M 106 - INTEGRAL CALCULUS

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Solution of the second mid-term exam

Second semester 1433-1434 H

Multiple choice questions (One mark for each question)

Q.1 If $\dfrac{3x+6}{x^2+5x+4} = \dfrac{A}{x+1} + \dfrac{B}{x+4}$ then A is equal to :

(a) -1 (b) 1 (c) $\dfrac{1}{3}$ (d) 3

Answer : $\dfrac{3x+6}{x^2+5x+4} = \dfrac{A}{x+1} + \dfrac{B}{x+4}$

$3x+6 = A(x+4) + B(x+1)$

Put $x=-1$ then $3(-1)+6 = A(-1+4) \Rightarrow 3 = 3A \Rightarrow A = 1$

The right answer is (b)

Q.2 To solve the integral $\int \sqrt{1-4x^2}\ dx$, we use the trigonometric substitution :

(a) $x = 4\sin\theta$ (b) $x = 4\sec\theta$ (c) $x = \dfrac{1}{2}\sin\theta$ (d) $3x = \dfrac{1}{2}\sec\theta$

Answer : $\int \sqrt{1-4x^2}\ dx = \int \sqrt{(1)^2 - (2x)^2}\ dx$

We use the trigonometric substitution $2x = \sin\theta \Rightarrow x = \dfrac{1}{2}\sin\theta$

The right answer is (c)

Q.3 The improper integral $\int_0^\infty xe^{-x^2}\ dx$

(a) diverges (b) converges to 1

(c) converges to $\frac{1}{2}$ (d) converges to -1

Answer : $\int_0^\infty xe^{-x^2}\ dx = \lim\limits_{t\to\infty} \int_0^t xe^{-x^2}\ dx = \lim\limits_{t\to\infty} \dfrac{-1}{2} \int_0^t e^{-x^2}\ (-2x)\ dx$

$= \lim\limits_{t\to\infty} \left(\dfrac{-1}{2} \left[e^{-x^2} \right]_0^t \right) = \lim\limits_{t\to\infty} \left(\dfrac{-1}{2} \left[e^{-t^2} - e^0 \right] \right) = \dfrac{-1}{2}\ [0-1] = \dfrac{1}{2}$

The right answer is (c)

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Q.4 The substitution $u = \tan\left(\frac{x}{2}\right)$ transforms the integral $\displaystyle\int \frac{1}{1+\sin x+\cos x}\ dx$ into :

(a) $\displaystyle\int \frac{1}{1+2u}\ du$ (b) $\displaystyle\int \frac{1}{1+u}\ du$

(c) $\displaystyle\int \frac{1}{1-u}\ du$ (d) $\displaystyle\int \frac{1}{u^2+u+1}\ du$

The answer : In this case $\sin x = \dfrac{2u}{1+u^2}$, $\cos x = \dfrac{1-u^2}{1+u^2}$ and $dx = \dfrac{2}{1+u^2}$

$$\int \frac{1}{1+\sin x+\cos x}\ dx = \int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\ \frac{2}{1+u^2}\ du$$

$$\int \frac{1}{\left(\frac{1+u^2+2u+1-u^2}{1+u^2}\right)}\ \frac{2}{1+u^2}\ du = \int \frac{1+u^2}{2u+2}\ \frac{2}{1+u^2}\ du$$

$$= \int \frac{2}{2(1+u)}\ du = \int \frac{1}{1+u}\ du$$

The right answer is (b)

Q.5 To evaluate $\displaystyle\int \sin^5 x\ dx$, we can use the following substitution :

(a) $u = \cos x$ (b) $u = \sin x$ (c) $u = \sin^5 x$ (d) $u = \cos^5 x$

The answer : $\displaystyle\int \sin^5 x\ dx = \int \sin^4 x\ \sin x\ dx = \int \left(\sin^2 x\right)^2\ \sin x\ dx$

$$= \int \left(1+\cos^2 x\right)^2\ \sin x\ dx$$

We can use the substitution $u = \cos x$, so $-du = \sin x\ dx$

The right answer is (a)

Full questions

Q.6 Evaluate the $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$ **[4 marks]**

The answer :

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \qquad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} \qquad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule again

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

Q.7 Evaluate the integral $\int x^3 \ln x \; dx$ **[4 marks]**

The answer : Using integration by parts

$u = \ln x \qquad dv = x^3 \; dx$

$du = \dfrac{1}{x} \; dx \qquad v = \dfrac{x^4}{4}$

$$\int x^3 \ln x \; dx = \frac{x^4}{4} \; \ln x - \int \frac{x^4}{4} \; \frac{1}{x} \; dx = \frac{x^4}{4} \; \ln x - \frac{1}{4} \int x^3 \; dx$$

$$= \frac{x^4}{4} \; \ln x - \frac{1}{4} \; \frac{x^4}{4} + c = \frac{x^4}{4} \; \ln x - \frac{x^4}{16} + c$$

Q.8 Evaluate the integral $\int \dfrac{6x^2 + x + 8}{x^3 + 4x} \; dx$ **[4 marks]**

The answer : Using Partial fractions

$$\frac{6x^2 + x + 8}{x^3 + 4x} = \frac{6x^2 + x + 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$6x^2 + x + 8 = A(x^2 + 4) + x(Bx + C) = Ax^2 + 4A + Bx^2 + Cx$$

$$6x^2 + x + 8 = (A + B)x^2 + Cx + 4A$$

By comparing the coefficients :

$4A = 8 \;\Rightarrow\; A = 2$

$C = 1$

$A + B = 6 \;\Rightarrow\; B = 6 - 2 = 4$

$$\frac{6x^2 + x + 8}{x^3 + 4x} = \frac{2}{x} + \frac{4x + 1}{x^2 + 4}$$

$$\int \frac{6x^2+x+8}{x^3+4x}\ dx = \int \left(\frac{2}{x}+\frac{4x+1}{x^2+4}\right)\ dx$$

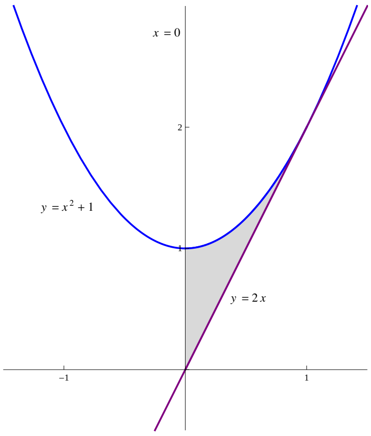
$$= 2\int \frac{1}{x}\ dx + \int \frac{4x}{x^2+4}\ dx + \int \frac{1}{x^2+4}\ dx$$

$$= 2\int \frac{1}{x}\ dx + 2\int \frac{2x}{x^2+4}\ dx + \int \frac{1}{x^2+2^2}\ dx$$

$$= 2\ln|x| + 2\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

Q.9 Sketch the region bounded by $y = x^2+1$, $y = 2x$, $x = 0$ and find its area. **[4 marks]**

The answer :



$y = x^2+1$ is a parabola opens upward and its vertex is $(0,1)$, $y = 2x$ is a straight line passing through the origin and $x = 0$ is the y-axis.

Points of intersection of $y = x^2+1$ and $y = 2x$:

$x^2+1 = 2x \ \Rightarrow\ x^2-2x+1 = 0 \ \Rightarrow\ (x-1)^2 = 0 \ \Rightarrow\ x = 1$

$$Area = \int_0^1 \int \left[(x^2+1)-2x\right]\ dx = \int_0^1 (x^2-2x+1)\ dx$$

$$Area = \int_0^1 (x-1)^2\ dx = \left[\frac{(x-1)^3}{3}\right]_0^1$$

$$Area = \frac{(1-1)^3}{3} - \frac{(0-1)^3}{3} = 0 + \frac{1}{3} = \frac{1}{3}$$

Q.10 Evaluate $\int \frac{1}{(x^2+16)^{\frac{3}{2}}}\ dx$

The answer : $\int \frac{1}{(x^2+16)^{\frac{3}{2}}}\ dx = \int \frac{1}{[x^2+4^2]^{\frac{3}{2}}}\ dx$

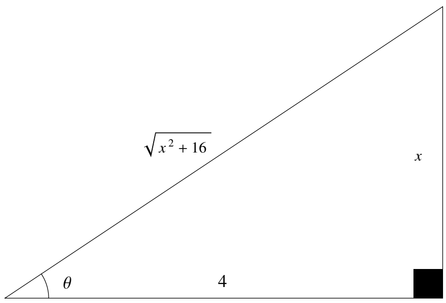
Using the trigonometric substitution $x = 4\tan\theta \quad \Rightarrow \quad \tan\theta = \dfrac{x}{4}$

$dx = 4\sec^2\theta \; d\theta$

$$\int \frac{1}{(x^2+16)^{\frac{3}{2}}}\; dx = \int \frac{4\sec^2\theta}{\left(16\tan^2\theta + 16\right)^{\frac{3}{2}}}\; d\theta$$

$$= \int \frac{4\sec^2\theta}{\left[16\left(\tan^2\theta+1\right)\right]^{\frac{3}{2}}}\; d\theta = \int \frac{4\sec^2\theta}{(16\sec^2\theta)^{\frac{3}{2}}}\; d\theta = \int \frac{4\sec^2\theta}{4^3\sec^3\theta}\; d\theta$$

$$= \frac{1}{4^2}\int \frac{1}{\sec\theta}\; d\theta = \frac{1}{16}\int \cos\theta\; d\theta = \frac{1}{16}\sin\theta + c$$



$$\int \frac{1}{(x^2+16)^{\frac{3}{2}}}\; dx = \frac{1}{16}\;\frac{x}{\sqrt{x^2+16}} + c$$