) - i\tilde{f}_1(ix)\right] \\
&= a + ib\left[\tilde{f}_1(x) - i\tilde{f}_1(ix)\right] = \alpha\tilde{f}(x).
\end{aligned}$$

Hence, $\tilde{f}$ is linear.

To prove (b)

1- for any x such that $\tilde{f}(x) = 0$ this holds, since $p(x) \geq 0$.

2- Let $x \in X$ such that $\tilde{f}(x) \neq 0$, then we can write $\tilde{f}$ by using polar form of complex quantities

$$\tilde{f}(x) = \left|\tilde{f}(x)\right| e^{i\theta}$$

$$\Rightarrow \left|\tilde{f}(x)\right| = \tilde{f}(x)e^{-i\theta} = \tilde{f}(e^{-i\theta}x)$$

since $\left|\tilde{f}(x)\right|$ is real, then $\tilde{f}(e^{-i\theta}x)$ is real

$$\Rightarrow \tilde{f}(e^{-i\theta}x) = \tilde{f}_1(e^{-i\theta}x)$$

Now,

$$\begin{aligned}
\left|\tilde{f}(x)\right| = \tilde{f}(e^{-i\theta}x) = \tilde{f}_1(e^{-i\theta}x) &\leq p(e^{-i\theta}x) && \textit{from } (7) \\
&= \left|e^{-i\theta}\right| p(x) && \textit{from } (2) \\
&= p(x)
\end{aligned}$$

Hence $\left|\tilde{f}(x)\right| \leq p(x) \qquad \forall x \in X.$

<u>Hahn-Banach theorem (Normed space)</u> (2.3.2)

Let f be a bounded linear functional on a subspace Z of a normed space X, then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm

$$\left\|\tilde{f}\right\|_X = \|f\|_Z$$

where

$$\left\|\tilde{f}\right\|_X = \sup_{\substack{x\in X \\ \|x\|=1}} \left|\tilde{f}(x)\right| \quad , \|f\|_Z = \sup_{\substack{x\in Z \\ \|x\|=1}} |f(x)|.$$

(and $\|f\|_Z = 0$ in the trivial case $Z = \{0\}$).

Proof:

If $Z = \{0\}$, then $f = 0$ and the extension is $\tilde{f} = 0$.
Now, let $Z \neq \{0\}$, we want to use theorem (2.3.1), for all $x \in Z$ we have

$$|f(x)| \leq \|f\|_Z \|x\|$$

This is of the from (3) in theorem (2.3.1)

$$p(x) = \|f\|_Z \|x\|$$

p Is defined on all of X , and p satisfies (1), since by the triangle inequality

$$p(x+y) = \|f\|_Z \|x+y\| \leq \|f\|_Z (\|x\| + \|y\|)$$
$$= \|f\|_Z \|x\| + \|f\|_Z \|y\| = p(x) + p(y).$$

p also satisfies(2) because

$$p(\alpha x) = \|f\|_Z \|\alpha x\| = |\alpha| \|f\|_Z \|x\| = |\alpha| p(x).$$

Hence, we can apply theorem (2.3.1), that mean there exists a linear functional $\tilde{f}$ on X which is an extension of f and satisfies

$$\left|\tilde{f}(x)\right| \leq p(x) = \|f\|_Z \|x\| \qquad x \in X$$

Taking the supremum over all $x \in X$ of norm 1, we obtain the inequality

$$\left\|\tilde{f}\right\|_X = \sup_{\substack{x\in X \\ \|x\|=1}} \left|\tilde{f}(x)\right| \leq \|f\|_Z \qquad (a)$$

and since under an extension the norm cannot decrease, we also have

$$\left\|\tilde{f}\right\|_X \geq \|f\|_Z \qquad (b)$$

hence, from(a) and (b) we get

$$\left\|\tilde{f}\right\|_X = \|f\|_Z.$$

Definition (2.3.3)

The *dual space* X^* of a normed space X consists of the bounded linear functionals on X .

Theorem (Bounded linear functionals) (2.3.4)

Let X be a normed space and $x_o \neq 0$ be any element of X , then there exists a bounded linear functional $\tilde{f}$ on X such that

$$\|\tilde{f}\| = 1 \qquad , \tilde{f}(x_o) = \|x_o\|.$$

Proof:

Let $Z = \{x | x = \alpha x_o\}$ where α is a scalar, Z subspace of X ,
we define a linear functional $f : Z \to \Re$, by

$$f(x) = f(\alpha x_o) = \alpha \|x_o\| \qquad (1)$$

f is bounded and has norm $\|f\| = 1$,because

$$|f(x)| = |f(\alpha x_o)| = |\alpha| \|x_o\| = \|\alpha x_o\| = \|x\|$$

$$\|f\| = \sup_{\substack{x \in Z \\ \|x\|=1}} |f(x)| = \sup_{\substack{x \in Z \\ \|x\|=1}} \|x\| = 1$$

and from theorem (2.3.2), f has linear extension $\tilde{f}$ from Z to X , of norm $\|\tilde{f}\| = \|f\| = 1$
and from (1) we see that

$$\tilde{f}(x_o) = f(x_o) = \|x_o\|.$$

Corollary (Norm, zero vector) (2.3.5)

For every x in a normed space X , we have

$$\|x\| = \sup_{\substack{f \in X^* \\ f \neq 0}} \frac{|f(x)|}{\|f\|}$$

Hence if x_o is such that $f(x_o) = 0$ for all $f \in X^*$, then $x_o = 0$.

Proof:

From theorem (2.3.4), we have, writing x for x_o

$$\sup_{\substack{f\in X^* \\ f\neq 0}} \frac{|f(x)|}{\|f\|} \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|} = \frac{\|x\|}{1} = \|x\| \qquad (1)$$

and from $|f(x)| \leq \|f\|\|x\|$ we obtain

$$\sup_{\substack{f\in X^* \\ f\neq 0}} \frac{|f(x)|}{\|f\|} \leq \|x\| \qquad (2)$$

so, from (1) and (2) we get

$$\|x\| = \sup_{\substack{f\in X^* \\ f\neq 0}} \frac{|f(x)|}{\|f\|}.$$

Applications:

A- Let p be defined on a vector space X and satisfy

$$p(x+y) \leq p(x) + p(y) \qquad \forall x, y \in X$$

and $\quad p(\alpha x) = |\alpha| p(x) \qquad$ for every scalar α

Then for any given $x_o \in X$ there is a linear functional $\tilde{f}$ on X such that $\tilde{f}(x_o) = p(x_o)$ and $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

Proof:

Let $x_o \in X$ fixed and $Z = \{x | x = \alpha x_o, \alpha \in C\}$,

and define $f : Z \to C$ by

$$f(\alpha x_o) = \alpha p(x_o)$$

clearly f is linear functional on Z, also

$$|f(x)| = |f(\alpha x_o)| = |\alpha p(x_o)| = |\alpha|\,|p(x_o)| \leq |\alpha| p(x_o) = p(\alpha x_o) = p(x)$$

$$\Rightarrow |f(x)| \leq p(x)$$

By theorem (2.3.1), f has linear extension $\tilde{f}$ on X such that

$$|\tilde{f}(x)| \leq p(x) \qquad \forall x \in X$$

and if $\alpha = 1$, we get $\tilde{f}(x_o) = f(x_o) = 1(p(x_o)) = p(x_o)$.

B- Let X be a normed space and X^* its dual space. If $X \neq 0$, then X^* cannot be $\{0\}$.

Proof:

Let $x_o \in X, x_o \neq 0$, then by theorem (2.3.4) there exists a bounded linear functional f on X such that

$$\|f\| = 1 \text{ and } f(x_o) = \|x_o\|$$

since $x_o \neq 0 \Rightarrow \|x_o\| \neq 0 \Rightarrow f(x_o) \neq 0 \qquad \forall x_o \in X$ (since $X \neq \{0\}$)

Hence $f \neq 0 \Rightarrow X^* \neq \{0\}$.

C- If $f(x) = f(y)$ for every bounded linear functional f on a normed space X, then $x = y$.

Proof:

Let $f(x) = f(y) \qquad \forall f \in X^*$

$$\Rightarrow f(x) - f(y) = 0 \qquad \forall f \in X^*$$

$$\Rightarrow f(x-y) = 0 \qquad \forall f \in X^* \text{ (since } f \text{ is linear)}$$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y.$$

D- Under the assumptions of theorem (2.3.4) there is a bounded linear functional $\hat{f}$ on X such that

$$\|\hat{f}\| = \|x_o\|^{-1} \text{ and } \hat{f}(x_o) = 1.$$

Proof:

Let $x_o \in X, x_o \neq 0$, then by theorem (2.3.4) there exists a bounded linear functional g on X such that

$$\|g\| = 1 \text{ and } g(x_o) = \|x_o\|$$

Now, let $\hat{f} = g\|x_o\|^{-1}$, then

$$\|\hat{f}\| = \|g\|\|x_o\|^{-1} = 1(\|x_o\|^{-1}) = \|x_o\|^{-1}$$

and $$\hat{f}(x_o) = g(x_o)\|x_o\|^{-1} = \|x_o\|\|x_o\|^{-1} = 1.$$

2.4 ***Open mapping theorem***

Definition (2.4.1)

Let X and Y be metric spaces, then $T : D(T) \to Y$ with domain $D(T) \subset X$ is called an *open mapping* if for every open set in $D(T)$ the image is an open set in Y.

Remark (Baire's category theorem) (2.4.2)

If a metric space $X \neq \phi$ is complete, it is nonmeager in itself, hence if $X = \bigcup_{k=1}^{\infty} A_k$, where A_k closed, Then at least one A_k contains a nonempty open subset.

Lemma (Open unit ball) (2.4.3)

A bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image $T(B_0)$ of the open unit ball $B_0 = B(0;1) \subset X$ contains an open ball about $0 \in Y$.

Proof:

Proceeding stepwise, we prove:

(a) $\overline{T(B_1)}$ contains an open ball, where $B_1 = B(0;\frac{1}{2})$.

(b) $\overline{T(B_n)}$ contains an open ball $\forall n \in N$, where $B_n = B(0;2^{-n})$.

(c) $T(B_0)$ contains an open ball about $0 \in Y$.

(a) We consider the open ball $B_1 = B(0;\frac{1}{2}) \subset X$, any fixed $x \in X$ is in kB_1 with real k, clearly $\bigcup_{k=1}^{\infty} kB_1 \subset X$ (1) since $kB_1 \subset X, \forall k \in N$

and let $x \in X$, $2\|x\| > 0$, then $\exists k_x > 2\|x\| \Rightarrow \|x\| < \frac{k_x}{2}$, then

$$x \in k_x B_1 \subset \bigcup_{k=1}^{\infty} kB_1 \qquad \forall x \in X \qquad (2)$$

Hence, from (1) and (2) we get $X = \bigcup_{k=1}^{\infty} kB_1$

since T is surjective and linear,

$$\Rightarrow \bigcup_{k=1}^{\infty} \overline{kT(B_1)} = \bigcup_{k=1}^{\infty} k\overline{T(B_1)} \subset Y = T(X)$$

$$= T(\bigcup_{k=1}^{\infty} kB_1) = \bigcup_{k=1}^{\infty} kT(B_1) \subset \bigcup_{k=1}^{\infty} k\overline{T(B_1)} = \bigcup_{k=1}^{\infty} \overline{kT(B_1)}$$

since Y is complete, it is nonmeager in itself, then by (2.4.2) $\exists k_o \in N$ such that $k_o\overline{T(B_1)}$ contain an open ball, say

$$B(y_o;\alpha) \subset k_o\overline{T(B_1)}$$

$$\Rightarrow B(y_o;\varepsilon) = \frac{1}{k_o}B(y_o;\alpha) \subset \overline{T(B_1)}, \varepsilon = \frac{\alpha}{k_o}.$$

(b) From (a) we shown that $\overline{T(B_1)}$ contain an open ball, say $B(y_o;\varepsilon) \subset \overline{T(B_1)}$ for some $y_o \in \overline{T(B_1)}, \varepsilon > 0$.

Hence, $B(0;\varepsilon) = B(y_o;\varepsilon) - y_o \subset \overline{T(B_1)} - y_o$

Now, let $y \in \overline{T(B_1)} - y_o$, then $y + y_o \in \overline{T(B_1)}$, then there are

$u_n \in B_1$ such that $Tu_n \to y + y_o$

and $v_n \in B_1$ such that $Tv_n \to y_o$

$$\Rightarrow \|u_n - v_n\| \le \|u_n\| + \|v_n\| < \frac{1}{2} + \frac{1}{2} = 1$$

$$\Rightarrow u_n - v_n \in B_0$$

since $T(u_n - v_n) = Tu_n - Tv_n \to y$

$$\Rightarrow y \in \overline{T(B_0)}$$

Hence, $B(0;\varepsilon) = B(y_o;\varepsilon) - y_o \subset \overline{T(B_1)} - y_o \subset \overline{T(B_0)}$ (3)

Now, let $B_n = B(0;2^{-n}) \subset X$, $B_n = B(0;2^{-n}) = 2^{-n}B(0;1) = 2^{-n}B_0$
since T is linear

$$\Rightarrow \overline{T(B_n)} = 2^{-n}\overline{T(B_0)}$$

from (3) we thus obtain $V_n = B(0;\frac{\varepsilon}{2^{-n}}) \subset \overline{T(B_n)}$ (4)

(c) We finally prove that $V_1 = B(0;\frac{\varepsilon}{2}) \subset T(B_0)$

Let $y \in V_1 \subset \overline{T(B_1)}$ *from* (4), $n = 1$

$y \in \overline{T(B_1)} \Rightarrow$ y is a limit point of $\overline{T(B_1)}$

$\Rightarrow$ every neighborhood of y contains a point of $\overline{T(B_1)}$

$\Rightarrow \exists x_1 \in B_1$ such that $\|y - Tx_1\| < \frac{\varepsilon}{2^2}$

this implies that $y - Tx_1$ belong to $V_2 = B(0; \frac{\varepsilon}{2^2}) \subset \overline{T(B_2)}$

$\Rightarrow$ $y - Tx_1$ is a limit point of $\overline{T(B_2)}$

$\Rightarrow$ every neighborhood of $y - Tx_1$ contains a point of $\overline{T(B_2)}$

$\Rightarrow \exists x_2 \in B_2$ such that $\|y - Tx_1 - Tx_2\| < \frac{\varepsilon}{2^3}$

this implies that $y - Tx_1 - Tx_2$ belong to $V_3 = B(0; \frac{\varepsilon}{2^3}) \subset \overline{T(B_3)}$

and so on ,in the n th step we can choose an $x_n \in B_n$ such that

$$\left\| y - \sum_{i=1}^{n} Tx_i \right\| < \frac{\varepsilon}{2^{n+1}} \qquad (5)$$

let $z_n = x_1 + ... + x_n$, since $x_k \in B_k$ we have $\|x_k\| < \frac{1}{2^k}$. This yield for $n > m$

$$\|z_n - z_m\| \le \sum_{k=m+1}^{n} \|x_k\| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \to 0$$

as $m \to \infty$. Hence (z_n) is Cauchy. (z_n) converse, say $z_n \to x$ because X is complete. Also $x \in B_0$ since B_0 has radius 1and

$$\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

since T is continuous, $Tz_n \to Tx$ and (5) shows that $Tx = y$. Hence $y \in T(B_0)$.

Open mapping theorem, Bounded inverse theorem (2.4.4)

A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous and thus bounded.

Proof:

We want to prove that for every open set $A \subset X$ the image $T(A)$ is open in Y, this we do by showing that for every $y = Tx \in T(A)$ the set $T(A)$ contains an open ball about $y = Tx$

Now, let $y = Tx, x \in A$, since A is open, then A contains an open ball about x, say

$$B(x;\varepsilon) \subset A$$
$$\Rightarrow B(0_X;\varepsilon) = B(x;\varepsilon) - x \subset A - x$$
$$\Rightarrow T(B(0_X;\varepsilon)) \subset T(A) - Tx$$
$$\Rightarrow T(\frac{1}{\varepsilon}B(0_X;1)) \subset T(A) - Tx$$
$$\Rightarrow \frac{1}{\varepsilon}T(B_0) \subset T(A) - Tx$$
$$\Rightarrow T(B_0) \subset \varepsilon(T(A) - Tx)$$

But from (2.4.3)

$T(B_0)$ contains a ball about 0_Y, say $B(0_Y;\delta)$

$$\Rightarrow B(0_Y;\delta) \subset T(B_0) \subset \varepsilon(T(A) - Tx)$$
$$\Rightarrow \frac{1}{\varepsilon}B(0_Y;\delta) \subset T(A) - Tx$$
$$\Rightarrow B(0_Y;\frac{\delta}{\varepsilon}) \subset T(A) - Tx$$
$$\Rightarrow B(0_Y;\frac{\delta}{\varepsilon}) + Tx \subset T(A)$$
$$\Rightarrow B(Tx;\frac{\delta}{\varepsilon}) \subset T(A)$$

Hence, $T(A)$ contains an open ball about $y = Tx$, so $T(A)$ is open in Y.

Finally, if $T^{-1}: Y \to X$ exists, it is continuous because T is open. Since T^{-1} is linear, then it is bounded.

Applications

A- Let X be the normed space whose points are sequences of complex numbers $x = (\xi_i)$ with only finitely many nonzero terms and norm defined by $\|x\| = \sup_i |\xi_i|$,

Let $T : X \to Y$ be defined by $y = Tx = (\xi_1, \frac{1}{2}\xi_2, ...) = (\frac{\xi_i}{i})_{i=1}^{\infty}$

Then T is linear and bounded but T^{-1} is unbounded.

Proof:

Let $x, y \in X$, $x = (\xi_i)$, $y = (\eta_i)$, α is any scalar

1- $T(x+y) = (\frac{\xi_i + \eta_i}{i})_{i=1}^{\infty} = (\frac{\xi_i}{i} + \frac{\eta_i}{i})_{i=1}^{\infty} = (\frac{\xi_i}{i})_{i=1}^{\infty} + (\frac{\eta_i}{i})_{i=1}^{\infty} = Tx + Ty$.

2- $T(\alpha x) = (\frac{\alpha \xi_i}{i})_{i=1}^{\infty} = \alpha (\frac{\xi_i}{i})_{i=1}^{\infty} = \alpha Tx$.

Hence, T is linear.

Also, $\|Tx\| = \sup_i \left| \frac{\xi_i}{i} \right| \le \sup_i |\xi_i| = \|x\|$, then T is bounded.

Let $x = (\xi_i) \in X \Rightarrow \xi_i = 0$ for all but finite number of ξ_i 's

Let $0 = Tx = (\frac{\xi_i}{i}) \Rightarrow \xi_i = 0, \forall i \Rightarrow x = 0$, hence T is one to one, then $T^{-1} : R(T) \to X$ exists.

Let $y = (\eta_i) \in X \Rightarrow \eta_i = 0$ for all but finite number of η_i 's

$\Rightarrow (i\eta_i) \in X$ and $T(i\eta_i) = (\eta_i)$, so T is surgective.

Now, let $T^{-1} : R(T) \to X$ is defined by $x = T^{-1}(y) = (i\eta_i)_{i=1}^{\infty}$

Let $y_n \in X$, $y_n = (\eta_1^{(n)}, \eta_2^{(n)}, ..., \eta_k^{(n)}, ...)$, where $\eta_k^{(n)} = \begin{cases} \frac{1}{n} & k = n \\ 0 & k \neq n \end{cases}$

$\Rightarrow \|y_n\| = \frac{1}{n}$

and $T^{-1}(y_n) = (0,0,....., 0,1,0,.....)$ where 1is the n th term

$\Rightarrow T(0,...0,1,0,...) = y_n$

$$\left\|T^{-1}\right\| = \sup_{\substack{y\in X \\ y\neq 0}} \frac{\left\|T^{-1}(y_n)\right\|}{\|y\|} \geq \frac{\left\|T^{-1}(y_n)\right\|}{\|y_n\|} = \frac{1}{1/n} = n \qquad \forall n \in N$$

$\Rightarrow T^{-1}$ is unbounded.

This example does not contradict the open mapping theorem, as X is not Banach space.

B- Let $T : X \rightarrow Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, then there are positive real number a and b such that

$$a\|x\| \leq \|T(x)\| \leq b\|x\| \qquad \text{for all } x \in X.$$

<u>Proof:</u>

Since T is bounded, then $\exists b$ such that $\|T(x)\| \leq b\|x\| \quad \forall x \in X \quad (1)$

And since T is bounded linear operator from a Banach space X onto a Banach space Y, then T^{-1} is bounded, so $\exists\alpha$ such that

$\left\|T^{-1}(y)\right\| \leq \alpha\|y\| \quad \forall y \in Y, y = Tx$

for all $x \in X \Rightarrow \|x\| = \left\|T^{-1}T(x)\right\| \leq \alpha\|T(x)\|$,

put $\alpha = \dfrac{1}{a} \Rightarrow a\|x\| \leq \|T(x)\| \quad \forall x \in X \quad (2)$

Hence, from (1) and (2) we get

$$a\|x\| \leq \|T(x)\| \leq b\|x\| \qquad \text{for all } x \in X.$$

C- Let X and Y be Banach spaces and $T : X \rightarrow Y$ an injective bounded linear operator, then $T^{-1} : R(T) \rightarrow X$ is bounded if and only if $R(T)$ is closed in Y.

<u>Proof:</u>

Suppose that $T^{-1} : R(T) \rightarrow X$ is bounded, and let $y \in \overline{R(T)}$, then there is the sequence (y_n) in $R(T)$ such that $y_n \rightarrow y$ .since $y_n \in R(T)$, $y_n = Tx_n, x_n \in X \Rightarrow x_n = T^{-1}y_n$

Now, since (y_n) is convergent, it is a Cauchy sequence. Hence

$\|x_n - x_m\| = \left\|T^{-1}y_n - T^{-1}y_m\right\| = \left\|T^{-1}(y_n - y_m)\right\| \leq \left\|T^{-1}\right\|\|y_n - y_m\|$ since T^{-1} is bounded therefore, if $\varepsilon > 0$ is given $\exists k_\varepsilon \in N$ such that $\forall n, m \geq k_\varepsilon$

$$\|y_n - y_m\| < \frac{\varepsilon}{\|T^{-1}\|}$$

which implies that $\|x_n - x_m\| < \varepsilon$, so (x_n) is Cauchy sequence in X , and hence is convergent since X is Banach space, say $x_n \to x$

$\Rightarrow y_n = Tx_n$ converges to Tx

By the uniqueness of the limit $Tx = y \Rightarrow y \in R(T) \Rightarrow R(T)$ is closed.

Conversely

Let $R(T)$ is closed in Y , then $R(T)$ is Banach space so that $T : X \to R(T)$ is a bijective bounded linear operator defined from a Banach space X onto a Banach space $R(T)$, hence by open mapping theorem T^{-1} is bounded.

2.5 ***Closed Linear Operators, Closed Graph Theorem***

Definition (Closed linear operator) (2.5.1)

Let X and Y be normed space and $T : D(T) \to Y$ a linear operator with domain $D(T) \subset X$, Then T is called a *closed linear operator* if its graph

$$\vartheta(T) = \{(x, y) : x \in D(T), y = Tx\}$$

is closed in the normed space $X \times Y$, where the two algebraic operations of a vector space in $X \times Y$ are defined as usual, that is

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\alpha(x, y) = (\alpha x, \alpha y)$$

(α a scalar) and the norm on $X \times Y$ is defined by

$$\|(x, y)\| = \|x\| + \|y\|.$$

Remark (2.5.2)

A subspace M of a complete X is itself complete if and only if M closed in X .

Closed Graph Theorem (2.5.3)

Let X and Y be Banach spaces and $T : D(T) \to Y$ a closed linear operator, where $D(T) \subset X$, then if $D(T)$ is closed in X , the operator T is bounded.

Proof:

We first show that $X \times Y$ with norm defined by $\|(x, y)\| = \|x\| + \|y\|$ is complete,

Let (z_n) be Cauchy in $X \times Y$, where $z_n = (x_n, y_n)$, then for every $\varepsilon > 0$, there is $k_\varepsilon \in N$ such that

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon \qquad m, n > k_\varepsilon \qquad (1)$$

Hence (x_n) and (y_n) are Cauchy in X and Y respectively, and converge. Say $x_n \to x$ and $y_n \to y$, because X and Y are complete.

This implies that $z_n \to z = (x, y)$ since from (1) with $m \to \infty$ we have $\|z_n - z\| \le \varepsilon$, for $n > k_\varepsilon$, Since the Cauchy sequence (z_n) was arbitrary, hence $X \times Y$ is complete.

By assumption, $\vartheta(T)$ is closed in $X \times Y$ and $D(T)$ is closed in X
Hence $\vartheta(T)$ and $D(T)$ are complete by (2.5.2),
We consider the mapping

$$p : \vartheta(T) \to D(T)$$
$$p(x, Tx) = x$$

p is linear, p is bounded because

$$\|p(x, Tx)\| = \|x\| \le \|x\| + \|Tx\| = \|(x, Tx)\|.$$

p is bijective; in fact the inverse mapping is

$$p^{-1} : D(T) \to \vartheta(T)$$
$$p^{-1}(x) = (x, Tx)$$

Since $\vartheta(T)$ and $D(T)$ are complete, we can apply the bounded inverse theorem (2.4.4) and see that p^{-1} is bounded, say

$$\|(x, Tx)\| \le b\|x\| \qquad \text{for some } b \text{ and all } x \in D(T)$$

Hence T is bounded because

$$\|Tx\| \le \|Tx\| + \|x\| = \|(x, Tx)\| \le b\|x\| \qquad \forall x \in D(T).$$

<u>Theorem (Closed linear operator)</u> (2.5.4)

Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$ and X and Y are normed spaces, then T is closed if and only if it has the following property:
If $x_n \to x$ where $x_n \in D(T)$, and $Tx_n \to y$, then $x \in D(T)$ and $Tx = y$.

<u>Lemma (Closed operator)</u> (2.5.5)

Let $T : D(T) \to Y$ be abounded linear operator with domain $D(T) \subset X$, where X and Y are normed spaces, then:
(a) If $D(T)$ is closed subset of X , Then T is closed.
(b) If T is closed and Y is complete, then $D(T)$ is a closed subset of X .

Proof:

(a) If (x_n) is in $D(T)$ and converges, say $x_n \to x$ and is such that (Tx_n) also converges, then $x \in \overline{D(T)} = D(T)$ since $D(T)$ is closed, and $Tx_n \to Tx$ since T is continuous , Hence T is closed by theorem (2.5.4)

(b) For $x \in \overline{D(T)}$ there is a sequence (x_n) in $D(T)$ such that $x_n \to x$, since T is bounded

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \le \|T\|\|x_n - x_m\|$$

This show that (Tx_n) is Cauchy, (Tx_n) converges, Say $Tx_n \to y \in Y$ because Y is complete. Since T is closed, $x \in D(T)$ by theorem (2.5.4) and $Tx = y$, Hence $D(T)$ is closed because $x \in \overline{D(T)}$ was arbitrary.

Remark (2.5.6)

Closedness does not imply boundedness of a linear operator.

***Example*:**

Let $X = C[0,1]$ and $T : D(T) \to X$ is defined by

$$T(x) = x'$$

where $x \in D(T) \subseteq X$, $D(T)$ is subspace of functions $x \in X$ which have continuous derivative, Then T is not bounded, but is closed.

Proof:

We see from (1.3) that T is not bounded.
To prove that T is closed by appling theorem (2.5.4)
Let (x_n) in $D(T)$ be such that both (x_n) and (Tx_n) converge, say

$$x_n \to x \quad \text{and} \quad Tx_n = x_n' \to y$$

Since convergence in the norm of $C[0,1]$ is uniform convergence on $[0,1]$, from $x_n' \to y$ we have

$$\int_0^t y(\tau)d\tau = \int_0^t \lim_{n\to\infty} x_n'(\tau)d\tau = \lim_{n\to\infty}\int_0^t x_n'(\tau)d\tau = x(t) - x(0)$$

That is $x(t) = x(0) + \int_0^t y(\tau)d\tau$

This show that $x \in D(T)$ and $x' = y$, by theorem (2.5.4) T is closed.

Remark (2.5.7)

Boundedness does not imply Closedness of a linear operator.

Example:

Let $T : D(T) \to D(T) \subseteq X$ be the identity operator on $D(T)$, where $D(T)$ is a proper dense subspace of a normed space X, then T is linear and bounded but T is not closed, this follows immediately from theorem (2.5.4) if we take $x \in X - D(T)$ and a sequence (x_n) in $D(T)$ which converges to x.

Lemma (2.5.8)

Let X and Y be normed spaces, and let $T : D(T) \to Y$ be a closed linear operator, $D(T) \subseteq X$. If $T^{-1} : R(T) \to X$ exists, it is a closed linear operator.

Proof:

We have see from theorem (1.2.5(b)) if $T^{-1} : R(T) \to X$ exists, it is linear.

To show that $T^{-1} : R(T) \to X$ is closed

Suppose that T is a closed operator, and let (y_n) be a sequence in $R(T)$ such that (y_n) converges to $y \in Y$, and $(T^{-1}(y_n))$ converges to $x \in X$, then $y_n = Tx_n$ for some $x_n \in D(T)$

Hence $(x_n) = (T^{-1}y_n)$ is sequence in $D(T)$ which converges to $x \in X$ since T is closed, and $(y_n) = (Tx_n)$ converges to y, we must have $y = Tx$. That is $y \in R(T) = D(T^{-1})$, hence $x = T^{-1}y$

This implies that T^{-1} is closed by theorem (2.5.4).

Applications

A- The Null space $N(T)$ of a closed linear operator $T : X \to Y$ is a closed subspace of X.

Proof:

Let $x \in \overline{N(T)}$ then there exist a sequence (x_n) in $N(T)$ such that $x_n \to x$
Now, $T(x_n) = 0, \forall n \in N$ so that $T(x_n) \to 0$
Since T is closed, then $x \in D(T)$, and $0 = T(x) \Rightarrow x \in N(T)$, then $N(T)$ is a closed subspace of X.

B- Let T be closed linear operator with domain $D(T)$ in a Banach space X and range $R(T)$ in a normed space Y .If T^{-1} exists and is bounded, then $R(T)$ is closed.

Proof:

Suppose that $T^{-1} : R(T) \to D(T)$ exists,
Since $T : D(T) \to Y$ is closed, then T^{-1} is closed linear operator by lemma (2.5.8), Since $T^{-1} : R(T) \to D(T)$ is bounded and closed linear operator, so $D(T^{-1}) = R(T)$ is closed by lemma (2.5.5(b)).

C- If $T : X \to Y$ is a closed linear operator, where X and Y are normed space, and Y is compact, then T is bounded.

Proof:

Since Y is compact, then Y is complete, so $T^{-1}(Y) = X = D(T)$ is closed by lemma (2.5.5),
Hence T is bounded by theorem (2.5.3).

References:

1- Introductory Functional Analysis with Applications (by Erwm Kreyszig).

2- Internet.