Problem Set (3)

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PHYS 453: Quantum mechanics

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Problem 3.1. Calculate the probability per nm for an electron in an infinite well of width $L = 5$ nm

Problem 3.2. A quantum system described by the wavefunction $\Psi(x) = 2i\,\psi_1(x) - N\psi_3(x)$, where $\psi_n(x)$ is the n th state of an infinite potential well of width a.

(i) Normalise $\Psi(x)$.

(ii) Write the time evolution of this system

(iii) What will happen if we transformed $x \to -x$?

Problem 3.3. For a classical harmonic oscillator, the particle can not go beyond the points where the total energy equals the potential energy. Identify these points for a quantum-mechanical harmonic oscillator in its ground state. Write an integral giving the probability that the particle will go beyond these classically-allowed points. (You need not evaluate the integral.)

Problem 3.4. Evaluate the average (expectation) values of potential energy and kinetic energy for the ground state of the harmonic oscillator. Then show that

$$\langle T \rangle = \langle V \rangle. \tag{1}$$

This result is an instance of the virial theorem.

Problem 3.5. Let $\hat{N}$ and $\hat{X}$ be 2 operators such that

$$[\hat{N}, \hat{X}] = c\hat{X}$$

where c is a constant, and let $|\, n\,\rangle$ be an eigenstate of $\hat{N}$;

$$\hat{N}|\, n\,\rangle = n|\, n\,\rangle$$

Show that:

$$\hat{N}\hat{X}|\, n\,\rangle = (n + c)\hat{X}|\, n\,\rangle$$

Compare this ‘ general’ result with the ladder operators and the Hamiltonian of the quantum SHO

Useful formulae

† **The complementary error function**

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt.$$

with $\operatorname{erfc}(0) = 1, \operatorname{erfc}(1) = 0.157, \operatorname{erfc}(2) = 0.004\ldots$

† **Gamma function**

$$\Gamma(n) = (n-1)! \quad \text{if n is integer}$$
$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx \quad z \in \mathbb{C}$$

Particular values

$$\Gamma\left(\tfrac{n}{2}\right) = \sqrt{\pi}\frac{(n-2)!!}{2^{\frac{n-1}{2}}}$$

† **Trigonometric identities**

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$
$$cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

† **Integrals involving exponential functions**

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\, dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \quad (a > 0)$$

$$\int_0^{\infty} x^n e^{-ax^2}\, dx = \begin{cases} \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}} & (n > -1,\ a > 0) \\[2ex] \dfrac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\dfrac{\pi}{a}} & (n = 2k,\ k \text{ integer},\ a > 0) \\[2ex] \dfrac{k!}{2a^{k+1}} & (n = 2k+1,\ k \text{ integer},\ a > 0) \end{cases}$$