M 106 - INTEGRAL CALCULUS
Solution of the first mid-term exam
First semester 1441 H
Dr. Tariq A. Alfadhel

Q1. (2+3+3 Marks)

(a) Find the number c so that $\displaystyle\sum_{k=11}^{30} (k^2 - c) = 0$.

Solution : $\displaystyle\sum_{k=11}^{30} (k^2 - c) = 0 \iff \sum_{k=11}^{30} k^2 - \sum_{k=11}^{30} c = 0$

$$\iff \sum_{k=11}^{30} c = \sum_{k=11}^{30} k^2 \iff c \sum_{k=11}^{30} 1 = \sum_{k=1}^{30} k^2 - \sum_{k=1}^{10} k^2$$

$$\iff 20c = \left(\frac{(30)(31)(61)}{6}\right) - \left(\frac{(10)(11)(21)}{6}\right)$$

$$\iff 20c = \frac{30}{6}\ ((31)(61) - (11)(7))$$

$$\iff c = \frac{5}{20}\ (1891 - 77) = \frac{1814}{4} = \frac{907}{2}$$

(b) Approximate the integral $\displaystyle\int_0^{2\pi} (\cos x)^4\ dx$ using Simpson's rule with $n = 8$.

Solution : $[a, b] = [0, 2\pi]$, $n = 8$, and $f(x) = (\cos x)^4$.

$$\Delta x = \frac{b-a}{n} = \frac{2\pi - 0}{8} = \frac{\pi}{4}$$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	1	1	1
1	$\frac{\pi}{4}$	$\frac{1}{4}$	4	1
2	$\frac{\pi}{2}$	0	2	0
3	$\frac{3\pi}{4}$	$\frac{1}{4}$	4	1
4	π	1	2	2
5	$\frac{5\pi}{4}$	$\frac{1}{4}$	4	1
6	$\frac{3\pi}{2}$	0	2	0
7	$\frac{7\pi}{4}$	$\frac{1}{4}$	4	1
8	2π	1	1	1
				8

$$\int_0^{2\pi} (\cos x)^4\ dx \approx \frac{2\pi - 0}{3(8)}(8) \approx \frac{2\pi}{3} \approx 2.0944$$

(c) Use Riemann sums to evaluate $\displaystyle\int_0^1 x^3\ dx$.

Solution : $[a,b]=[0,1]$, $f(x)=x^3$.

$\Delta_x = \dfrac{b-a}{n} = \dfrac{1-0}{n} = \dfrac{1}{n}$

$x_k = a + k\ \Delta_x = 0 + k\left(\dfrac{1}{n}\right) = \dfrac{k}{n}$

Using the right-end point of the sub-intervals.

$$R_n = \sum_{k=1}^{n} f\left(x_k\right)\ \Delta_x = \sum_{k=1}^{n}\left[\left(\frac{k}{n}\right)^3\ \frac{1}{n}\right] = \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{1}{n^4}\ \frac{n^2(n+1)^2}{4} = \frac{1}{4}\ \frac{(n+1)^2}{n^2} = \frac{1}{4}\ \frac{n^2+2n+1}{n^2}$$

$$\int_0^1 x^3\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{1}{4}\ \frac{n^2+2n+1}{n^2}\right] = \frac{1}{4}\ (1) = \frac{1}{4}$$

Q2. (3+2+3 Marks)

(a) Evaluate the integral $\displaystyle\int \frac{5^{\tan x}}{(\cos x)^2}\ dx$

Solution : $\displaystyle\int \frac{5^{\tan x}}{(\cos x)^2}\ dx = \int 5^{\tan x}\ \sec^2 x\ dx = \frac{5^{\tan x}}{\ln 5} + c$

Using the formula $\displaystyle\int a^{f(x)}\ f'(x)\ dx = \frac{a^{f(x)}}{\ln a} + c$, where $a>0$

(b) If $y = 2^{(\sin x)^2} + x^{\pi}\pi^{x}$, find y'.

Solution : $y' = 2^{(\sin x)^2}\ 2\sin x\ \cos x\ \ln 2 + \left(\pi\ x^{\pi-1}\right)\ \pi^x + x^{\pi}\ \left(\pi^x\ \ln\pi\right)$

$y' = 2\ln 2\ 2^{(\sin x)^2}\ \sin x\ \cos x + \pi\ x^{\pi-1}\ \pi^x + x^{\pi}\ \pi^x\ \ln\pi$

(c) Compute $\displaystyle\int \frac{dx}{\sqrt{x}\ (2+x)}$

Solution : $\displaystyle\int \frac{dx}{\sqrt{x}\ (2+x)} = 2\int \frac{dx}{2\sqrt{x}\ \left[(\sqrt{2})^2+(\sqrt{x})^2\right]}$

$$= 2\int \frac{\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{2})^2+(\sqrt{x})^2}\ dx = 2\ \frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{2}}\right) + c$$

Using the formula $\displaystyle\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\ dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right)+c$, where $a>0$

Q3. (3+3+3 Marks)

(a) Find $\displaystyle\int \frac{(\ln x + 1)}{\sqrt{16(x\ln x)^2 - 9}}\, dx$

Solution : $\displaystyle\int \frac{(\ln x + 1)}{\sqrt{16(x\ln x)^2 - 9}}\, dx = \frac{1}{4}\int \frac{4(\ln x + 1)}{\sqrt{[4(x\ln x)]^2 - (3)^2}}\, dx$

$\displaystyle = \cosh^{-1}\left(\frac{4(x\ln x)}{3}\right) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\, dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

(b) Evaluate the integral $\displaystyle\int \frac{dx}{x\sqrt{x^5 - 4}}$

Solution : $\displaystyle\int \frac{dx}{x\sqrt{x^5 - 4}} = \int \frac{dx}{x\sqrt{\left(x^{\frac{5}{2}}\right)^2 - (2)^2}} = \int \frac{\left(x^{\frac{3}{2}}\right)}{x\,\left(x^{\frac{3}{2}}\right)\sqrt{\left(x^{\frac{5}{2}}\right)^2 - (2)^2}}\, dx$

$\displaystyle = \frac{2}{5}\int \frac{\left(\frac{5}{2}\, x^{\frac{3}{2}}\right)}{x^{\frac{5}{2}}\sqrt{\left(x^{\frac{5}{2}}\right)^2 - (2)^2}}\, dx = \frac{2}{5}\,\frac{1}{2}\ \sec^{-1}\left(\frac{x^{\frac{5}{2}}}{2}\right) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\,\sqrt{[f(x)]^2 - a^2}}\, dx = \frac{1}{a}\ \sec^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

(c) Compute $\displaystyle\int \frac{2e^{-3x}}{1 - e^{-6x}}\, dx$

Solution : $\displaystyle\int \frac{2e^{-3x}}{1 - e^{-6x}}\, dx = 2\int \frac{e^{-3x}}{(1)^2 - \left(e^{-3x}\right)^2}\, dx = \frac{2}{-3}\int \frac{-3\ e^{-3x}}{(1)^2 - \left(e^{-3x}\right)^2}\, dx$

$\displaystyle = -\frac{2}{3}\ \tanh^{-1}\left(e^{-3x}\right) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{a^2 - [f(x)]^2}\, dx = \frac{1}{a}\ \tanh^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

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Solution of the second mid-term exam
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Q1. (2+3+3 Marks)

(a) Find $\lim_{x\to 0}\left(1+8x^2\right)^{\frac{1}{x^2}}$

Solution : $\lim_{x\to 0}\left(1+8x^2\right)^{\frac{1}{x^2}} \qquad (1^\infty)$

Put $y=\left(1+8x^2\right)^{\frac{1}{x^2}} \iff \ln|y| = \ln\left|\left(1+8x^2\right)^{\frac{1}{x^2}}\right| = \dfrac{\ln\left|1+8x^2\right|}{x^2}$

$$\lim_{x\to 0}\ln|y| = \lim_{x\to 0}\frac{\ln\left|1+8x^2\right|}{x^2} \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule

$$\lim_{x\to 0}\ln|y| = \lim_{x\to 0}\frac{\ln\left|1+8x^2\right|}{x^2} = \lim_{x\to 0}\frac{\left(\frac{16x}{1+8x^2}\right)}{2x} = \lim_{x\to 0}\frac{8}{1+8x^2} = \frac{8}{1} = 8$$

Therefore, $\lim_{x\to 0}\left(1+8x^2\right)^{\frac{1}{x^2}} = e^8$

(b) Compute the integral $\int e^{4x}\sin x\, dx$

Solution : Using integration by parts twice.

$$u=\sin x \qquad dv = e^{4x}\,dx$$
$$du=\cos x\,dx \qquad v=\frac{1}{4}\,e^{4x}$$

$$\int e^{4x}\sin x\,dx = \frac{1}{4}e^{4x}\sin x - \int\frac{1}{4}e^{4x}\cos x\,dx = \frac{1}{4}e^{4x}\sin x - \frac{1}{4}\int e^{4x}\cos x\,dx$$

$$u=\cos x \qquad dv = e^{4x}\,dx$$
$$du=-\sin x\,dx \qquad v=\frac{1}{4}\,e^{4x}$$

$$\int e^{4x}\sin x\,dx = \frac{1}{4}e^{4x}\sin x - \frac{1}{4}\left[\frac{1}{4}e^{4x}\cos x - \int\frac{1}{4}e^{4x}(-\sin x)\,dx\right]$$

$$\int e^{4x}\sin x\,dx = \frac{1}{4}e^{4x}\sin x - \frac{1}{16}e^{4x}\cos x - \frac{1}{16}\int e^{4x}\sin x\,dx$$

$$\int e^{4x}\sin x\,dx + \frac{1}{16}\int e^{4x}\sin x\,dx = \frac{1}{4}e^{4x}\sin x - \frac{1}{16}e^{4x}\cos x + c$$

$$\frac{17}{16}\int e^{4x}\sin x\,dx = \frac{1}{4}e^{4x}\sin x - \frac{1}{16}e^{4x}\cos x + c$$

$$\int e^{4x}\sin x\;dx = \frac{16}{17}\left[\frac{1}{4}e^{4x}\sin x - \frac{1}{16}e^{4x}\cos x + c\right]$$

(c) Evaluate $\displaystyle\int (\sin x)^2(\cos x)^2\;dx$

Solution : $\displaystyle\int (\sin x)^2(\cos x)^2\;dx = \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\;dx$

$$= \frac{1}{4}\int\left(1-\cos^2 2x\right)\;dx = \frac{1}{4}\int\left[1-\left(\frac{1+\cos 4x}{2}\right)\right]\;dx$$

$$= \frac{1}{4}\int\left(1-\frac{1}{2}-\frac{\cos 4x}{2}\right)\;dx = \frac{1}{4}\int\left(\frac{1}{2}-\frac{\cos 4x}{2}\right)\;dx$$

$$= \frac{1}{4}\int\frac{1}{2}\;dx - \frac{1}{4}\int\frac{\cos 4x}{2}\;dx = \frac{1}{8}\int dx - \frac{1}{4}\,\frac{1}{2}\,\frac{1}{4}\int\cos 4x\;(4)\;dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

Q2. (3+3+2 Marks)

(a) Evaluate the integral $\displaystyle\int\frac{\sqrt{x^2-25}}{x}\;dx$

Solution : Using Trigonometric substitutions.

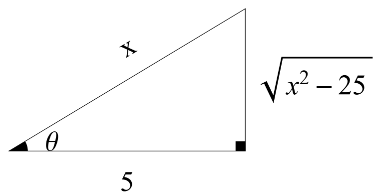
Put $x = 5\sec\theta \implies \sec\theta = \dfrac{x}{5}$

$dx = 5\sec\theta\;\tan\theta\;d\theta$

$$\int\frac{\sqrt{x^2-25}}{x}\;dx = \int\frac{\sqrt{25\sec^2\theta-25}\;5\sec\theta\;\tan\theta}{5\sec\theta}\;d\theta$$

$$= \int\sqrt{25\left(\sec^2\theta-1\right)}\;\tan\theta\;d\theta = \int 5\sqrt{\tan^2\theta}\;\tan\theta\;d\theta = 5\int\tan^2\theta\;d\theta$$

$$= 5\int\left(\sec^2\theta-1\right)\;d\theta = 5\left(\tan\theta-\theta\right)+c = 5\tan\theta - 5\theta + c$$



From the triangle : $\tan\theta = \dfrac{\sqrt{x^2-25}}{5}$

Note that $\sec\theta = \dfrac{x}{5} \implies \theta = \sec^{-1}\left(\dfrac{x}{5}\right)$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = 5\frac{\sqrt{x^2-25}}{5} - 5\sec^{-1}\left(\frac{x}{5}\right) + c$$

$$= \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + c$$

(b) Find $\displaystyle\int \frac{3x^2+7x+2}{(x+1)^2(x+3)}\,dx$

Solution : Using the method of partial fractions.

$$\frac{3x^2+7x+2}{(x+1)^2(x+3)} = \frac{A_1}{x+3} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}$$

$$3x^2+7x+2 = A_1(x+1)^2 + A_2(x+1)(x+3) + A_3(x+3)$$

$$3x^2+7x+2 = A_1(x^2+2x+1) + A_2(x^2+4x+3) + A_3(x+3)$$

$$3x^2+7x+2 = A_1x^2 + 2A_1x + A_1 + A_2x^2 + 4A_2x + 3A_2 + A_3x + 3A_3$$

$$3x^2+7x+2 = (A_1+A_2)\,x^2 + (2A_1+4A_2+A_3)\,x + (A_1+3A_2+3A_3)$$

By comparing the coefficients of the two polynomials in both sides :

$$\begin{array}{ll} A_1 + A_2 = 3 & \longrightarrow (1) \\ 2A_1 + 4A_2 + A_3 = 7 & \longrightarrow (2) \\ A_1 + 3A_2 + 3A_3 = 2 & \longrightarrow (3) \end{array}$$

Multiplying Eq(2) by 3 and subtracting Eq (3) : $5A_1 + 9A_2 = 19 \quad \longrightarrow (4)$

Multiplying Eq(1) by 5 and subtracting Eq (4) : $-4A_2 = -4 \implies A_2 = 1$

From Eq(1) : $A_1 = 2$

From Eq(2) : $2(2) + 4(1) + A_3 = 7 \implies A_3 = 7 - 8 = -1$

$$\int \frac{3x^2+7x+2}{(x+1)^2(x+3)}\,dx = \int \left(\frac{2}{x+3} + \frac{1}{x+1} + \frac{-1}{(x+1)^2}\right)\,dx$$

$$= 2\int \frac{1}{x+3}\,dx + \int \frac{1}{x+1}\,dx - \int (x+1)^{-2}\,dx$$

$$= 2\ln|x+3| + \ln|x+1| - \frac{(x+1)^{-1}}{-1} + c = 2\ln|x+3| + \ln|x+1| + \frac{1}{x+1} + c$$

(c) Compute $\displaystyle\int \frac{dx}{\sqrt{x(x+1)+1}}$

Solution : $\displaystyle\int \frac{dx}{\sqrt{x(x+1)+1}} = \int \frac{dx}{\sqrt{x^2+x+1}}$

$$= \int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx = \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$= \sinh\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c = \sinh\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Q3. (3+3+3 Marks)

(a) $\displaystyle\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$

Solution : Put $u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5\ du$

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\ dx = \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{1}{3}}}\ du = \int \frac{6u^5}{u^3 + u^2}\ du$$

$$= \int \frac{6u^5}{u^2(u+1)}\ du = \int \frac{6u^3}{u+1}\ du$$

Using long division of polynomials

$$\int \frac{6u^3}{u+1}\ du = \int \left(6u^2 - 6u + 6 - \frac{6}{u+1)}\right)\ du$$

$$= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + c$$

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\ dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$

(b) Does the integral $\displaystyle\int_0^\infty \frac{x\ dx}{1+x^4}$ converge? Find its value if it does.

Solution : $\displaystyle\int_0^\infty \frac{x\ dx}{1+x^4} = \lim_{t\to\infty} \int_0^t \frac{x\ dx}{1+x^4} = \lim_{t\to\infty}\left(\frac{1}{2}\int_0^t \frac{2x}{1+(x^2)^2}\ dx\right)$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\left[\tan^{-1}(x^2)\right]_0^t\right) = \lim_{t\to\infty}\left(\frac{1}{2}\left[\tan^{-1}(t^2) - \tan^{-1}(0)\right]\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

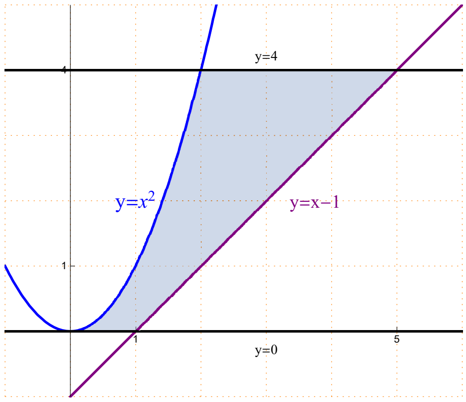
(c) Compute the area of the region bounded by the curves : $y = x^2$, $y = x - 1$, $y = 0$ and $y = 4$.

Solution : $y = x^2$ is a parabola with vertex $(0,0)$ and open upwards.

$y = x - 1$ is a straight line passing through $(0,-1)$ and its slope is 1.

$y = 0$ is the x-axis.

$y = 4$ is a straight line parallel to the x-axis and passing through $(0,4)$



$y = x - 1 \implies x = y + 1$

$y = x^2 \implies x = \pm\sqrt{y}$, but since the desired region is in the first quadrant then $x = \sqrt{y}$

$$Area = \int_0^4 \int [(y+1) - \sqrt{y}] \; dy = \int_0^4 \int \left[y - y^{\frac{1}{2}} + 1\right] \; dy$$

$$= \left[\frac{y^2}{2} - \frac{2}{3}\; y^{\frac{3}{2}} + y\right]_0^4 = \left(\frac{(4)^2}{2} - \frac{2}{3}\;(4)^{\frac{3}{2}} + 4\right) - (0 - 0 + 0)$$

$$= 8 - \frac{16}{3} + 4 = 12 - \frac{16}{3} = \frac{36-16}{3} = \frac{20}{3}$$

M 106 - INTEGRAL CALCULUS
Solution of the final exam
First semester 1441 H
Dr. Tariq A. Alfadhel

Q1. (2+2+3 Marks)

(a) Find the number c in the mean value theorem for $f(x) = -x^2 + 4x$ on $[0,3]$

Solution : Using the formula $(b-a)\ f(c) = \int_a^b f(x)\ dx$

$$(3-0)(-c^2+4c) = \int_0^3 (-x^2+4x)\ dx = \left[-\frac{x^3}{3} + 2x^2\right]_0^3$$

$$3(-c^2+4c) = \left(-\frac{3^3}{3} + 2(3^2)\right) - (0+0) = -9 + 18 = 9$$

$$\Longrightarrow\ -c^2+4c = 3 \Longrightarrow c^2 - 4c + 3 = 0$$

$$\Longrightarrow (c-1)(c-3) = 0 \Longrightarrow c = 1\ ,\ c = 3$$

Note that $1 \in (0,3)$ while $3 \notin (0,3)$.

The number that satisfies the mean value theorem is $c = 1$.

(b) Compute the integral $\displaystyle\int \frac{dx}{\sqrt{5^x - 16}}$

Solution : $\displaystyle\int \frac{dx}{\sqrt{5^x-16}} = \int \frac{1}{\sqrt{\left(5^{\frac{x}{2}}\right)^2 - (4)^2}}\ dx$

$$= 2\int \frac{\frac{1}{2}\ 5^{\frac{x}{2}}}{5^{\frac{x}{2}}\sqrt{\left(5^{\frac{x}{2}}\right)^2-(4)^2}}\ dx = 2\ \frac{1}{4}\ \sec^{-1}\left(\frac{5^{\frac{x}{2}}}{4}\right) + c = \frac{1}{2}\ \sec^{-1}\left(\frac{5^{\frac{x}{2}}}{4}\right) + c$$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}}\ dx = \frac{1}{a}\ \sec^{-1}\left(\frac{f(x)}{a}\right) + c$

where $a > 0$ and $|f(x)| > a$

(c) Evaluate $\displaystyle\int \frac{\cot x}{\sqrt{9 - (\sin x)^4}}\ dx$

Solution : $\displaystyle\int \frac{\cot x}{\sqrt{9-(\sin x)^4}}\ dx = \int \frac{\cos x}{\sin x\ \sqrt{(3)^2 - \left(\sin^2 x\right)^2}}\ dx$

$$= \frac{1}{2} \int \frac{2 \sin x \ \cos x}{\sin^2 x \ \sqrt{(3)^2 - \left(\sin^2 x\right)^2}} \ dx = \frac{1}{2} \left(-\frac{1}{3}\right) \ sech^{-1} \left(\frac{\sin^2 x}{3}\right) + c$$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2 - [f(x)]^2}} \ dx = -\frac{1}{a} \ sech^{-1} \left(\frac{f(x)}{a}\right) + c$

where $a > 0$ and $|f(x)| < a$

Q2. (3+3+3 Marks)

(a) Compute $\displaystyle\lim_{x \to 3^+} \left(\frac{1}{x-3} - \frac{1}{\ln(x-2)}\right)$

Solution : $\displaystyle\lim_{x \to 3^+} \left(\frac{1}{x-3} - \frac{1}{\ln(x-2)}\right) \qquad (\infty - \infty)$

$$= \lim_{x \to 3^+} \left(\frac{\ln(x-2) - (x-3)}{(x-3) \ \ln(x-2)}\right) \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule

$$\lim_{x \to 3^+} \left(\frac{\ln(x-2) - (x-3)}{(x-3) \ \ln(x-2)}\right) = \lim_{x \to 3^+} \left(\frac{\frac{1}{x-2} - 1}{\ln(x-2) + (x-3) \ \frac{1}{x-2}}\right)$$

$$= \lim_{x \to 3^+} \left(\frac{\frac{1-(x-2)}{x-2}}{\frac{(x-2) \ \ln(x-2) \ +(x-3)}{x-2}}\right)$$

$$= \lim_{x \to 3^+} \left(\frac{-x+1}{(x-2) \ \ln(x-2) + (x-3)}\right) = -\infty$$

Note that $-x+1 \to -2$ and $(x-2)\ln(x-2)+(x-3) \to 0^+$ when $x \to 3^+$

(b) Find $\displaystyle\int x^2 \ \tan^{-1} x \ dx$

Solution : Using integration by parts.

$$u = \tan^{-1} x \qquad dv = x^2 \ dx$$
$$du = \frac{1}{1+x^2} \ dx \qquad v = \frac{x^3}{3}$$

$$\int x^2 \ \tan^{-1} x \ dx = \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{3} \ \frac{1}{1+x^2} \ dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \ dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{(x^3+x)-x}{1+x^2} \ dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \left[\int \frac{x^3+x}{1+x^2} \ dx - \int \frac{x}{1+x^2} \ dx\right]$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x(x^2+1)}{1+x^2}\ dx + \frac{1}{3}\int \frac{x}{1+x^2}\ dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\ dx + \frac{1}{3}\ \frac{1}{2}\int \frac{2x}{1+x^2}\ dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\ \frac{x^2}{2} + \frac{1}{6}\ \ln\left|1+x^2\right| + c$$

(c) Evaluate the integral $\int (\tan x)^4(\sec x)^6\ dx$

Solution : Put $u = \tan x \implies du = \sec^2 x\ dx$

$$\int (\tan x)^4(\sec x)^6\ dx = \int \tan^4 x\ \sec^4 x\ \sec^2 x\ dx$$

$$= \int \tan^4 x\ \left(\sec^2 x\right)^2\ \sec^2 x\ dx = \int \tan^4 x\ \left(1+\tan^2 x\right)^2\ \sec^2 x\ dx$$

$$= \int u^4\left(1+u^2\right)^2\ du = \int u^4\left(1+2u^2+u^4\right)\ du = \int \left(u^4+2u^6+u^8\right)\ du$$

$$= \frac{u^5}{5} + 2\ \frac{u^7}{7} + \frac{u^9}{9} + c = \frac{\tan^5 x}{5} + 2\ \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + c$$

Q3. (3+3+3 Marks)

(a) Compute the integral $\int \frac{x^2}{(x^2+9)^{\frac{3}{2}}}\ dx$

Solution : $\int \frac{x^2}{(x^2+9)^{\frac{3}{2}}}\ dx = \int \frac{x^2}{(x^2+3^2)^{\frac{3}{2}}}\ dx$

Using Trigonometric substitutions.

Put $x = 3\tan\theta \implies \tan\theta = \frac{x}{3}$

$dx = 3\sec^2\theta\ d\theta$

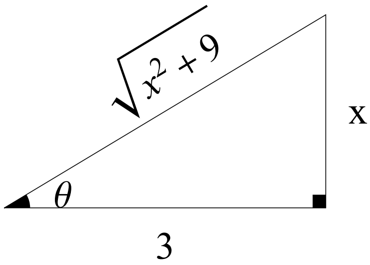
$$\int \frac{x^2}{(x^2+9)^{\frac{3}{2}}}\ dx = \int \frac{(3\tan\theta)^2\ 3\sec^2\theta}{((3\tan\theta)^2+9)^{\frac{3}{2}}}\ d\theta = \int \frac{(9\tan^2\theta)\ (3\sec^2\theta)}{\left(9\tan^2\theta+9\right)^{\frac{3}{2}}}\ d\theta$$

$$= 27\int \frac{\tan^2\theta\ \sec^2\theta}{\left[9\left(\tan^2\theta+1\right)\right]^{\frac{3}{2}}}\ d\theta = 27\int \frac{\tan^2\theta\ \sec^2\theta}{(9\sec^2\theta)^{\frac{3}{2}}}\ d\theta = 27\int \frac{\tan^2\theta\ \sec^2\theta}{(9)^{\frac{3}{2}}(\sec^2\theta)^{\frac{3}{2}}}\ d\theta$$

$$= 27\int \frac{\tan^2\theta\ \sec^2\theta}{27\ \sec^3\theta}\ d\theta = \int \frac{\tan^2\theta}{\sec\theta}\ d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\ d\theta$$

$$= \int \left(\frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta}\right)\ d\theta = \int (\sec\theta - \cos\theta)\ d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c$$



From the triangle : $\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$ and $\sec\theta = \dfrac{\sqrt{x^2+9}}{3}$

$$\int \frac{x^2}{(x^2+9)^{\frac{3}{2}}}\ dx = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| - \frac{x}{\sqrt{x^2+9}} + c$$

(b) Find the integral $\displaystyle\int \frac{3x-2}{(x^2+4)(x+2)}\ dx$

Solution : Using the method of partial fractions.

$$\frac{3x-2}{(x^2+4)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

$$3x-2 = A(x^2+4) + (Bx+C)(x+2)$$

$$3x-2 = Ax^2 + 4A + Bx^2 + 2Bx + Cx + 2C$$

$$3x-2 = (A+B)x^2 + (2B+C)x + (4A+2C)$$

By comparing the coefficients of the two polynomials in both sides :

$$\begin{array}{ll} A+B=0 & \longrightarrow\ (1) \\ 2B+C=3 & \longrightarrow\ (2) \\ 4A+2C=-2 & \longrightarrow\ (3) \end{array}$$

Dividing Eq(3) by 2 : $2A+C=-1 \quad \longrightarrow\ (4)$

Subtracting Eq(2) from Eq(4) : $2A-2B=-4 \implies A-B=-2 \quad \longrightarrow\ (5)$

Adding Eq(1) to Eq(5) : $2A=-2 \implies A=-1$

From Eq(1) : $-1+B=0 \implies B=1$

From Eq (2) : $2+C=3 \implies C=1$

$$\int \frac{3x-2}{(x^2+4)(x+2)}\ dx = \int\left(\frac{-1}{x+2} + \frac{x+1}{x^2+4}\right)\ dx$$

$$= \int \frac{-1}{x+2}\ dx + \int \frac{x}{x^2+4}\ dx + \int \frac{1}{x^2+4}\ dx$$

$$= -\int \frac{1}{x+2}\ dx + \frac{1}{2}\int \frac{2x}{x^2+4}\ dx + \int \frac{1}{x^2+2^2}\ dx$$

$$= -\ln|x+2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

(c) Evaluate the integral $\displaystyle\int \frac{dx}{3-\sin x+\cos x}$

Solution : Using half-angle substitution.

Put $u=\tan\left(\frac{x}{2}\right)$

$$\sin x=\frac{2u}{1+u^2}\ ,\quad \cos x=\frac{1-u^2}{1+u^2}\ ,\ dx=\frac{2}{1+u^2}\ du$$

$$\int \frac{dx}{3-\sin x+\cos x}=\int \frac{\left(\frac{2}{1+u^2}\right)}{3-\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\ du$$

$$=\int \frac{\left(\frac{2}{1+u^2}\right)}{\left(\frac{3(1+u^2)-2u+(1-u^2)}{1+u^2}\right)}\ du=\int \frac{2}{3+3u^2-2u+1-u^2}\ du$$

$$=\int \frac{2}{2u^2-2u+4}\ du=\int \frac{2}{2(u^2-u+2)}\ du=\int \frac{1}{u^2-u+2}\ du$$

$$=\int \frac{1}{\left(u^2-u+\frac{1}{4}\right)+\frac{7}{4}}\ du=\int \frac{1}{\left(u-\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\ du$$

$$=\frac{1}{\left(\frac{\sqrt{7}}{2}\right)}\ \tan^{-1}\left(\frac{\left(u-\frac{1}{2}\right)}{\left(\frac{\sqrt{7}}{2}\right)}\right)+c=\frac{2}{\sqrt{7}}\ \tan^{-1}\left(\frac{2u-1}{\sqrt{7}}\right)+c$$

$$=\frac{2}{\sqrt{7}}\ \tan^{-1}\left(\frac{2\tan\left(\frac{x}{2}\right)-1}{\sqrt{7}}\right)+c$$

Q4. (3+2+1 Marks)

(a) Sketch the region bounded by the curves : $y=4-x^2$, $y=x+2$, $x=-3$, $x=0$ and find its area.

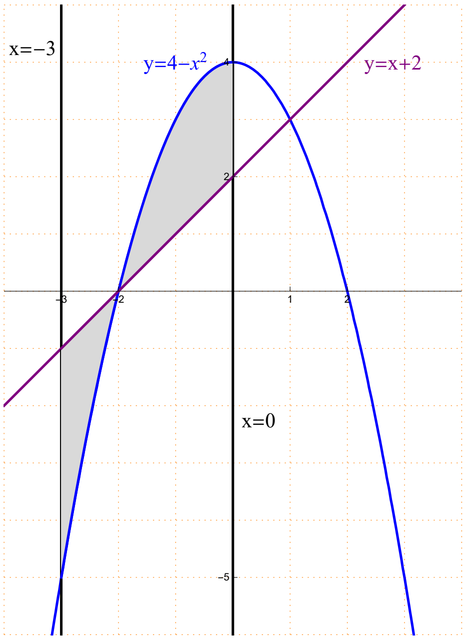
Solution :

$y=4-x^2$ is a parabola opens downwards with vertex $(0,4)$

$y=x+2$ is a straight line passing through $(0,2)$ with slope equals 1.

$x=-3$ is a straight line parallel to the y-axis and passing through $(-3,0)$

$x=0$ is the y-axis.



Points of intersection of $y = 4 - x^2$ and $y = x + 2$:

$x + 2 = 4 - x^2 \implies x^2 + x - 2 = 0 \implies (x+2)(x-1) = 0$

$\implies x = -2\ ,\ x = 1$

Note that $-2 \in [-3, 0]$ while $1 \notin [-3, 0]$

$$Area = \int_{-3}^{-2} \left[(x+2) - (4 - x^2)\right]\ dx + \int_{-2}^{0} \left[(4 - x^2) - (x+2)\right]\ dx$$

$$= \int_{-3}^{-2} \left(x^2 + x - 2\right)\ dx + \int_{-2}^{0} \left(-x^2 - x + 2\right)\ dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x\right]_{-3}^{-2} + \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{0}$$

$$= \left[\left(\frac{-8}{3} + \frac{4}{2} + 4\right) - \left(\frac{-27}{3} + \frac{9}{2} + 6\right)\right] + \left[(0 - 0 + 0) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)\right]$$

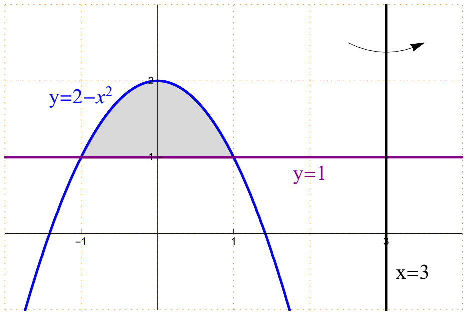
$$= \frac{-8}{3} + 6 + 9 - \frac{9}{2} - 6 - \frac{8}{3} + 6 = 15 - \frac{16}{3} - \frac{9}{2} = \frac{90 - 32 - 27}{6} = \frac{31}{6}$$

(b) Find the volume obtained by revolving the region bounded by the curves $y = -x^2 + 2$, $y = 1$ about the line of equation $x = 3$.

Solution :

$y = -x^2 + 2$ is a parabola opens downwards with vertex $(0, 2)$

$y = 1$ is a straight line parallel to the x-axis and passing through $(0, 1)$



Points of intersections of $y = 2 - x^2$ and $y = 1$:

$1 = 2 - x^2 \implies x^2 - 1 = 0 \implies (x-1)(x+1) = 0 \implies x = \pm 1$

Using Cylindrical shells method :

$$Volume = 2\pi \int_{-1}^{1} (3-x)\left[(2-x^2) - 1\right]\ dx = 2\pi \int_{-1}^{1} (3-x)(1-x^2)\ dx$$

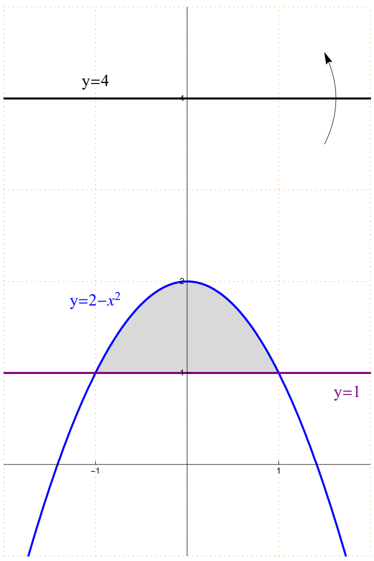
$$= 2\pi \int \left(3 - 3x^2 - x + x^3\right)\ dx = 2\pi \int \left(x^3 - 3x^2 - x + 3\right)\ dx$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 - \frac{x^2}{2} + 3x\right]_{-1}^{1} = 2\pi \left[\left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right) - \left(\frac{1}{4} + 1 - \frac{1}{2} - 3\right)\right]$$

$$= 2\pi \left[\left(2 - \frac{1}{4}\right) - \left(-2 - \frac{1}{4}\right)\right] = 2\pi \left(2 - \frac{1}{4} + 2 + \frac{1}{4}\right) = 8\pi$$

(c) Setup an integral for the volume obtained by revolving the region in part (b) about the line of equation $y = 4$.

Solution :



Using Washer Method :

$$\mathit{Volume} = \pi \int_{-1}^{1} \left[(4-1)^2 - \left(4-(2-x^2)\right)^2 \right] \; dx = \pi \int_{-1}^{1} \left[(3)^2 - \left(x^2+2\right)^2 \right] \; dx$$

$$= \pi \int_{-1}^{1} \left[9 - \left(x^4 + 2x^2 + 4\right) \right] \; dx = \pi \int_{-1}^{1} \left(-x^4 - 2x^2 + 5 \right) \; dx$$

Q5. (3+3+3 Marks)

(a) Find the length of the curve given by $r = \left(\cos \left(\frac{\theta}{2} \right) \right)^2$, $0 \le \theta \le \pi$.

Solution :

$$\frac{dr}{d\theta} = 2\cos\left(\frac{\theta}{2}\right) \; \left(-\sin\left(\frac{\theta}{2}\right) \; \frac{1}{2} \right) = -\cos\left(\frac{\theta}{2}\right) \; \sin\left(\frac{\theta}{2}\right)$$

$$L = \int_0^{\pi} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta$$

$$= \int_0^{\pi} \sqrt{\left(\cos^2\left(\frac{\theta}{2}\right)\right)^2 + \left(-\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2} \; d\theta$$

$$= \int_0^{\pi} \sqrt{\cos^4\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\theta}{2}\right)} \; d\theta$$

$$= \int_0^{\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right) \left[\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right]} \; d\theta = \int_0^{\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right)} \; d\theta$$

$$= \int_0^{\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| \; d\theta = \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) \; d\theta = 2\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) \; \left(\frac{1}{2}\right) \; d\theta$$

$$= 2\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 2\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] 2(1-0) = 2$$

(b) Sketch the region R that lies inside the curve $r = 1 - \sin\theta$ and outside the curve $r = 1$ and find its area.

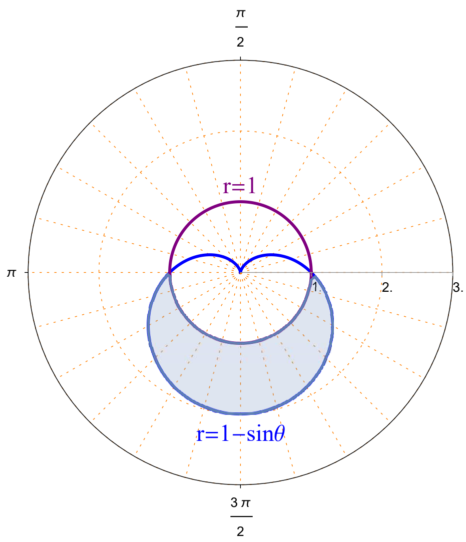
Solution :

$r = r = 1 - \sin\theta$ is a cardioid , symmetric with respect to the line $\theta = \frac{\pi}{2}$

$r = 1$ is the unit circle.

Points of intersection of $r = 1 - \sin\theta$ and $r = 1$:

$1 = 1 - \sin\theta \implies \sin\theta = 0 \implies \theta = \pi \; , \; \theta = 2\pi$



$$Area = \frac{1}{2}\int_{\pi}^{2\pi}\left[(1-\sin\theta)^2-(1)^2\right]\ d\theta = \frac{1}{2}\int_{\pi}^{2\pi}\left[1-2\sin\theta+\sin^2\theta-1\right]\ d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi}\left[\sin^2\theta-2\sin\theta\right]\ d\theta = \frac{1}{2}\int_{\pi}^{2\pi}\left[\left(\frac{1-\cos 2\theta}{2}\right)-2\sin\theta\right]\ d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi}\left[\frac{1}{2}-\frac{\cos 2\theta}{2}-2\sin\theta\right]\ d\theta = \frac{1}{2}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}+2\cos\theta\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[(\pi-0+2)-\left(\frac{\pi}{2}-0-2\right)\right] = \frac{1}{2}\left(\pi+2-\frac{\pi}{2}+2\right)$$

$$= \frac{1}{2}\left(4+\frac{\pi}{2}\right) = 2+\frac{\pi}{4}$$

(c) Find the area of the surface obtained by revolving the curve $r = 2\cos\theta$, $0 \le \theta \le \frac{\pi}{4}$ about the y-axis.

Solution : $\dfrac{dr}{d\theta} = -2\sin\theta$

$$S.A = 2\pi\int_{0}^{\frac{\pi}{4}} |2\cos\theta\ \ \cos\theta|\sqrt{(2\cos\theta)^2+(-2\sin\theta)^2}\ d\theta$$

$$= 2\pi\int_{0}^{\frac{\pi}{4}} \left|2\cos^2\theta\right|\sqrt{4\cos^2\theta+4\sin^2\theta}\ d\theta = 2\pi\int_{0}^{\frac{\pi}{4}} 2\cos^2\theta\sqrt{4\left(\cos^2\theta+\sin^2\theta\right)}\ d\theta$$

$$= 2\pi\int_{0}^{\frac{\pi}{4}} 2\cos^2\theta\ (2)\ d\theta = 4\pi\int_{0}^{\frac{\pi}{4}} 2\left(\frac{1+\cos 2\theta}{2}\right)\ d\theta = 4\pi\int_{0}^{\frac{\pi}{4}}(1+\cos 2\theta)\ d\theta$$

$$= 4\pi\left[\theta+\frac{\sin 2\theta}{2}\right]_{0}^{\frac{\pi}{4}} = 4\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right)-(0+0)\right] = \pi^2+2\pi$$

M 106 - INTEGRAL CALCULUS
Solution of Home Work No. 1
Second semester 1441 H
Dr. Tariq A. Alfadhel

Q.1 If $\displaystyle\int_0^x g(t)\ dt = \frac{4x}{x+1}$ and g is continuous on $[0,\infty)$, find $g(1)$.

Solution :

Using Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_0^x g(t)\ dt = \frac{d}{dx}\left(\frac{4x}{x+1}\right)$$

$$g(x) = \frac{4(x+1)-(1)4x}{(x+1)^2} = \frac{4x+4-4x}{(x+1)^2} = \frac{4}{(x+1)^2}$$

$$g(1) = \frac{4}{(1+1)^2} = \frac{4}{4} = 1$$

Q.2 Find the value of α so that $\displaystyle\sum_{k=5}^{15}\left(k^2-\alpha k\right) = 990$.

Solution :

$$\sum_{k=5}^{15}\left(k^2-\alpha k\right) = 990 \implies \sum_{k=1}^{15}\left(k^2-\alpha k\right) - \sum_{k=1}^{4}\left(k^2-\alpha k\right) = 990$$

$$\implies \sum_{k=1}^{15}k^2 - \alpha\sum_{k=1}^{15}k - \left(\sum_{k=1}^{4}k^2 - \alpha\sum_{k=1}^{4}k\right) = 990$$

$$\implies \left(\sum_{k=1}^{15}k^2 - \sum_{k=1}^{4}k^2\right) + \alpha\left(\sum_{k=1}^{4}k - \sum_{k=1}^{15}k\right) = 990$$

$$\implies \left(\frac{15\times16\times31}{6} + \frac{4\times5\times9}{6}\right) - \left(\frac{15\times16}{2} - \frac{4\times5}{2}\right)\alpha = 990$$

$$\implies (1240-30) + (10-120)\alpha = 990$$

$$\implies 1210 - 110\alpha = 990 \implies -110\alpha = -220 \implies \alpha = \frac{-220}{-110} = 2$$

Q.3 Find the value of c in the mean value theorem for $f(x) = \sqrt{x+1}$ on $[-1,8]$.

Solution :

Using the formula $(b-a)\ f(c) = \displaystyle\int_a^b f(x)\ dx$

where $f(x) = \sqrt{x+1}$ and $[a,b] = [-1,8]$

$$(8-(-1))\ \sqrt{c+1} = \int_{-1}^{8} \sqrt{x+1}\ dx = \int_{-1}^{8} (x+1)^{\frac{1}{2}}\ dx$$

$$9\ \sqrt{c+1} = \left[\frac{2}{3}\ (x+1)^{\frac{3}{2}}\right]_{-1}^{8} = \frac{2}{3}\ (9)^{\frac{3}{2}} - 0 = \frac{2}{3}(27) = 18$$

$$\implies 9\ \sqrt{c+1} = 18 \implies \sqrt{c+1} = 2 \implies c+1 = 4 \implies c = 3 \in (-1,8)$$

Q.4 If $y = e^x\ (\cos x)^{x^2}$, find y' .

Solution :

$$\ln|y| = \ln\left|e^x\ (\cos x)^{x^2}\right| = \ln|e^x| + \ln\left|(\cos x)^{x^2}\right|$$

$$\ln|y| = x\ln e + x^2\ln|\cos x| = x + x^2\ln|\cos x|$$

Differentiating both sides with respect to x :

$$\frac{y'}{y} = 1 + (2x)\ln|\cos x| + x^2\left(\frac{-\sin x}{\cos x}\right) = 1 + 2x\ln|\cos x| - \frac{x^2\sin x}{\cos x}$$

$$y' = y\ \left[1 + 2x\ln|\cos x| - x^2\ \tan x\right]$$

$$y' = e^x\ (\cos x)^{x^2}\ \left[1 + 2x\ln|\cos x| - x^2\ \tan x\right]$$

Q.5 Evaluate the following integrals :

(a) $\displaystyle\int \frac{x^2}{\sqrt[3]{7-6x^3}}\ dx$

Solution :

$$\int \frac{x^2}{\sqrt[3]{7-6x^3}}\ dx = \int x^2\ \left(7-6x^3\right)^{-\frac{1}{3}}\ dx = \frac{1}{-18}\int \left(7-6x^3\right)^{-\frac{1}{3}}(-18x^2)\ dx$$

$$= -\frac{1}{18}\ \frac{\left(7-6x^3\right)^{\frac{2}{3}}}{\frac{2}{3}} + c = -\frac{\left(7-6x^3\right)^{\frac{2}{3}}}{12} + c$$

(b) $\displaystyle\int 5^x 2^{5^x}\ dx$

Solution :

$$\int 5^x 2^{5^x}\ dx = \frac{1}{\ln 5}\int 2^{5^x}\ (5^x\ln 5)\ dx = \frac{1}{\ln 5}\ \frac{2^{5^x}}{\ln 2} + c$$

(c) $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+4x^3}}\ dx$

Solution :

$$\int \frac{\sqrt{x}}{\sqrt{1+4x^3}}\ dx = \int \frac{x^{\frac{1}{2}}}{\sqrt{(1)^2+\left(2x^{\frac{3}{2}}\right)^2}}\ dx = \frac{1}{3}\int \frac{3\ x^{\frac{1}{2}}}{\sqrt{(1)^2+\left(2x^{\frac{3}{2}}\right)^2}}\ dx$$

$$= \frac{1}{3}\ \sinh^{-1}\left(2x^{\frac{3}{2}}\right) + c$$

(d) $\displaystyle\int \frac{dx}{x\sqrt{x^6-16}}$

Solution :

$$\int \frac{dx}{x\sqrt{x^6-16}} = \int \frac{1}{x\ \sqrt{(x^3)^2-(4)^2}}\ dx = \int \frac{x^2}{x^2.x\ \sqrt{(x^3)^2-(4)^2}}\ dx$$

$$= \frac{1}{3}\int \frac{3x^2}{x^3\ \sqrt{(x^3)^2-(4)^2}}\ dx = \frac{1}{3}\ \frac{1}{4}\ \sec^{-1}\left(\frac{x^3}{4}\right)+c = \frac{1}{12}\ \sec^{-1}\left(\frac{x^3}{4}\right)+c$$

(e) $\displaystyle\int \frac{dx}{\sqrt{25+e^{4x}}}$

Solution :

$$\int \frac{dx}{\sqrt{25+e^{4x}}} = \int \frac{1}{\sqrt{(5)^2+(e^{2x})^2}}\ dx = \int \frac{e^{2x}}{e^{2x}\ \sqrt{(5)^2+(e^{2x})^2}}\ dx$$

$$= \frac{1}{2}\int \frac{2\ e^{2x}}{e^{2x}\ \sqrt{(5)^2+(e^{2x})^2}}\ dx = \frac{1}{2}\left(-\frac{1}{5}\ csch^{-1}\left(\frac{e^{2x}}{5}\right)\right) + c$$

$$= -\frac{1}{10}\ csch^{-1}\left(\frac{e^{2x}}{5}\right) + c$$

M 106 - INTEGRAL CALCULUS
Solution of Home Work No.2
Second semester 1441 H
Dr. Tariq A. Alfadhel

Q.1 Find $\lim_{x\to 0^+} (\sin x)^{2x}$

Solution : $\lim_{x\to 0^+} (\sin x)^{2x} \qquad (0^0)$

Put $y = (\sin x)^{2x} \iff \ln|y| = \ln\left|(\sin x)^{2x}\right| = 2x \ \ln|\sin x|$

$\lim_{x\to 0^+} \ln|y| = \lim_{x\to 0^+} 2x \ \ln|\sin x| \qquad (0.(-\infty))$

$$\lim_{x\to 0^+} \ln|y| = \lim_{x\to 0^+} 2x \ \ln|\sin x| = \lim_{x\to 0^+} \frac{2\ln|\sin x|}{\left(\frac{1}{x}\right)} \qquad \left(\frac{-\infty}{\infty}\right)$$

Using L'Hôpital's rule

$$\lim_{x\to 0^+} \ln|y| = \lim_{x\to 0^+} \frac{2\left(\frac{\cos x}{\sin x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \frac{-2x^2 \ \cos x}{\sin x}$$

$$= \lim_{x\to 0^+} \left(-2x \ \frac{x}{\sin x} \ \cos x\right) = -2(0).(1).(1) = 0$$

Therefore, $\lim_{x\to 0^+} (\sin x)^{2x} = e^0 = 1$

Q.2 Evaluate the following integrals :

(1) $\int x^2 \ \ln|x| \ dx$

Solution : Using integration by parts.

$u = \ln|x| \qquad dv = x^2 \ dx$
$du = \frac{1}{x} \ dx \qquad v = \frac{x^3}{3}$

$$\int x^2 \ \ln|x| \ dx = \frac{x^3}{3} \ \ln|x| - \int \frac{x^3}{3} \ \frac{1}{x} \ dx = \frac{x^3}{3} \ \ln|x| - \frac{1}{3}\int x^2 \ dx$$

$$= \frac{x^3}{3} \ \ln|x| - \frac{1}{3}\frac{x^3}{3} + c = \frac{x^3}{3} \ \ln|x| - \frac{x^3}{9} + c$$

(2) $\int \sin^4 x \ \cos^5 x \ dx$

Solution : Using the substitution $u = \sin x$

$du = \cos x \ dx$

$$\int \sin^4 x \ \cos^5 x \ dx = \int \sin^4 x \ \cos^4 x \ \cos x \ dx = \int \sin^4 x \ \left(\cos^2 x\right)^2 \ \cos x \ dx$$

$$= \int \sin^4 x \ \left(1 - \sin^2 x\right)^2 \ \cos x \ dx = \int u^4 \left(1 - u^2\right)^2 \ du = \int u^4 \left(1 - 2u^2 + u^4\right) \ du$$

$$= \int \left(u^4 - 2u^6 + u^8\right) \ du = \frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9} + c = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$$

(3) $\displaystyle\int \frac{x^2}{\sqrt{9 - x^2}} \ dx$

Solution : Using Trigonometric substitutions.

Put $x = 3\sin\theta \implies \sin\theta = \dfrac{x}{3}$

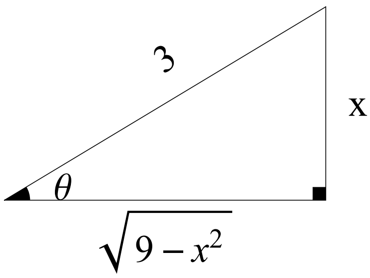
$dx = 3\cos\theta \ d\theta$

$$\int \frac{x^2}{\sqrt{9 - x^2}} \ dx = \int \frac{(3\sin\theta)^2 \ 3\cos\theta}{\sqrt{9 - 9\sin^2\theta}} \ d\theta = \int \frac{3^3 \sin^2\theta \ \cos\theta}{\sqrt{9(1 - \sin^2\theta)}} \ d\theta$$

$$= \int \frac{3^3 \sin^2\theta \ \cos\theta}{\sqrt{9\cos^2\theta}} \ d\theta = \int \frac{3^3 \sin^2\theta \ \cos\theta}{3\cos\theta} \ d\theta = 9\int \sin^2\theta \ d\theta$$

$$= 9\int \frac{1}{2}(1 - \cos 2\theta) \ d\theta = \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c = \frac{9}{2}\left[\theta - \frac{2\sin\theta \ \cos\theta}{2}\right] + c$$

$$= \frac{9}{2}(\theta - \sin\theta \ \cos\theta) + c$$



From the triangle : $\cos\theta = \dfrac{\sqrt{9 - x^2}}{3}$

Note that $\sin\theta = \dfrac{x}{3} \implies \theta = \sin^{-1}\left(\dfrac{x}{3}\right)$

$$\int \frac{x^2}{\sqrt{9 - x^2}} \ dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3} \ \frac{\sqrt{9 - x^2}}{3}\right] + c$$

(4) $\displaystyle\int \frac{x^2 + 3x + 8}{x^3 + 4x} \ dx$

Solution : Using the method of partial fractions.

$$\frac{x^2+3x+8}{x^3+4x}=\frac{x^2+3x+8}{x\left(x^2+4\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

$$x^2+3x+8=A(x^2+4)+(Bx+C)x=Ax^2+4A+Bx^2+Cx$$

$$x^2+3x+8=(A+B)x^2+Cx+4A$$

By comparing the coefficients of the two polynomials in both sides :

$$\begin{array}{ll} A+B=1 & \longrightarrow (1) \\ C=3 & \longrightarrow (2) \\ 4A=8 & \longrightarrow (3) \end{array}$$

From Equation (3) : $4A=8 \implies A=2$

From Equation (1) : $A+B=1 \implies 2+B=1 \implies B=-1$

$$\int \frac{x^2+3x+8}{x^3+4x}\ dx=\int\left(\frac{2}{x}+\frac{-x+3}{x^2+4}\right)\ dx$$

$$=\int \frac{2}{x}\ dx+\int\left(\frac{-x}{x^2+4}+\frac{3}{x^2+4}\right)\ dx$$

$$=2\int \frac{1}{x}-\frac{1}{2}\int \frac{2x}{x^2+4}\ dx+3\int \frac{1}{(x)^2+(2)^2}\ dx$$

$$=2\ln|x|-\frac{1}{2}\ln(x^2+4)+\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$$

(5) $\displaystyle\int \frac{dx}{\sqrt{-x^2+6x}}$

Solution : By completing the square.

$$\int \frac{dx}{\sqrt{-x^2+6x}}=\int \frac{dx}{\sqrt{-(x^2-6x+9)+9}}=\int \frac{dx}{\sqrt{9-(x-3)^2}}$$

$$=\int \frac{1}{\sqrt{(3)^2-(x-3)^2}}\ dx=\sin^{-1}\left(\frac{x-3}{3}\right)+c$$

(6) $\displaystyle\int \frac{dx}{4+5\cos x}$

Solution : Using Half-angle substitution.

Put $u=\tan\left(\frac{x}{2}\right)$, then $\cos x=\dfrac{1-u^2}{1+u^2}$ and $dx=\dfrac{2}{1+u^2}\ du$

$$\int \frac{dx}{4+5\cos x}=\int \frac{1}{4+5\left(\frac{1-u^2}{1+u^2}\right)}\ \frac{2}{1+u^2}\ du=\int \frac{1+u^2}{4(1+u^2)+5(1-u^2)}\ \frac{2}{1+u^2}\ du$$

$$=\int \frac{2}{4+4u^2+5-5u^2}\ du=\int \frac{2}{9-u^2}\ du=2\int \frac{1}{(3)^2-(u)^2}\ du$$

$$= 2\left[\frac{1}{3}\tanh^{-1}\left(\frac{u}{3}\right)\right] + c = \frac{2}{3}\tanh^{-1}\left(\frac{1}{3}\ \tan\left(\frac{x}{2}\right)\right) + c$$

Q.3 Does the integral $\displaystyle\int_0^{\infty} \frac{x}{(1+x^2)^3}\, dx$ converge? Find its value if it does.

Solution : $\displaystyle\int_0^{\infty} \frac{x}{(1+x^2)^3}\, dx = \lim_{t\to\infty} \int_0^{t} \frac{x}{(1+x^2)^3}\, dx$

$$= \lim_{t\to\infty} \left(\frac{1}{2}\int_0^t 2x\ (1+x^2)^{-3}\right) = \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{(1+x^2)^{-2}}{-2}\right]_0^t\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{4}\left[\frac{1}{(1+x^2)^2}\right]_0^t\right) = \lim_{t\to\infty}\left(-\frac{1}{4}\left[\frac{1}{(1+t^2)^2} - \frac{1}{(1+0)^2}\right]\right)$$

$$= -\frac{1}{4}\,(0-1) = \frac{1}{4}$$

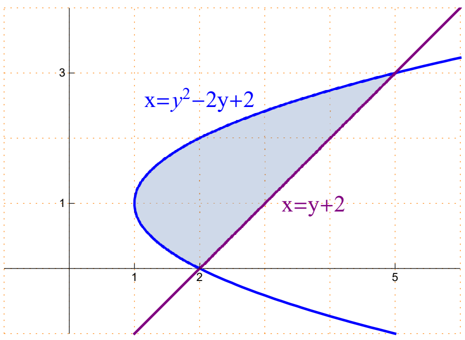
Hence, the improper integral $\displaystyle\int_0^{\infty} \frac{x}{(1+x^2)^3}\, dx$ converges to $\frac{1}{4}$.

Q.4 Sketch the region bounded by the graph of the curves $x = y^2 - 2y + 2$, $x = y + 2$ and calculate its area.

Solution :

$x = y^2 - 2y + 2 = (y^2 - 2y + 1) + 1 = (y-1)^2 + 1$ is a parabola opens to the right with vertex $(1, 1)$

$x = y + 2 \implies y = x - 2$ is a straight line passing through $(2, 0)$ with slope equals 1 .



Points of intersection of $x = y^2 - 2y + 2$ and $x = y + 2$:

$y^2 - 2y + 2 = y + 2 \implies y^2 - 3y = 0 \implies y(y-3) = 0 \implies y = 0\ ,\ y = 3$

$$Area = \int_0^3 \left[(y+2) - (y^2 - 2y + 2)\right]\ dy = \int_0^3 \left(3y - y^2\right)\ dy$$

$$= \left[3\,\frac{y^2}{2} - \frac{y^3}{3}\right]_0^3 = \left(3\left(\frac{9}{2}\right) - \frac{27}{3}\right) - (0-0) = \frac{27}{2} - 9 = \frac{9}{2}$$