M 106 - INTEGRAL CALCULUS
Solution of the first mid-term exam
First semester 1439-1440 H
Dr. Tariq A. Alfadhel

Q1. (3+3+2 Marks)

(a) If $F(x) = \ln|2x| \int_1^{4x^2} \left(1+t^2\right)^{10}\ dt$. Find $F'\left(\frac{1}{2}\right)$

Solution :

$$F'(x) = \left(\frac{2}{2x}\right) \int_1^{4x^2} \left(1+t^2\right)^{10}\ dt + \ln|2x| \left(\frac{d}{dx} \int_1^{4x^2} \left(1+t^2\right)^{10}\ dt\right)$$

$$= \frac{1}{x} \int_1^{4x^2} \left(1+t^2\right)^{10}\ dt + \ln|2x| \left[\left(1+(4x^2)^2\right)^{10}\ (8x)\right]$$

$$= \frac{1}{x} \int_1^{4x^2} \left(1+t^2\right)^{10}\ dt + 8x \ln|2x| \left(1+16x^4\right)^{10}$$

$$F'\left(\frac{1}{2}\right) = 2 \int_1^{1} \left(1+t^2\right)^{10}\ dt + (4) \ln|1|\, (2)^{10} = 0$$

Note that $\int_1^1 \left(1+t^2\right)^{10}\ dt = 0$ and $\ln(1) = 0$

(b) Use Riemann sums to find the value of $\int_0^2 3x^2\ dx$

Solution : $[a,b] = [0,2]$, $f(x) = 3x^2$.

$$\Delta_x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = a + k\ \Delta_x = 0 + k\left(\frac{2}{n}\right) = \frac{2k}{n}$$

Using the right-end point of the sub-intervals.

$$R_n = \sum_{k=1}^{n} f\left(x_k\right)\ \Delta_x = \sum_{k=1}^{n} \left[3\left(\frac{2k}{n}\right)^2\ \frac{2}{n}\right] = \frac{2}{n}\sum_{k=1}^{n} 3\left(\frac{4k^2}{n^2}\right) = \frac{24}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{24}{n^3}\ \frac{n(n+1)(2n+1)}{6} = 4\ \frac{(n+1)(2n+1)}{n^2}$$

$$\int_0^2 3x^2\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[4\ \frac{(n+1)(2n+1)}{n^2}\right] = (4)\ (2) = 8$$

(c) Use Trapezoid rule with $n = 4$ to approximate the integral $\int_0^{\pi} \sin^4 x \; dx$

Solution : $[a, b] = [0, \pi]$, $n = 4$, and $f(x) = \sin^4 x$.

$\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	$\frac{\pi}{4}$	$\frac{1}{4}$	2	$\frac{1}{2}$
2	$\frac{\pi}{2}$	1	2	2
3	$\frac{3\pi}{4}$	$\frac{1}{4}$	2	$\frac{1}{2}$
4	π	0	1	0
				3

$$\int_0^{\pi} \sin^4 x \; dx \approx \frac{\pi - 0}{(2)(4)}(3) \approx \frac{3\pi}{8} \approx 1.178$$

Q2. (2+3+3 Marks)

(a) If $f(x) = \log_2\left(\sin^{-1} x\right)$, $x > 0$. Find $f'(x)$.

Solution :

$$f'(x) = \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{\sin^{-1} x} \; \frac{1}{\ln 2} = \frac{1}{\ln(2) \; \sin^{-1} x \; \sqrt{1-x^2}}$$

(b) Compute the integral $\displaystyle\int \frac{4^{-\ln|x|}}{x} \; dx$

Solution : $\displaystyle\int \frac{4^{-\ln|x|}}{x} \; dx = \int 4^{-\ln|x|} \; \frac{1}{x} \; dx$

$$= -\int 4^{-\ln|x|} \; \left(-\frac{1}{x}\right) \; dx = -\frac{4^{-\ln|x|}}{\ln 4} + c$$

Using the formula $\displaystyle\int a^{f(x)} f'(x) \; dx = \frac{a^{f(x)}}{\ln a} + c$, where $a > 0$

(c) If $y = x^{2x^2}(x-1)^{\frac{3}{2}}$, $x > 1$. Find y'.

Solution : $y = x^{2x^2}(x-1)^{\frac{3}{2}} \implies \ln|y| = \ln\left|x^{2x^2}(x-1)^{\frac{3}{2}}\right|$

$$\implies \ln|y| = \ln\left|x^{2x^2}\right| + \ln\left|(x-1)^{\frac{3}{2}}\right| = 2x^2 \ln|x| + \frac{3}{2}\ln|x-1|$$

Differentiating both sides with respect to x :

$$\frac{y'}{y} = \left(4x \ln|x| + 2x^2 \ \frac{1}{x}\right) + \frac{3}{2} \ \frac{1}{x-1}$$

$$y' = y \ \left(4x \ln|x| + 2x + \frac{3}{2(x-1)}\right)$$

$$y' = x^{2x^2}(x-1)^{\frac{3}{2}} \ \left(4x \ln|x| + 2x + \frac{3}{2(x-1)}\right)$$

Q3. (3+3+3 Marks)

(a) Evaluate the integral $\int \frac{2x+3}{\sqrt{4-x^2}} \ dx$

Solution : $\int \frac{2x+3}{\sqrt{4-x^2}} \ dx = \int \left(\frac{2x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}}\right) \ dx$

$$= \int 2x(4-x^2)^{-\frac{1}{2}} \ dx + 3\int \frac{1}{\sqrt{4-x^2}} \ dx$$

$$= -\int (4-x^2)^{-\frac{1}{2}}(-2x) \ dx + 3\int \frac{1}{\sqrt{(2)^2-(x)^2}} \ dx$$

$$= -\ \frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + 3\sin^{-1}\left(\frac{x}{2}\right) + c = -2\sqrt{4-x^2} + 3\sin^{-1}\left(\frac{x}{2}\right) + c$$

Using the formula $\int [f(x)]^n \ f'(x) \ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

and the formula $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} \ dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| < a$.

(b) Find $\int \frac{e^{\frac{x}{2}}}{7+e^x} \ dx$

Solution : $\int \frac{e^{\frac{x}{2}}}{7+e^x} \ dx = \int \frac{e^{\frac{x}{2}}}{\left(\sqrt{7}\right)^2 + \left(e^{\frac{x}{2}}\right)^2} \ dx$

$$= 2\int \frac{e^{\frac{x}{2}} \ \left(\frac{1}{2}\right)}{\left(\sqrt{7}\right)^2 + \left(e^{\frac{x}{2}}\right)^2} \ dx = 2 \ \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{e^{\frac{x}{2}}}{\sqrt{7}}\right) + c$$

Using the formula $\int \frac{f'(x)}{a^2 + [f(x)]^2} \ dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

(c) Compute the integral $\int \frac{\sin x}{\sqrt{e^{\cos x} - 1}} \ dx$

Solution : $\displaystyle\int \frac{\sin x}{\sqrt{e^{\cos x} - 1}}\, dx = \int \frac{\sin x}{\sqrt{\left(e^{\frac{1}{2}\cos x}\right)^2 - (1)^2}}\, dx$

$$= \int \frac{e^{\frac{1}{2}\cos x}\ \sin x}{e^{\frac{1}{2}\cos x}\ \sqrt{\left(e^{\frac{1}{2}\cos x}\right)^2 - (1)^2}}\, dx = -2\int \frac{e^{\frac{1}{2}\cos x}\ \left(-\frac{1}{2}\ \sin x\right)}{e^{\frac{1}{2}\cos x}\ \sqrt{\left(e^{\frac{1}{2}\cos x}\right)^2 - (1)^2}}\, dx$$

$$= -2\sec^{-1}\left(e^{\frac{1}{2}\cos x}\right) + c$$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\ \sqrt{[f(x)]^2 - a^2}}\, dx = \frac{1}{a}\ \sec^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| > a$.

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Solution of the second mid-term exam
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Q1. (2+3+3 Marks)

(a) Evaluate the integral $\displaystyle\int \frac{dx}{e^{-x}\sqrt{e^{2x}-1}}$

Solution : $\displaystyle\int \frac{dx}{e^{-x}\sqrt{e^{2x}-1}} = \int \frac{1}{e^{-x}\sqrt{(e^x)^2-(1)^2}}\,dx$

$\displaystyle= \int \frac{e^x}{\sqrt{(e^x)^2-(1)^2}}\,dx = \cosh^{-1}(e^x) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}\,dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| > a$.

(b) Evaluate the integral $\displaystyle\int \frac{\tan x}{\sqrt{4-\cos^4 x}}\,dx$

Solution : $\displaystyle\int \frac{\tan x}{\sqrt{4-\cos^4 x}}\,dx = \int \frac{\tan x}{\sqrt{(2)^2-(\cos^2 x)^2}}\,dx$

$\displaystyle= \int \frac{\sin x}{\cos x\,\sqrt{(2)^2-(\cos^2 x)^2}}\,dx = \int \frac{\cos x\ \sin x}{\cos^2 x\,\sqrt{(2)^2-(\cos^2 x)^2}}\,dx$

$\displaystyle= -\frac{1}{2}\int \frac{2\cos x\ (-\sin x)}{\cos^2 x\,\sqrt{(2)^2-(\cos^2 x)^2}}\,dx = -\frac{1}{2}\left(-\frac{1}{2}\ sech^{-1}\left(\frac{\cos^2 x}{2}\right)\right) + c$

$\displaystyle= \frac{1}{4}\ sech^{-1}\left(\frac{\cos^2 x}{2}\right) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-[f(x)]^2}}\,dx = -\frac{1}{a}\ sech^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| < a$.

(c) Find $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Solution : $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) \qquad \left(\frac{\infty}{\infty}\right)$

$$\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to0^+}\frac{\sin x-x}{x\ \sin x}\qquad\left(\frac{0}{0}\right)$$

Using L'Hôpital's rule.

$$\lim_{x\to0^+}\frac{\sin x-x}{x\ \sin x}=\lim_{x\to0^+}\frac{\cos x-1}{\sin x\ +x\ \cos x}\qquad\left(\frac{0}{0}\right)$$

Using L'Hôpital's rule.

$$\lim_{x\to0^+}\frac{\cos x-1}{\sin x\ +x\ \cos x}=\lim_{x\to0^+}\frac{-\sin x}{\cos x\ +(\cos x-x\ \sin x)}$$

$$=\lim_{x\to0^+}\frac{-\sin x}{2\cos x-x\ \sin x}=\frac{0}{2-0}=0$$

Therefore, $\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=0$

Q2. (3+2+3 Marks)

(a) Evaluate the integral $\int e^{2x}\sin x\ dx$

Solution : Using integration by parts twice.

$$u=\sin x \qquad dv=e^{2x}\ dx$$
$$du=\cos x\ dx \qquad v=\frac{1}{2}\ e^{2x}$$

$$\int e^{2x}\sin x\ dx=\frac{1}{2}e^{2x}\sin x-\int\frac{1}{2}e^{2x}\cos x\ dx=\frac{1}{2}e^{2x}\sin x-\frac{1}{2}\int e^{2x}\cos x\ dx$$

$$u=\cos x \qquad dv=e^{2x}\ dx$$
$$du=-\sin x\ dx \qquad v=\frac{1}{2}\ e^{2x}$$

$$\int e^{2x}\sin x\ dx=\frac{1}{2}e^{2x}\sin x-\frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x-\int\frac{1}{2}e^{2x}(-\sin x)\ dx\right]$$

$$\int e^{2x}\sin x\ dx=\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x-\frac{1}{4}\int e^{2x}\sin x\ dx$$

$$\int e^{2x}\sin x\ dx+\frac{1}{4}\int e^{2x}\sin x\ dx=\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x+c$$

$$\frac{5}{4}\int e^{2x}\sin x\ dx=\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x+c$$

$$\int e^{2x}\sin x\ dx=\frac{4}{5}\left[\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x+c\right]$$

(b) Evaluate the integral $\int \sec^4 x\ \tan^7 x\ dx$

Solution : Put $u=\tan x \implies du=\sec^2 x\ dx$

$$\int \sec^4 x \ \tan^7 x \ dx = \int \sec^2 x \ \tan^7 x \ \sec^2 x \ dx$$

$$= \int \left(1 + \tan^2 x\right) \ \tan^7 x \ \sec^2 x \ dx = \int \left(1 + u^2\right) u^7 \ du$$

$$= \int \left(u^7 + u^9\right) \ du = \frac{u^8}{8} + \frac{u^{10}}{10} + c = \frac{\tan^8 x}{8} + \frac{\tan^{10} x}{10} + c$$

(c) Evaluate the integral $\int \frac{dx}{x^3\sqrt{x^2 - 4}}$

Solution : Using Trigonometric substitutions.

Put $x = 2\sec\theta \implies \sec\theta = \frac{x}{2}$

$dx = 2\sec\theta \ \tan\theta \ d\theta$

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \int \frac{2\sec\theta \ \tan\theta}{(2\sec\theta)^3\sqrt{(2\sec\theta)^2 - 4}} \ d\theta$$

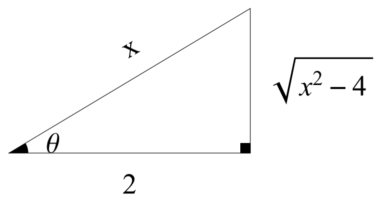
$$= \int \frac{2\sec\theta \ \tan\theta}{8\sec^3\theta\sqrt{4\sec^2\theta - 4}} \ d\theta = \frac{1}{4}\int \frac{\tan\theta}{\sec^2\theta\sqrt{4\left(\sec^2\theta - 1\right)}} \ d\theta$$

$$= \frac{1}{4}\int \frac{\tan\theta}{\sec^2\theta\sqrt{4\tan^2\theta}} \ d\theta = \frac{1}{4}\int \frac{\tan\theta}{\sec^2\theta(2\tan\theta)} \ d\theta = \frac{1}{8}\int \frac{1}{\sec^2\theta} \ d\theta$$

$$= \frac{1}{8}\int \cos^2\theta \ d\theta = \frac{1}{8}\int \left(\frac{1+\cos 2\theta}{2}\right) \ d\theta = \frac{1}{16}\int \left(1 + \cos 2\theta\right) \ d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] + c = \frac{1}{16}\left[\theta + \frac{2\sin\theta \ \cos\theta}{2}\right] + c = \frac{1}{16}\left[\theta + \sin\theta \ \cos\theta\right] + c$$

Note that $x = 2\sec\theta \implies \sec\theta = \frac{x}{2} \implies \theta = \sec^{-1}\left(\frac{x}{2}\right)$, also $\cos\theta = \frac{2}{x}$



From the triangle : $\sin\theta = \frac{\sqrt{x^2-4}}{x}$

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \frac{1}{16}\left[\sec^{-1}\left(\frac{x}{2}\right) + \frac{\sqrt{x^2-4}}{x} \ \frac{2}{x}\right] + c$$

$$= \frac{1}{16}\left[\sec^{-1}\left(\frac{x}{2}\right) + \frac{2\sqrt{x^2-4}}{x^2}\right] + c$$

Q3. (3+3+3 Marks)

(a) Evaluate the integral $\displaystyle\int \frac{6x^2+x+8}{x^3+4x}\,dx$

Solution : Using the method of partial fractions.

$$\frac{6x^2+x+8}{x^3+4x} = \frac{6x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$6x^2+x+8 = A(x^2+4)+(Bx+C)x = Ax^2+4A+Bx^2+Cx$$

$$6x^2+x+8 = (A+B)x^2+Cx+4A$$

By comparing the coefficients of the two polynomials in both sides :

$$\begin{array}{ll} A+B=6 & \longrightarrow (1) \\ C=1 & \longrightarrow (2) \\ 4A=8 & \longrightarrow (3) \end{array}$$

From Eq(3) : $A=2$

From Eq(1) : $B=4$

$$\int \frac{6x^2+x+8}{x^3+4x}\,dx = \int \left(\frac{2}{x} + \frac{4x+1}{x^2+4}\right)\,dx = \int \frac{2}{x}\,dx + \int \frac{4x+1}{x^2+4}\,dx$$

$$= \int \frac{2}{x}\,dx + \int \frac{4x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= 2\int \frac{1}{x}\,dx + 2\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{(x)^2+(2)^2}\,dx$$

$$= 2\ln|x| + 2\ln\left(x^2+4\right) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

(b) Evaluate the integral $\displaystyle\int \frac{dx}{(x+1)^{\frac{5}{6}}-(x+1)^{\frac{1}{2}}}$

Solution : Put $x+1=u^6 \implies dx = 6u^5\,du$

$$\int \frac{dx}{(x+1)^{\frac{5}{6}}-(x+1)^{\frac{1}{2}}} = \int \frac{6u^5}{(u^6)^{\frac{5}{6}}-(u^6)^{\frac{1}{2}}}\,du = \int \frac{6u^5}{u^5-u^3}\,du$$

$$= \int \frac{6u^5}{u^3(u^2-1)}\,du = 6\int \frac{u^2}{u^2-1}\,du = 6\int \frac{(u^2-1)+1}{u^2-1}\,du$$

$$= 6\int \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right)\,du = 6\int \left(1 - \frac{1}{1-u^2}\right)\,du$$

$$= 6\left[u - \tanh^{-1}u\right] + c = 6\left[(x+1)^{\frac{1}{6}} - \tanh^{-1}\left((x+1)^{\frac{1}{6}}\right)\right] + c$$

(c) Evaluate the integral $\int \frac{dx}{2+\cos x}$

Solution : Using half-angle substitution.

Put $u = \tan\left(\frac{x}{2}\right)$

$$\cos x = \frac{1-u^2}{1+u^2} \ , \ dx = \frac{2}{1+u^2}\ du$$

$$\int \frac{dx}{2+\cos x} = \int \frac{\left(\frac{2}{1+u^2}\right)}{\left(2+\frac{1-u^2}{1+u^2}\right)}\ du = \int \frac{\left(\frac{2}{1+u^2}\right)}{\left(\frac{2(1+u^2)+1-u^2}{1+u^2}\right)}\ du$$

$$= \int \frac{2}{2+2u^2+1-u^2}\ du = \int \frac{2}{u^2+3}\ du = 2\int \frac{1}{(u)^2+\left(\sqrt{3}\right)^2}\ du$$

$$= 2\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\ \tan\left(\frac{x}{2}\right)\right) + c$$

M 106 - INTEGRAL CALCULUS
Solution of the final exam
First semester 1439-1440 H
Dr. Tariq A. Alfadhel

Q1. (2+2 Marks)

(a) Approximate the integral $\int_0^4 \sqrt{x^3+8}\ dx$ using Simpson's Rule with $n=4$

Solution : $[a,b]=[0,4]$, $n=4$, and $f(x)=\sqrt{x^3+8}$.

$\Delta x = \dfrac{b-a}{n} = \dfrac{4-0}{4} = 1$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	$2\sqrt{2}$	1	2.82843
1	1	3	4	12
2	2	4	2	8
3	3	$\sqrt{35}$	4	23.6643
4	4	$6\sqrt{2}$	1	8.48528
				54.978

$$\int_0^4 \sqrt{x^3+8}\ dx \approx \frac{4-0}{3(4)}(54.978) \approx \frac{54.978}{3} \approx 18.326$$

(b) If $F(x)=(2+\sin x)^{e^x}$, find $F'(x)$

Solution : $\ln|F(x)| = \ln\left|(2+\sin x)^{e^x}\right| = e^x \ln|2+\sin x|$

Differentiate both sides with respect to x

$$\frac{F'(x)}{F(x)} = e^x \ln|2+\sin x| + e^x\left(\frac{\cos x}{2+\sin x}\right)$$

$$F'(x) = F(x)\left(e^x \ln|2+\sin x| + \frac{e^x\cos x}{2+\sin x}\right)$$

$$F'(x) = (2+\sin x)^{e^x}\left(e^x \ln|2+\sin x| + \frac{e^x\cos x}{2+\sin x}\right)$$

Q2. (3+3+3 Marks)

(a) Evaluate the integral $\int \left(3^x+3^{-x}+2\right)\ dx$

Solution : $\int \left(3^x+3^{-x}+2\right)\ dx = \int 3^x\ dx + \int 3^{-x}\ dx + \int 2\ dx$

$$= \int 3^x \; dx - \int 3^{-x} \; (-1) \; dx + \int 2 \; dx = \frac{3^x}{\ln 3} - \frac{3^{-x}}{\ln 3} + 2x + c$$

Using the formula $\int a^{f(x)} \; f'(x) \; dx = \frac{a^{f(x)}}{\ln a} + c$, where $a > 0$

(b) Evaluate the integral $\int \frac{dx}{\sqrt{2^{2x} - 1}}$

Solution : $\int \frac{dx}{\sqrt{2^{2x} - 1}} = \int \frac{1}{\sqrt{(2^x)^2 - (1)^2}} \; dx = \int \frac{2^x}{2^x \sqrt{(2^x)^2 - (1)^2}} \; dx$

$$= \frac{1}{\ln 2} \int \frac{2^x \; \ln 2}{2^x \sqrt{(2^x)^2 - (1)^2}} \; dx = \frac{1}{\ln 2} \sec^{-1} (2^x) + c$$

Using the formula $\int \frac{f'(x)}{f(x) \; \sqrt{[f(x)]^2 - a^2}} \; dx = \frac{1}{a} \; \sec^{-1} \left(\frac{f(x)}{a} \right) + c$,

where $a > 0$ and $|f(x)| > a$

(c) Evaluate the integral $\int \frac{dx}{\sqrt{x} \; \sqrt{1 + x}}$

Solution : $\int \frac{dx}{\sqrt{x} \; \sqrt{1 + x}} = \int \frac{1}{\sqrt{x} \; \sqrt{(1)^2 + (\sqrt{x})^2}} \; dx$

$$= 2 \int \frac{\left(\frac{1}{2\sqrt{x}} \right)}{\sqrt{(1)^2 + (\sqrt{x})^2}} \; dx = 2 \sinh^{-1} \left(\sqrt{x} \right) + c$$

Using the formula $\int \frac{f'(x)}{\sqrt{a^2 + [f(x)]^2}} \; dx = \sinh^{-1} \left(\frac{f(x)}{a} \right) + c$, where $a > 0$

Q3. (3+3+3 Marks)

(a) Evaluate the integral $\int \frac{dx}{x\sqrt{4 - x^6}}$

Solution : $\int \frac{dx}{x\sqrt{4 - x^6}} = \int \frac{1}{x\sqrt{(2)^2 - (x^3)^2}} \; dx$

$$= \int \frac{x^2}{x^3 \sqrt{(2)^2 - (x^3)^2}} \; dx = \frac{1}{3} \int \frac{3x^2}{x^3 \sqrt{(2)^2 - (x^3)^2}} \; dx$$

$$= \frac{1}{3} \left(-\frac{1}{2} \; sech^{-1} \left(\frac{x^3}{2} \right) \right) + c = -\frac{1}{6} \; sech^{-1} \left(\frac{x^3}{2} \right) + c$$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\,\sqrt{a^2-[f(x)]^2}}\,dx = -\frac{1}{a}\;sech^{-1}\left(\frac{f(x)}{a}\right)+c$,

where $a>0$ and $|f(x)|<a$

(b) Evaluate the integral $\displaystyle\int \frac{dx}{(x^2-1)^{\frac{3}{2}}}$

Solution : Using trigonometric substitutions.

Put $x=\sec\theta$

$dx=\sec\theta\;\tan\theta\;d\theta$

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} = \int \frac{\sec\theta\;\tan\theta}{(\sec^2\theta-1)^{\frac{3}{2}}}\,d\theta = \int \frac{\sec\theta\;\tan\theta}{\left(\tan^2\theta\right)^{\frac{3}{2}}}\,d\theta = \int \frac{\sec\theta\;\tan\theta}{\tan^3\theta}\,d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \sec\theta\;\cot^2\theta\;d\theta = \int \frac{1}{\cos\theta}\;\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int (\sin\theta)^{-2}\cos\theta\;d\theta = \frac{(\sin\theta)^{-1}}{-1}+c = -\frac{1}{\sin\theta}+c$$

Note that $x=\sec\theta \implies \cos\theta=\dfrac{1}{x} \implies \sin\theta=\sqrt{1-\left(\dfrac{1}{x}\right)^2}=\sqrt{1-\dfrac{1}{x^2}}$

$$\implies \sin\theta=\sqrt{\frac{x^2-1}{x^2}}=\frac{\sqrt{x^2-1}}{x} \implies \frac{1}{\sin\theta}=\frac{x}{\sqrt{x^2-1}}$$

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2-1}}+c$$

(c) Evaluate the integral $\displaystyle\int \frac{4x^2}{(x-1)^2(x+1)}\,dx$

Solution : Using the method of partial fractions.

$$\frac{4x^2}{(x-1)^2(x+1)} = \frac{A_1}{x+1}+\frac{A_2}{x-1}+\frac{A_3}{(x-1)^2}$$

$4x^2 = A_1(x-1)^2 + A_2(x-1)(x+1) + A_3(x+1)$

$4x^2 = A_1\left(x^2-2x+1\right) + A_2\left(x^2-1\right) + A_3(x+1)$

$4x^2 = A_1x^2 - 2A_1x + A_1 + A_2x^2 - A_2 + A_3x + A_3$

$4x^2 = (A_1+A_2)\,x^2 + (-2A_1+A_3)\,x + (A_1-A_2+A_3)$

By comparing the coefficients of the two polynomials in both sides :

$A_1+A_2=4 \qquad \longrightarrow (1)$
$-2A_1+A_3=0 \qquad \longrightarrow (2)$
$A_1-A_2+A_3=0=0 \qquad \longrightarrow (3)$

Adding the three equations together: $2A_3 = 4 \implies A_3 = 2$

From Eq(2) : $-2A_1 + 2 = 0 \implies A_1 = 1$

From Eq(1) : $1 + A_2 = 4 \implies A_2 = 3$

$$\int \frac{4x^2}{(x-1)^2(x+1)}\, dx = \int \left(\frac{1}{x+1} + \frac{3}{x-1} + \frac{2}{(x-1)^2} \right) dx$$

$$= \int \frac{1}{x+1}\, dx + \int \frac{3}{x-1}\, dx + \int \frac{2}{(x-1)^2}\, dx$$

$$= \int \frac{1}{x+1}\, dx + 3\int \frac{1}{x-1}\, dx + 2\int (x-1)^{-2}\, dx$$

$$= \ln|x+1| + 3\ln|x-1| + 2\,\frac{(x-1)^{-1}}{-1} + c = \ln|x+1| + 3\ln|x-1| - \frac{2}{x-1} + c$$

Q4. (3+3+2 Marks)

(a) Does the integral $\displaystyle\int_0^{\infty} (1+2x)e^{-x}\, dx$ converge? Find its value if it does.

Solution : Solving the integral $\displaystyle\int (1+2x)e^{-x}\, dx$ by parts :

$u = 1 + 2x \quad dv = e^{-x}\, dx$
$du = 2\, dx \quad v = -e^{-x}$

$$\int (1+2x)e^{-x}\, dx = (1+2x)\left(-e^{-x}\right) - \int 2\,\left(-e^{-x}\right)\, dx$$

$$= -(1+2x)e^{-x} - 2\int \left(-e^{-x}\right)\, dx = -(1+2x)e^{-x} - 2e^{-x} + c$$

$$= -e^{-x}\left[(1+2x) + 2\right] + c = -\frac{2x+3}{e^x} + c$$

$$\int_0^{\infty} (1+2x)e^{-x}\, dx = \lim_{t\to\infty} \int_0^{t} (1+2x)e^{-x}\, dx = \lim_{t\to\infty} \left(\left[-\frac{2x+3}{e^x} \right]_0^t \right)$$

$$= \lim_{t\to\infty} \left(-\frac{2t+3}{e^t} - \left(-\frac{2(0)+3}{e^0} \right) \right) = 0 - (-3) = 3$$

The integral $\displaystyle\int_0^{\infty} (1+2x)e^{-x}\, dx$ converges, and its value is 3.

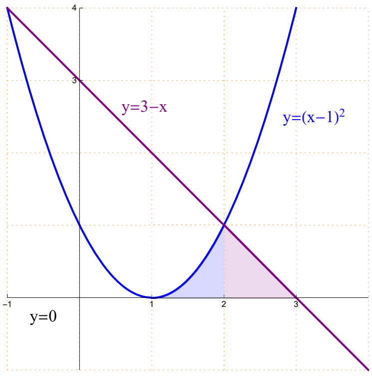
Note that $\displaystyle\lim_{t\to\infty} \left(-\frac{2t+3}{e^t} \right) = 0$ by L'Hôpital's rule.

(b) Sketch the region bounded by the curves $y = (x-1)^2$, $y = 3 - x$, $y = 0$ and find its area.

Solution : $y = 0$ is the x-axis.

$y = (x-1)^2$ is a parabola with vertex $(1,0)$ and opens upwards.

$y = 3 - x$ is a straight line passing through $(0,3)$ and its slope is -1.



Points of intersection of $y = (x-1)^2$ and $y = 3 - x$:

$(x-1)^2 = 3 - x \implies x^2 - 2x + 1 = 3 - x \implies x^2 - x - 2 = 0$

$\implies (x-2)(x+1) = 0 \implies x = -1 \ , \ x = 2$

Point of intersection of $y = 0$ and $y = 3 - x$: $3 - x = 0 \implies x = 3$

Point of intersection of $y = 0$ and $y = (x-1)^2$: $(x-1)^2 = 0 \implies x = 1$

$$Area = \int_1^2 (x-1)^2 \ dx + \int_2^3 (3-x) \ dx = \left[\frac{(x-1)^3}{3}\right]_1^2 + \left[3x - \frac{x^2}{2}\right]_2^3$$

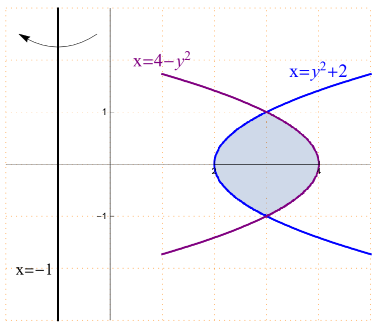
$$= \left[\frac{1}{3} - 0\right] + \left[\left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)\right] = \frac{1}{3} + 3 - \frac{5}{2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

(c) Set up an integral for the volume obtained by revolving the region bounded by the curves $x = y^2 + 2$, $x = 4 - y^2$ about the line of equation $x = -1$

Solution :

$x = y^2 + 2$ is a parabola with vertex $(2,0)$ and opens to the right.

$x = 4 - y^2$ is a parabola with vertex $(4,0)$ and opens to the left.



Points of intersection of $x = y^2 + 2$ and $x = 4 - y^2$:

$$y^2+2=4-y^2 \implies 2y^2-2=0 \implies y^2-1=0$$

$$\implies (y-1)(y+1)=0 \implies y=-1\ ,\ y=1$$

Using the washer method :

$$V=\pi\int_{-1}^{1}\left[\left((4-y^2)+1\right)^2-\left((y^2+2)+1\right)^2\right]\ dy$$

Q5. (3+3+3 Marks)

(a) Find the length of the curve given by the equations $x=\dfrac{t^3}{3}$, $y=\dfrac{2}{9}t^{\frac{9}{2}}$, $0\le t\le 1$

Solution : $x=\dfrac{t^3}{3} \implies \dfrac{dx}{dt}=t^2$

$$y=\frac{2}{9}t^{\frac{9}{2}} \implies \frac{dy}{dt}=t^{\frac{7}{2}}$$

$$L=\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt=\int_0^1\sqrt{(t^2)^2+\left(t^{\frac{7}{2}}\right)^2}\ dt$$

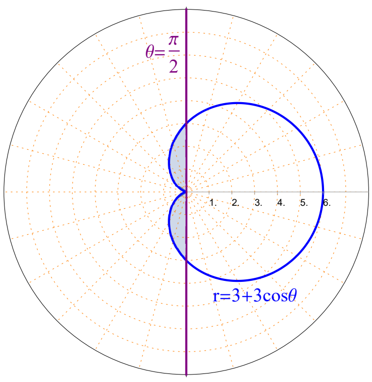
$$=\int_0^1\sqrt{t^4+t^7}\ dt=\int_0^1\sqrt{t^4\left(1+t^3\right)}\ dt=\int_0^1\left|t^2\right|\sqrt{1+t^3}\ dt$$

$$=\frac{1}{3}\int_0^1\left(1+t^3\right)^{\frac{1}{2}}\ \left(3t^2\right)\ dt=\frac{1}{3}\left[\frac{\left(1+t^3\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1=\frac{1}{3}\ \frac{2}{3}\ \left[\left(1+t^3\right)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{9}\ \left[\left(1+t^3\right)^{\frac{3}{2}}\right]_0^1=\frac{2}{9}\left[(2)^{\frac{3}{2}}-1\right]=\frac{2}{9}\left(2\sqrt{2}-1\right)$$

(b) Sketch the region that lies inside the curve $r=3+3\cos\theta$ and to the left of the line $\theta=\dfrac{\pi}{2}$ and find its area.

Solution :



$r = 3 + 3\cos\theta$ is a cardioid symmetric with respect to the polar axis.

$\theta = \dfrac{\pi}{2}$ is a straight line passing through the pole and perpendicular to the polar axis.

Note that the desired region is symmetric with respect to the polar axis.

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (3+3\cos\theta)^2 \, d\theta\right) = \int_{\frac{\pi}{2}}^{\pi} [3(1+\cos\theta)]^2 \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left[9(1+2\cos\theta+\cos^2\theta)\right] \, d\theta = \int_{\frac{\pi}{2}}^{\pi} 9\left[1+2\cos\theta+\left(\frac{1+\cos 2\theta}{2}\right)\right] \, d\theta$$

$$= 9\int_{\frac{\pi}{2}}^{\pi} \left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right) \, d\theta = 9\left[\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= 9\left[\left(\frac{3}{2}\,\pi+0+0\right)-\left(\frac{3}{2}\,\frac{\pi}{2}+2+0\right)\right] = 9\left(\frac{3\pi}{4}-2\right)$$

(c) Find the area of the surface obtained by revolving the curve $r = 8\cos\theta$, $0 \le \theta \le \dfrac{\pi}{2}$ about the y-axis.

Solution : $\dfrac{dr}{d\theta} = -8\sin\theta$

$$S.A = 2\pi \int_0^{\frac{\pi}{2}} |8\cos\theta \;\cos\theta| \sqrt{(8\cos\theta)^2 + (-8\sin\theta)^2} \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \left|8\cos^2\theta\right| \sqrt{64\cos^2\theta + 64\sin^2\theta} \, d\theta = 2\pi \int_0^{\frac{\pi}{2}} 8\cos^2\theta \sqrt{64\left(\cos^2\theta+\sin^2\theta\right)} \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 8\cos^2\theta \;(8) \, d\theta = 128\pi \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = 64\pi \int_0^{\frac{\pi}{2}} (1+\cos 2\theta) \, d\theta$$

$$= 64\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 64\pi \left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = 32\pi^2$$

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Solution of the first mid-term exam
Second semester 1439-1440 H
Dr. Tariq A. Alfadhel

Q1. (2+3+3 Marks)

(a) Find the value of $F'(0)$ if $F(x) = (\cos x) \displaystyle\int_0^{\tan x} \sqrt{1+t^2}\ dt$

Solution : $F'(x) = \dfrac{d}{dx}\left[(\cos x)\displaystyle\int_0^{\tan x} \sqrt{1+t^2}\ dt\right]$

$= (-\sin x)\displaystyle\int_0^{\tan x} \sqrt{1+t^2}\ dt + (\cos x)\ \dfrac{d}{dx}\displaystyle\int_0^{\tan x} \sqrt{1+t^2}\ dt$

$= (-\sin x)\displaystyle\int_0^{\tan x} \sqrt{1+t^2}\ dt + (\cos x)\left(\sqrt{1+\tan^2 x}\ \sec^2 x\right)$

$F'(0) = (-\sin(0))\displaystyle\int_0^{\tan(0)} \sqrt{1+t^2}\ dt + (\cos(0))\left(\sqrt{1+\tan^2(0)}\ \sec^2(0)\right)$

$= (0)(0) + (1)\sqrt{1+0}(1) = 1$

(b) Evaluate the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2}$, $x > 0$

Solution : $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2} = \int \left(1+\sqrt{x}\right)^{-2}\ \frac{1}{\sqrt{x}}\ dx$

$= 2\displaystyle\int \left(1+\sqrt{x}\right)^{-2}\ \frac{1}{2\sqrt{x}}\ dx = 2\ \frac{(1+\sqrt{x})^{-1}}{-1} + c = \frac{-2}{1+\sqrt{x}} + c$

Using the formula $\displaystyle\int [f(x)]^n\ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

(c) Approximate the integral $\displaystyle\int_0^5 \frac{2^x}{1+x}\ dx$ using Trapezoid rule with $n = 5$.

Solution : $[a,b] = [0,5]$, $n = 5$, and $f(x) = \dfrac{2^x}{1+x}$.

$\Delta x = \dfrac{b-a}{n} = \dfrac{5-0}{5} = 1$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	1	1	1
1	1	1	2	2
2	2	1.333	2	2.667
3	3	2	2	4
4	4	3.2	2	6.4
5	5	5.333	1	5.333
				21.4

$$\int_0^5 \frac{2^x}{1+x}\,dx \approx \frac{5-0}{(2)(5)}(21.4) \approx \frac{21.4}{2} \approx 10.7$$

Q2. (3+2+3 Marks)

(a) Evaluate the integral $\int (\ln|x|+1)\ 3^{x\ln|x|}\ dx$

Solution : Note that $\frac{d}{dx}(x\ln|x|) = (1)\ln|x| + x\ \frac{1}{x} = \ln|x|+1$

$$\int (\ln|x|+1)\ 3^{x\ln|x|}\ dx = \int 3^{x\ln|x|}\ (\ln|x|+1)\ dx = \frac{3^{x\ln|x|}}{\ln 3} + c$$

Using the formula $\int a^{f(x)}\ f'(x)\ dx = \frac{a^{f(x)}}{\ln a} + c$, where $a > 1$

(b) If $y = \frac{x(x^2+1)^3}{\sqrt[4]{2x-1}}$, find y'.

Solution : $\ln|y| = \ln\left|\frac{x(x^2+1)^3}{\sqrt[4]{2x-1}}\right| = \ln\left|\frac{x(x^2+1)^3}{(2x-1)^{\frac{1}{4}}}\right|$

$$\ln|y| = \ln\left|x(x^2+1)^3\right| - \ln\left|(2x-1)^{\frac{1}{4}}\right|$$

$$\ln|y| = \ln|x| + \ln\left|(x^2+1)^3\right| - \ln\left|(2x-1)^{\frac{1}{4}}\right|$$

$$\ln|y| = \ln|x| + 3\ln\left|x^2+1\right| - \frac{1}{4}\ln|2x-1|$$

Differentiating both sides with respect to x :

$$\frac{y'}{y} = \frac{1}{x} + 3\left(\frac{2x}{x^2+1}\right) - \frac{1}{4}\left(\frac{2}{2x-1}\right)$$

$$y' = y\left[\frac{1}{x} + \frac{6x}{x^2+1} - \frac{1}{2(2x-1)}\right]$$

$$y' = \frac{x(x^2+1)^3}{\sqrt[4]{2x-1}}\left[\frac{1}{x} + \frac{6x}{x^2+1} - \frac{1}{4x-2}\right]$$

(c) Compute $\int \frac{\sec^2 x}{\sqrt{9-(\tan x)^2}}\, dx$

Solution : $\int \frac{\sec^2 x}{\sqrt{9-(\tan x)^2}}\, dx = \int \frac{\sec^2 x}{\sqrt{(3)^2-(\tan x)^2}}\, dx$

$= \sin^{-1}\left(\frac{\tan x}{3}\right) + c$

Using the formula $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\, dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| < a$

Q3. (3+3+3 Marks)

(a) Find $\int \frac{1}{x\sqrt{16x^4-1}}\, dx$

Solution : $\int \frac{1}{x\sqrt{16x^4-1}}\, dx = \int \frac{1}{x\sqrt{(4x^2)^2-1}}\, dx$

$= \int \frac{4x}{x(4x)\sqrt{(4x^2)^2-1}}\, dx = \frac{1}{2}\int \frac{4x(2)}{4x^2\sqrt{(4x^2)^2-1}}\, dx$

$= \frac{1}{2}\int \frac{8x}{4x^2\sqrt{(4x^2)^2-1}}\, dx = \frac{1}{2}\sec^{-1}\left(4x^2\right) + c$

Using the formula $\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| > a$

(b) Evaluate the integral $\int \frac{x^2}{\sqrt{x^6-25}}\, dx$

Solution : $\int \frac{x^2}{\sqrt{x^6-25}}\, dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{(x^3)^2-(5)^2}}\, dx$

$= \frac{1}{3}\cosh^{-1}\left(\frac{x^3}{5}\right) + c$

Using the formula $\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}\, dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| > a$

(c) Compute $\displaystyle\int \frac{dx}{x\ln|x|\sqrt{1-(\ln|x|)^4}}$

Solution : $\displaystyle\int \frac{dx}{x\ln|x|\sqrt{1-(\ln|x|)^4}} = \int \frac{\left(\frac{1}{x}\right)}{\ln|x|\sqrt{1-\left(\ln^2|x|\right)^2}}\ dx$

$\displaystyle= \int \frac{\ln|x|\left(\frac{1}{x}\right)}{\ln^2|x|\sqrt{1-\left(\ln^2|x|\right)^2}}\ dx = \frac{1}{2}\int \frac{2\ln|x|\left(\frac{1}{x}\right)}{\ln^2|x|\sqrt{1-\left(\ln^2|x|\right)^2}}\ dx$

$\displaystyle= \frac{1}{2}\left(-sech^{-1}\left(\ln^2|x|\right)\right) + c = -\frac{1}{2}sech^{-1}\left(\ln^2|x|\right) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-[f(x)]^2}}\ dx = -\frac{1}{a}sech^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| < a$

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Q1. (2+3+3 Marks)

(a) Find $\lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^x \sin(t^2)\ dt}{x^3}$

Solution : $\lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^x \sin(t^2)\ dt}{x^3} \qquad \left(\dfrac{0}{0}\right)$

Using L'Hôpital's rule.

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \sin(t^2)\ dt}{x^3} = \lim_{x\to 0} \frac{\sin\left(x^2\right)}{3x^2} \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule.

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{3x^2} = \lim_{x\to 0} \frac{2x\ \cos\left(x^2\right)}{6x} = \lim_{x\to 0} \frac{\cos(x^2)}{3} = \frac{\cos(0)}{3} = \frac{1}{3}$$

Therefore, $\lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^x \sin(t^2)\ dt}{x^3} = \dfrac{1}{3}$

(b) Evaluate the indefinite integral $\displaystyle\int \cosh^{-1} x\ dx$

Solution : Using integration by parts.

$$u = \cosh^{-1} x \qquad dv = dx$$
$$du = \frac{1}{\sqrt{x^2-1}}\ dx \qquad v = x$$

$$\int \cosh^{-1} x\ dx = x\cosh^{-1} x - \int x\ \frac{1}{\sqrt{x^2-1}}\ dx$$

$$= x\cosh^{-1} x - \frac{1}{2}\int \left(x^2-1\right)^{-\frac{1}{2}} (2x)\ dx$$

$$= x\cosh^{-1} x - \frac{1}{2}\ \frac{\left(x^2-1\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x\cosh^{-1} x - \sqrt{x^2-1} + c$$

(c) Compute the integral $\displaystyle\int (\sin x)^3 (\cos x)^7\ dx$

Solution : Put $u = \cos x \implies -du = \sin x\ dx$

$$\int (\sin x)^3 (\cos x)^7 \; dx = \int \sin^2 x \;\; \cos^7 x \;\; \sin x \; dx$$

$$= \int \left(1 - \cos^2 x\right) \;\; \cos^7 x \;\; \sin x \; dx = - \int \left(1 - u^2\right) u^7 \; du$$

$$= - \int \left(u^7 - u^9\right) \; du = -\left(\frac{u^8}{8} - \frac{u^{10}}{10}\right) + c = -\frac{\cos^8 x}{8} + \frac{\cos^{10} x}{10} + c$$

Q2. (3+3+2 Marks)

(a) Evaluate the integral $\int \frac{dx}{\left(x^2+4\right)^2}$

Solution : Using Trigonometric substitutions.

Put $x = 2\tan\theta \implies \tan\theta = \frac{x}{2}$

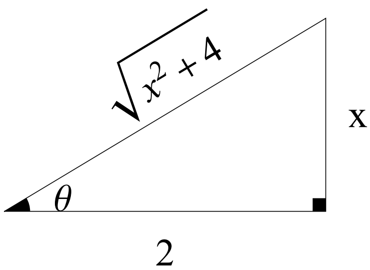
$dx = 2\sec^2\theta \; d\theta$

$$\int \frac{dx}{\left(x^2+4\right)^2} = \int \frac{2\sec^2\theta}{\left(4\tan^2\theta + 4\right)^2} \; d\theta = \int \frac{2\sec^2\theta}{\left[4\left(\tan^2\theta + 1\right)\right]^2} \; d\theta$$

$$= \int \frac{2\sec^2\theta}{\left[4\sec^2\theta\right]^2} \; d\theta = \int \frac{2\sec^2\theta}{16\sec^4\theta} \; d\theta = \frac{1}{8}\int \frac{1}{\sec^2\theta} \; d\theta = \frac{1}{8}\int \cos^2\theta \; d\theta$$

$$= \frac{1}{8}\int \left(\frac{1+\cos 2\theta}{2}\right) \; d\theta = \frac{1}{16}\int \left(1+\cos 2\theta\right) \; d\theta = \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{1}{16}\left(\theta + \frac{2\sin\theta \;\; \cos\theta}{2}\right) + c = \frac{1}{16}\left(\theta + \sin\theta \;\; \cos\theta\right) + c$$



From the triangle : $\sin\theta = \frac{x}{\sqrt{x^2+4}}$, $\cos\theta = \frac{2}{\sqrt{x^2+4}}$

Note that $\tan\theta = \frac{x}{2} \implies \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$\int \frac{dx}{\left(x^2+4\right)^2} = \frac{1}{16}\left[\tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}} \;\; \frac{2}{\sqrt{x^2+4}}\right] + c$$

$$= \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\;\frac{x}{x^2+4} + c$$

(b) Find $\displaystyle\int \frac{dx}{x^4+x^2}$

Solution : Using the method of partial fractions.

$$\frac{1}{x^4+x^2} = \frac{1}{x^2(x^2+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{x^2+1}$$

$$1 = A_1x(x^2+1) + A_2(x^2+1) + (Bx+C)x^2$$

$$1 = A_1x^3 + A_1x + A_2x^2 + A_2 + Bx^3 + Cx^2$$

$$1 = (A_1+B)\,x^3 + (A_2+C)\,x^2 + A_1x + A_2$$

By comparing the coefficients of the two polynomials in both sides :

$$\begin{array}{ll} A_1 + B = 0 & \longrightarrow (1) \\ A_2 + C = 0 & \longrightarrow (2) \\ A_1 = 0 & \longrightarrow (3) \\ A_2 = 1 & \longrightarrow (4) \end{array}$$

From Eq(1) : $B = 0$

From Eq(2) : $C = -1$

$$\int \frac{dx}{x^4+x^2} = \int \left(\frac{1}{x^2} + \frac{-1}{x^2+1}\right) dx = \int x^{-2}\, dx - \int \frac{1}{x^2+1}\, dx$$

$$= \frac{x^{-1}}{-1} - \tan^{-1} x + c = -\frac{1}{x} - \tan^{-1} x + c$$

(c) Compute $\displaystyle\int \frac{dx}{x^{\frac{1}{6}} + x^{\frac{1}{3}}}$

Solution : Put $x = u^6 \implies u = x^{\frac{1}{6}}$

$dx = 6u^5\, du$

$$\int \frac{dx}{x^{\frac{1}{6}} + x^{\frac{1}{3}}} = \int \frac{6u^5}{(u^6)^{\frac{1}{6}} + (u^6)^{\frac{1}{3}}}\, du = \int \frac{6u^5}{u+u^2}\, du$$

$$= \int \frac{6u^5}{u(1+u)}\, du = \int \frac{6u^4}{u+1}\, du = 6\int \frac{u^4}{u+1}\, du$$

Using long division of polynomials

$$6\int \frac{u^4}{u+1}\, du = 6\int \left(u^3 - u^2 + u - 1 + \frac{1}{u+1}\right) du$$

$$= 6\left(\frac{u^4}{4} - \frac{u^3}{3} + \frac{u^2}{2} - u + \ln|u+1|\right) + c$$

$$= \frac{3}{2}u^4 - 2u^3 + 3u^2 - 6u + 6\ln|u+1| + c$$

$$= \frac{3}{2}\left(x^{\frac{1}{6}}\right)^4 - 2\left(x^{\frac{1}{6}}\right)^3 + 3\left(x^{\frac{1}{6}}\right)^2 - 6x^{\frac{1}{6}} + 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$

$$= \frac{3}{2}x^{\frac{2}{3}} - 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$

Q3. (3+3+3 Marks)

(a) Find $\displaystyle\int \frac{dx}{3 + \cos x + 2\sin x}$

Solution : Using half-angle substitution.

Put $u = \tan\left(\frac{x}{2}\right)$

$$\sin x = \frac{2u}{1+u^2}\ ,\quad \cos x = \frac{1-u^2}{1+u^2}\ ,\quad dx = \frac{2}{1+u^2}\ du$$

$$\int \frac{dx}{3 + \cos x + 2\sin x} = \int \frac{\left(\frac{2}{1+u^2}\right)}{3 + \frac{1-u^2}{1+u^2} + 2\left(\frac{2u}{1+u^2}\right)}\ du$$

$$= \int \frac{\left(\frac{2}{1+u^2}\right)}{\left(\frac{3+1-u^2+4u}{1+u^2}\right)}\ du = \int \frac{2}{-u^2+4u+4}\ du = \int \frac{2}{4-(u^2-4u)}\ du$$

$$= 2\int \frac{1}{8-(u^2-4u+4)}\ du = 2\int \frac{1}{\left(\sqrt{8}\right)^2 - (u-2)^2}\ du$$

$$= 2\ \frac{1}{\sqrt{8}}\tanh^{-1}\left(\frac{u-2}{\sqrt{8}}\right) + c = \frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{\tan\left(\frac{x}{2}\right) - 2}{\sqrt{8}}\right) + c$$

(b) Show that the integral $\displaystyle\int_1^{\infty} \frac{\ln|x|}{x^2}\ dx$ converges and find its value.

Solution : $\displaystyle\int_1^{\infty} \frac{\ln|x|}{x^2}\ dx = \lim_{t\to\infty}\int_1^{t} \frac{\ln|x|}{x^2}\ dx = \lim_{t\to\infty}\int_1^{t} x^{-2}\ln|x|\ dx$

Using integration by parts.

$$u = \ln|x| \qquad dv = x^{-2}dx$$
$$du = \frac{1}{x}\ dx \qquad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$$

$$\lim_{t\to\infty}\int_1^{t} x^{-2}\ln|x|\ dx = \lim_{t\to\infty}\left(\left[-\frac{1}{x}\ln|x|\right]_1^t - \int_1^t \frac{1}{x}\left(-\frac{1}{x}\right)\ dx\right)$$

$$= \lim_{t\to\infty}\left(\left[-\frac{\ln|x|}{x}\right]_1^t + \int_1^t x^{-2}\ dx\right) = \lim_{t\to\infty}\left(\left[-\frac{\ln|x|}{x}\right]_1^t + \left[\frac{x^{-1}}{-1}\right]_1^t\right)$$

$$= \lim_{t\to\infty}\left(\left[-\frac{\ln|x|}{x}\right]_1^t + \left[\frac{-1}{x}\right]_1^t\right) = \lim_{t\to\infty}\left[-\frac{\ln|x|}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left[\frac{-\ln|x| - 1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{-\ln|t|-1}{t} - \left(\frac{-\ln(1)-1}{1}\right)\right) = \lim_{t\to\infty}\left(\frac{-\ln|t|-1}{t} - (-1)\right)$$

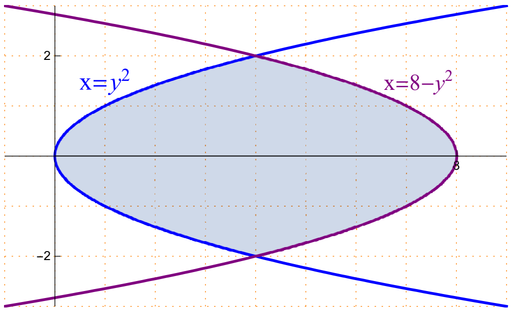
$$= \lim_{t\to\infty}\left(\frac{-\ln|t|-1}{t} + 1\right) = 0+1 = 1$$

Note that $\lim_{t\to\infty}\left(\frac{-\ln|t|-1}{t}\right) = 0$, using L'Hôpital's rule.

(c) Sketch the region bounded by the curves $x = y^2$, $x = 8 - y^2$, and find its area.

Solution : $x = y^2$ is a parabola with vertex $(0,0)$ and opens to the right.

$x = 8 - y^2$ is a parabola with vertex $(8,0)$ and opens to the left.



Points of intersection of $x = y^2$ and $x = 8 - y^2$:

$y^2 = 8 - y^2 \implies 2y^2 - 8 = 0 \implies y^2 - 4 = 0$

$\implies (y-2)(y+2) = 0 \implies y = -2\ ,\ y = 2$

$$Area = \int_{-2}^{2} \left[(8-y^2) - y^2\right]\ dy = \int_{-2}^{2} \left(8 - 2y^2\right)\ dy = \left[8y - \frac{2}{3}y^3\right]_{-2}^{2}$$

$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = 32 - \frac{32}{3} = \frac{64}{3}$$

M 106 - INTEGRAL CALCULUS
Solution of the final exam
Second semester 1439-1440 H
Dr. Tariq A. Alfadhel

Q1. (2+2 Marks)

(a) Approximate the integral $\int_0^6 \sqrt{1+x^2}\ dx$ using Simpson's Rule with $n=6$

Solution : $[a,b]=[0,6]$, $n=6$, and $f(x)=\sqrt{1+x^2}$.

$\Delta x = \frac{b-a}{n} = \frac{6-0}{6} = 1$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	1	1	1
1	1	1.41421	4	5.65685
2	2	2.23607	2	4.47214
3	3	3.16228	4	12.6491
4	4	4.12311	2	8.24621
5	5	5.09902	4	20.3961
6	6	6.08276	1	6.08276
				58.5032

$$\int_0^6 \sqrt{1+x^2}\ dx \approx \frac{6-0}{3(6)}(58.5032) \approx \frac{58.5032}{3} \approx 19.5011$$

(b) Find the number c in the mean value theorem for $f(x)=\frac{8}{x^2}$ on $[2,4]$

Solution : Using the formula $(b-a)\ f(c) = \int_a^b f(x)\ dx$

$$(4-2)\frac{8}{c^2} = \int_2^4 \frac{8}{x^2}\ dx = 8\int_2^4 x^{-2}\ dx$$

$$(2)\frac{8}{c^2} = 8\left[\frac{x^{-1}}{-1}\right]_2^4 = 8\left[\frac{-1}{x}\right]_2^4 = 8\left[-\frac{1}{4}-\left(\frac{1}{2}\right)\right] = 8\left(\frac{1}{4}\right) = 2$$

$$(2)\frac{8}{c^2} = 2 \implies \frac{8}{c^2} = 1 \implies c^2 = 8 \implies c = \pm\sqrt{8}$$

Note that $\sqrt{8} \in (2,4)$, while $-\sqrt{8} \notin (2,4)$

Q2. (3+3+3 Marks)

(a) Evaluate the integral $\int \frac{dx}{\sqrt{e^{6x}-25}}$

Solution : $\int \frac{dx}{\sqrt{e^{6x}-25}} = \int \frac{1}{\sqrt{(e^{3x})^2-(5)^2}}\ dx$

$= \int \frac{e^{3x}}{e^{3x}\sqrt{(e^{3x})^2-(5)^2}}\ dx = \frac{1}{3}\int \frac{e^{3x}(3)}{e^{3x}\sqrt{(e^{3x})^2-(5)^2}}\ dx$

$= \frac{1}{3}\ \frac{1}{5}\sec^{-1}\left(\frac{e^{3x}}{5}\right) + c = \frac{1}{15}\sec^{-1}\left(\frac{e^{3x}}{5}\right) + c$

Using the formula $\int \frac{f'(x)}{f(x)\ \sqrt{[f(x)]^2-a^2}}\ dx = \frac{1}{a}\ \sec^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| > a$

(b) Compute the integral $\int \frac{dx}{x\sqrt{1-x^8}}$

Solution : $\int \frac{dx}{x\sqrt{1-x^8}} = \int \frac{1}{x\sqrt{(1)^2-(x^4)^2}}\ dx$

$= \int \frac{x^3}{x^4\sqrt{(1)^2-(x^4)^2}}\ dx = \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{(1)^2-(x^4)^2}}\ dx$

$= \frac{1}{4}\left(-sech^{-1}\left(x^4\right)\right) + c = -\frac{1}{4}sech^{-1}\left(x^4\right) + c$

Using the formula $\int \frac{f'(x)}{f(x)\ \sqrt{a^2-[f(x)]^2}}\ dx = -\frac{1}{a}\ sech^{-1}\left(\frac{f(x)}{a}\right) + c$,

where $a > 0$ and $|f(x)| < a$

(c) Find $\int x\tan^{-1}x\ dx$

Solution : Using integration by parts.

$u = \tan^{-1}x \qquad dv = x\ dx$

$du = \frac{1}{1+x^2}\ dx \qquad v = \frac{x^2}{2}$

$\int x\ \tan^{-1}x\ dx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2}\ \frac{1}{1+x^2}\ dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{1+x^2}\ dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)\ dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)\ dx$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2}(x) + \frac{1}{2} \tan^{-1} x + c = \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2} + c$$

Q3. (3+3+3 Marks)

(a) Compute the following integral $\int \tan^5 x \ \sec^3 x \ dx$

Solution : Put $u = \sec x \implies du = \sec x \ \tan x \ dx$

$$\int \tan^5 x \ \sec^3 x \ dx = \int \tan^4 x \ \sec^2 x \ \sec x \ \tan x \ dx$$

$$= \int \left(\tan^2 x\right)^2 \ \sec^2 x \ \sec x \ \tan x \ dx$$

$$= \int \left(\sec^2 x - 1\right)^2 \ \sec^2 x \ \sec x \ \tan x \ dx = \int \left(u^2 - 1\right)^2 u^2 \ du$$

$$= \int \left(u^4 - 2u^2 + 1\right) u^2 \ du = \int \left(u^6 - 2u^4 + u^2\right) \ du$$

$$= \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + c = \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + c$$

(b) Find the integral $\int \cos(7x) \ \cos(5x) \ dx$

Solution : Using $\cos(ax) \ \cos(bx) = \frac{1}{2} [\cos(ax - bx) + \cos(ax + bx)]$

$$\int \cos(7x) \ \cos(5x) \ dx = \int \frac{1}{2} [\cos(7x - 5x) + \cos(7x + 5x)] \ dx$$

$$= \frac{1}{2} \int [\cos(2x) + \cos(12x)] \ dx = \frac{1}{2} \int \cos(2x) \ dx + \frac{1}{2} \int \cos(12x) \ dx$$

$$= \frac{1}{2} \ \frac{1}{2} \int \cos(2x) \ (2) \ dx + \frac{1}{2} \ \frac{1}{12} \int \cos(12x) \ (12) \ dx$$

$$= \frac{1}{4} \sin(2x) + \frac{1}{24} \cos(12x) + c$$

(c) Evaluate the integral $\int \frac{dx}{(x^2 - 1)^{\frac{3}{2}}}$

Solution : Using trigonometric substitutions.

Put $x = \sec\theta$

$dx = \sec\theta \ \tan\theta \ d\theta$

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} = \int \frac{\sec\theta\ \tan\theta}{(\sec^2\theta - 1)^{\frac{3}{2}}}\ d\theta = \int \frac{\sec\theta\ \tan\theta}{\left(\tan^2\theta\right)^{\frac{3}{2}}}\ d\theta = \int \frac{\sec\theta\ \tan\theta}{\tan^3\theta}\ d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\ d\theta = \int \sec\theta\ \cot^2\theta\ d\theta = \int \frac{1}{\cos\theta}\ \frac{\cos^2\theta}{\sin^2\theta}\ d\theta = \int \frac{\cos\theta}{\sin^2\theta}\ d\theta$$

$$= \int (\sin\theta)^{-2}\cos\theta\ d\theta = \frac{(\sin\theta)^{-1}}{-1} + c = -\frac{1}{\sin\theta} + c$$

Note that $x = \sec\theta \implies \cos\theta = \dfrac{1}{x} \implies \sin\theta = \sqrt{1 - \left(\dfrac{1}{x}\right)^2} = \sqrt{1 - \dfrac{1}{x^2}}$

$$\implies \sin\theta = \sqrt{\frac{x^2-1}{x^2}} = \frac{\sqrt{x^2-1}}{x} \implies \frac{1}{\sin\theta} = \frac{x}{\sqrt{x^2-1}}$$

$$\int \frac{dx}{(x^2-1)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2-1}} + c$$

Q4. (3+3+3 Marks)

(a) Evaluate the integral $\displaystyle\int \frac{2x-1}{x^2+4x+20}\ dx$

Solution : $\displaystyle\int \frac{2x-1}{x^2+4x+20}\ dx = \int \frac{(2x+4)-5}{x^2+4x+20}\ dx$

$$= \int \frac{2x+4}{x^2+4x+20}\ dx - \int \frac{5}{x^2+4x+20}\ dx$$

$$= \ln\left|x^2+4x+20\right| - 5\int \frac{1}{(x^2+4x+4)+16}\ dx$$

$$= \ln\left|x^2+4x+20\right| - 5\int \frac{1}{(x+2)^2+(4)^2}\ dx$$

$$= \ln\left|x^2+4x+20\right| - 5\ \frac{1}{4}\tan^{-1}\left(\frac{x+2}{4}\right) + c$$

(b) Sketch the region bounded by the curves $y = 2 - x^2$, $y = x$, $x = 0$, $x = 2$ and find its area.

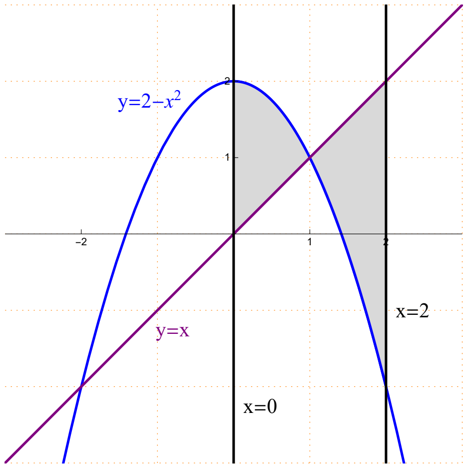
Solution :

$y = 2 - x^2$ is a parabola with vertex $(0, 2)$ and opens downwards.

$y = x$ is a straight line passing through the origin with slope equals 1.

$x = 2$ is a straight line parallel to the y-axis and passing through $(2, 0)$

$x = 0$ is the y-axis.



Points of intersection of $y = 2 - x^2$ and $y = x$:

$x = 2 - x^2 \implies x^2 + x - 2 = 0 \implies (x+2)(x-1) = 0$

$\implies x = -2 \ , \ x = 1$

Note that $1 \in [0, 2]$, while $-2 \notin [0, 2]$

$$Area = \int_0^1 \big[(2 - x^2) - x\big] \ dx + \int_1^2 \big[x - (2 - x^2)\big] \ dx$$

$$= \int_0^1 \big(-x^2 - x + 2\big) \ dx + \int_1^2 \big(x^2 + x - 2\big) \ dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x\right]_1^2$$

$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - (0 - 0 + 0)\right] + \left[\left(\frac{8}{3} + 2 - 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)\right]$$

$$= \frac{7}{6} + \frac{11}{6} = \frac{18}{6} = 3$$

(c) Set up an integral for the volume obtained by revolving the region bounded by the curves $y = x^2$, $y = 4$ about the line of equation

(i) $y = 6$

(ii) $x = -3$

Solution :

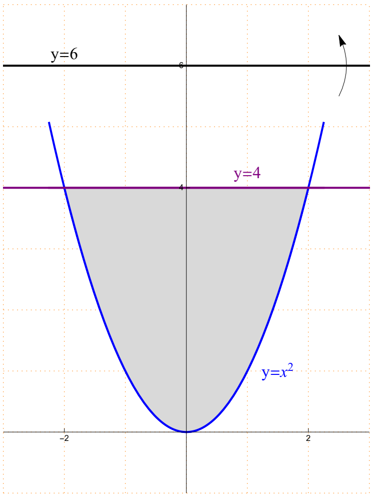
$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = 4$ is a straight line parallel to the x-axis and passing through $(0, 4)$.

Points of intersection of $y = x^2$ and $y = 4$:

$x^2 = 4 \implies x = \pm 2$

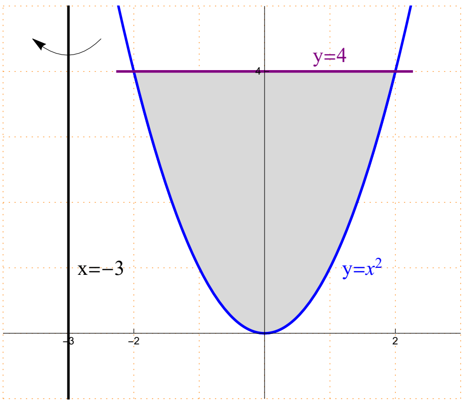
(i) About the line $y = 6$



Using Washer method :

$$V = \pi \int_{-2}^{2} \left[\left(6 - x^2\right)^2 - (6-4)^2 \right] \; dx = \pi \int_{-2}^{2} \left[\left(6 - x^2\right)^2 - (2)^2 \right] \; dx$$

(ii) About the line $x = -3$



Using Cylindrical shells method :

$$V = 2\pi \int_{-2}^{2} (3 + x)(4 - x^2) \; dx$$

Q5. (3+3+3 Marks)

(a) Sketch the region R that lies inside the curve $r = 2\sin\theta$ and outside the curve $r = 2 - 2\sin\theta$ and find its area.

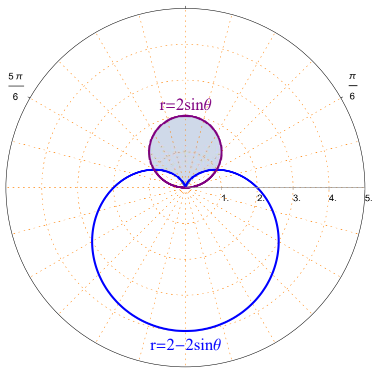
Solution :

$r = 2 - 2\sin\theta$ is a cardioid , symmetric with respect to the line $\theta = \dfrac{\pi}{2}$

$r = 2\sin\theta$ is a circle with center $(r, \theta) = \left(1, \dfrac{\pi}{2}\right)$ and radius equals 1.

Points of intersection of $r = 2 - 2\sin\theta$ and $r = 2\sin\theta$:

$2\sin\theta = 2 - 2\sin\theta \implies 4\sin\theta = 2 \implies \sin\theta = \dfrac{1}{2} \implies \theta = \dfrac{\pi}{6}$, $\theta = \dfrac{5\pi}{6}$



Note that the desired area is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

$$Area = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[(2\sin\theta)^2 - (2 - 2\sin\theta)^2\right]\, d\theta\right)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[4\sin^2\theta - \left(4 - 8\sin\theta + 4\sin^2\theta\right)\right]\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[4\sin^2\theta - 4 + 8\sin\theta - 4\sin^2\theta\right]\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [8\sin\theta - 4]\, d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [2\sin\theta - 1]\, d\theta = 4\left[-2\cos\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 4\left[\left(0 - \frac{\pi}{2}\right) - \left(-2\,\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)\right] = 4\left(\sqrt{3} + \frac{\pi}{6} - \frac{\pi}{2}\right) = 4\left(\sqrt{3} - \frac{\pi}{3}\right)$$

(b) Find the area of the surface obtained by revolving the curve $r = 4\cos\theta$, $0 \le \theta \le \dfrac{\pi}{2}$ about the y-axis.

Solution : $\dfrac{dr}{d\theta} = -4\sin\theta$

$$S.A = 2\pi \int_0^{\frac{\pi}{2}} |4\cos\theta\ \cos\theta| \sqrt{(4\cos\theta)^2 + (-4\sin\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \left|4\cos^2\theta\right| \sqrt{16\cos^2\theta + 16\sin^2\theta}\, d\theta = 2\pi \int_0^{\frac{\pi}{2}} 4\cos^2\theta \sqrt{16\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 4\cos^2\theta\ (4)\ d\theta = 8\pi \int_0^{\frac{\pi}{2}} 4\left(\frac{1+\cos 2\theta}{2}\right)\ d\theta = 16\pi \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\ d\theta$$

$$= 16\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 16\pi \left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = 8\pi^2$$

(c) Find the length of the curve given by the equations $x = \dfrac{t^4}{4}$, $y = \dfrac{t^6}{6}$, $0 \le t \le 1$

Solution : $x = \dfrac{t^4}{4} \implies \dfrac{dx}{dt} = t^3$

$y = \dfrac{t^6}{6} \implies \dfrac{dy}{dt} = t^5$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 \sqrt{(t^3)^2 + (t^5)^2}\ dt$$

$$= \int_0^1 \sqrt{t^6 + t^{10}}\ dt = \int_0^1 \sqrt{t^6\,(1+t^4)}\ dt = \int_0^1 \left|t^3\right| \sqrt{1+t^4}\ dt$$

$$= \frac{1}{4}\int_0^1 \left(1+t^4\right)^{\frac{1}{2}}\ (4t^3)\ dt = \frac{1}{4}\left[\frac{\left(1+t^4\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{1}{4}\,\frac{2}{3}\ \left[\left(1+t^4\right)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{6}\ \left[\left(1+t^4\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{6}\left[(2)^{\frac{3}{2}} - 1\right] = \frac{1}{6}\left(2\sqrt{2} - 1\right)$$