M 106 - INTEGRAL CALCULUS

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Solution of the second mid-term exam

First semester 1433-1434 H

Multiple choice questions (One mark for each question)

Q.1 The definite integral $\int \sin^2\left(\frac{x}{2}\right)\ dx$ is equal to

(a) $\frac{1}{2}x + \frac{1}{2}\sin x + c$ (b) $\frac{1}{2}(x - \sin x) + c$

(c) $\frac{1}{2}\cos(\frac{x}{2}) + c$ (d) $-\frac{1}{2}\cos(\frac{x}{2}) + c$

Answer : Note that $\sin^2 x = \frac{1}{2}[1 - \cos 2x]$

So, $\sin^2(\frac{x}{2}) = \frac{1}{2}[1 - \cos x]$

$$\int \sin^2\left(\frac{x}{2}\right)\ dx = \int \frac{1}{2}[1 - \cos x]\ dx = \frac{1}{2}(x - \sin x) + c$$

The right answer is (b)

Q.2 The substitution $u = \tan(\frac{x}{2})$ transforms the integral $\int \frac{1}{1+\sin x}\ dx$ into

(a) $\int du$ (b) $\int 2\ du$ (c) $\int \frac{1}{(u+1)^2}\ du$ (d) $\int \frac{2}{(u+1)^2}\ du$

Answer : $u = \tan(\frac{x}{2})$, $\sin x = \frac{2u}{1+u^2}$, $dx = \frac{2}{1+u^2}\ du$

$$\int \frac{1}{1+\sin x}\ dx = \int \frac{1}{1+\frac{2u}{1+u^2}}\ \frac{2}{1+u^2}du = \int \frac{1+u^2}{1+u^2+2u}\ \frac{2}{1+u^2}du$$

$$= \int \frac{2}{u^2+2u+1}du = \int \frac{2}{(u+1)^2}du$$

The right answer is (d)

Q.3 To evaluate the integral $\int \sqrt{2x^2+4}\ dx$, we use the substitution

(a) $x = \sqrt{2}\sec\theta$ (b) $x = 2\tan\theta$

(c) $x = 2\sec\theta$ (d) $x = \sqrt{2}\tan\theta$

Answer : $\int \sqrt{2x^2+4}\ dx = \int \sqrt{(\sqrt{2}x)^2 + 2^2}\ dx$

We use the trigonometric substitution $\sqrt{2}x = 2\tan\theta$

$\Rightarrow\quad x = \frac{2}{\sqrt{2}}\tan\theta \quad\Rightarrow\quad x = \sqrt{2}\tan\theta$

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The right answer is (d)

Q.4 $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$ is equal to

(a) 1 (b) e (c) 0 (d) ∞)

The answer : Put $y=(1+x)^{\frac{1}{x}} \quad \Rightarrow \quad \ln y = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$

$\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\frac{\ln(1+x)}{x} \qquad \left(\frac{0}{0}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = \frac{1}{1+0} = 1$$

Therefore , $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e^1 = e$

The right answer is (b)

Q.5 If $\dfrac{x^3+1}{(3x^2+1)^2} = \dfrac{Ax+B}{3x^2+1} + \dfrac{Cx+D}{(3x^2+1)^2}$ then the value of A is equal to

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

The answer :

$$\frac{x^3+1}{(3x^2+1)^2} = \frac{Ax+B}{3x^2+1} + \frac{Cx+D}{(3x^2+1)^2} = \frac{(Ax+B)(3x^2+1)+Cx+D}{(3x^2+1)^2}$$

$$x^3+1 = (Ax+B)(3x^2+1)+Cx+D$$

$$x^3+1 = 3Ax^3+Ax+3Bx^2+B+Cx+D \quad \Rightarrow \quad 3A=1 \quad \Rightarrow \quad A=\frac{1}{3}$$

The right answer is (d)

Q.6 The indefinite integral $\int \sin^3 x\ dx$ is equal to

(a) $-\cos x + \dfrac{\cos^3 x}{3} + c$ (b) $\cos x - \dfrac{\cos^3 x}{3} + c$

(c) $-\cos x - \dfrac{\cos^3 x}{3} + c$ (d) $-\cos x + \dfrac{\cos^2 x}{2} + c$

The answer : $\int \sin^3 x\ dx = \int \sin^2 x\ \sin x\ dx = \int (1-\cos^2 x)\ \sin x\ dx$

Put $u=\cos x$, then $-du = \sin x\ dx$

$$\int \sin^3 x\ dx = -\int (1-u^2)\ du = -\left[u - \frac{u^3}{3}\right] + c$$

$$= -u + \frac{u^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c$$

The right answer is (a)

Q.7 The improper integral $\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\ dx$

(a) converges to 0 (b) diverges

(c) converges to $\frac{\pi}{4}$ (d) converges to $\frac{\pi}{2}$

The answer : $\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\ dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{e^x}{1+e^{2x}}\ dx$

$= \lim_{t\to-\infty} \int_{t}^{0} \frac{e^x}{(1)^2+(e^x)^2}\ dx = \lim_{t\to-\infty} \left[\tan^{-1}(e^x)\right]_t^0$

$= \lim_{t\to-\infty} \left[tan^{-1}(e^0) - \tan^{-1}(e^t)\right] = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

The right answer is (c)

Q.8 The definite integral $\int x\ \cos x\ dx$ is equal to

(a) $\cos x - x\sin x + c$ (b) $-cosx + x\sin x + c$

(c) $\cos x + x\sin x + c$ (d) $x\sin x + c$

The answer : Using Integration by parts

$u = x$ $dv = \cos x\ dx$

$du = dx$ $v = \sin x$

$\int x\ \cos x\ dx = x\ \sin x - \int \sin x\ dx = x\ \sin x - (-\cos x) + c$

$= x\ \sin x + \cos x + c = \cos x + x\ \sin x + c$

The right answer is (c)

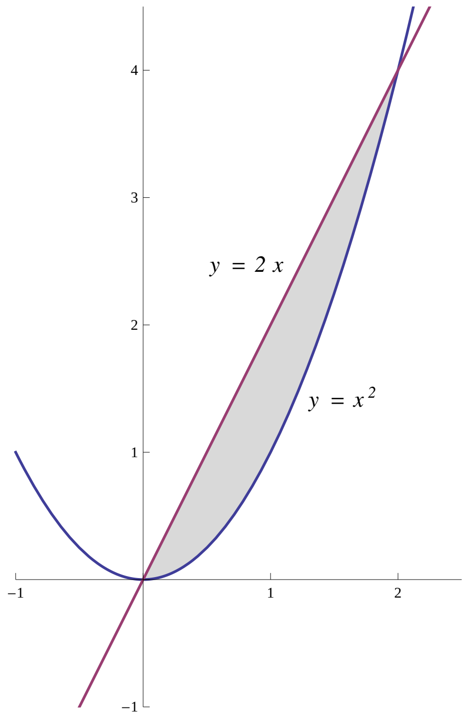
Q.9 The area of the region bounded by the graphs of the equations of $y = x^2$ and $y = 2x$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{2}$

The answer :

$y = x^2$ is a parabola opens upward with vertex $(0, 0)$.

$y = 2x$ is a straight line paasing through the origin .



Points of intersection between $y = x^2$ and $y = 2x$:

$x^2 = 2x \quad \Rightarrow \quad x^2 - 2x = 0 \quad \Rightarrow x(x-2) = 0 \quad \Rightarrow x = 0 \ , \ x = 2$

$$Area = \int_0^2 (2x - x^2)\ dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$Area = \left[\left(4 - \frac{8}{3}\right) - (0-0)\right] = \frac{12-8}{3} = \frac{4}{3}$$

The right answer is (c)

Q.10 To evaluate the integral $\int \frac{\sqrt[3]{x}}{1+\sqrt[4]{x}}\ dx$ we put

(a) $u^3 = x$ (b) $x = u^4$ (c) $u = \sqrt{x}$ (d) $u = x^{\frac{1}{12}}$

The answer : $\int \frac{\sqrt[3]{x}}{1+\sqrt[4]{x}}\ dx = \int \frac{x^{\frac{1}{3}}}{1+x^{\frac{1}{4}}}\ dx$

Note that the least common multiple of 3 and 4 is 12

Therefore, we use the substitution $x = u^{12} \quad \Rightarrow \quad u = x^{\frac{1}{12}}$

The right answer is (d)

Full questions

Q.11 Evaluate $\int \frac{-x^2+2x+1}{(x-1)(x^2+1)}\ dx$. **[3 marks]**

The answer : Using Partial fractions method

$$\frac{-x^2+2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$-x^2+2x+1 = A(x^2+1)+(Bx+C)(x-1)$$

$$-x^2+2x+1 = Ax^2+A+Bx^2-Bx+Cx-C$$

$$-x^2+2x+1 = (A+B)x^2+(C-B)x+(A-C)$$

$A+B=-1 \quad \longrightarrow \ (1)$

$C-B=2 \quad \longrightarrow \ (2)$

$A-C=1 \quad \longrightarrow \ (3)$

Adding equation (1) to equation (2) : $A+C=1 \quad \longrightarrow \ (4)$

Adding equation (3) to equation (4) : $2A=2 \ \Rightarrow A=1$

From equation (1) : $B=-2$

From equation (3) : $C=0$

$$\frac{-x^2+2x+1}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{-2x}{x^2+1}$$

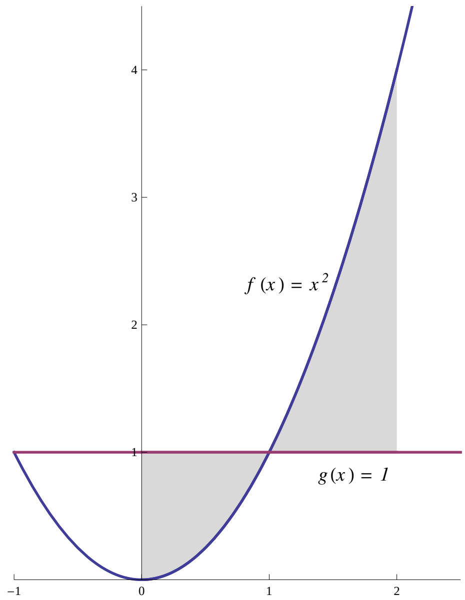
$$\int \frac{-x^2+2x+1}{(x-1)(x^2+1)}\ dx = \int \frac{1}{x-1}\ dx - \int \frac{2x}{x^2+1}\ dx$$

$$\int \frac{-x^2+2x+1}{(x-1)(x^2+1)}\ dx = \ln|x-1| - \ln(x^2+1) + c$$

Q.12 **Sketch** and **find** the area between the curves $f(x)=x^2$ and $g(x)=1$ on the interval $[0,2]$. **[4 marks]**

The answer :

$y=x^2$ is a parabola opens upward with vertex $(0,0)$, and $y=1$ is a straight line parallel to the $x-axis$ and passing through the point $(0,1)$.



Points of intersection between $y = x^2$ and $y = 1$: $x^2 = 1 \Rightarrow x = \pm 1$

Note that $x = -1 \notin [0, 2]$

$$Area = \int_0^1 (1 - x^2)\ dx + \int_1^2 (x^2 - 1)\ dx$$

$$Area = \left[x - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^2$$

$$Area = \left[\left(1 - \frac{1}{3}\right) - (0 - 0)\right] + \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right] = \frac{2}{3} + \frac{4}{3} = 2$$

Q.13 Evaluate $\int \frac{1}{(1 + x^2)^2}\ dx$. **[6 marks]**

The answer : Using trigonometric substitutions

Put $x = \tan\theta$.

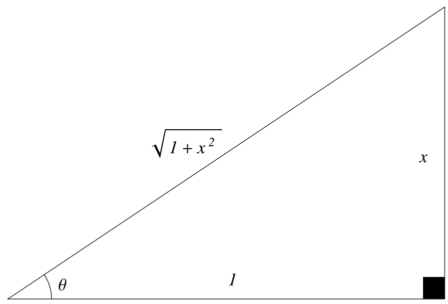
$dx = \sec^2\theta\ d\theta$

$$\int \frac{1}{(1 + x^2)^2}\ dx = \int \frac{\sec^2\theta}{(1 + \tan^2\theta)^2}\ d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\ d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^4\theta}\ d\theta = \int \frac{1}{\sec^2\theta}\ d\theta = \int \cos^2\theta\ d\theta$$

$$= \int \frac{1}{2}\left[1 + \cos 2\theta\right]\ d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c = \frac{1}{2}\left[\theta + \frac{2\sin\theta\ \cos\theta}{2}\right] + c$$

$$= \frac{1}{2}\left[\theta + \sin\theta\ \cos\theta\right] + c$$



$$\int \frac{1}{(1+x^2)^2}\ dx = \frac{1}{2}\left[\tan^{-1} x + \frac{x}{\sqrt{1+x^2}}\ \frac{1}{\sqrt{1+x^2}}\right] + c$$

$$\int \frac{1}{(1+x^2)^2}\ dx = \frac{1}{2}\left[\tan^{-1} x + \frac{x}{1+x^2}\right] + c$$

Q.14 Evaluate $\lim\limits_{x\to 0} \dfrac{x-\ln(x+1)}{x\ln(x+1)}$. **[2 marks]**

The answer : $\lim\limits_{x\to 0} \dfrac{x-\ln(x+1)}{x\ln(x+1)}$ $\left(\dfrac{0}{0}\right)$

Apply L'Hôpital's rule

$$\lim_{x\to 0} \frac{x-\ln(x+1)}{x\ln(x+1)} = \lim_{x\to 0} \frac{1-\frac{1}{x+1}}{\frac{x}{x+1}+\ln(x+1)}$$

$$= \lim_{x\to 0} \frac{\frac{(x+1)-1}{x+1}}{\frac{x+(x+1)\ln(x+1)}{x+1}} = \lim_{x\to 0} \frac{x}{x+(x+1)\ln(x+1)} \qquad \left(\frac{0}{0}\right)$$

Apply L'Hôpital's rule

$$\lim_{x\to 0} \frac{x}{x+(x+1)\ln(x+1)} = \lim_{x\to 0} \frac{1}{1+\ln(x+1)+(x+1)\frac{1}{x+1}}$$

$$= \lim_{x\to 0} \frac{1}{1+\ln(x+1)+1} = \frac{1}{1+\ln(1)+1} = \frac{1}{1+0+1} = \frac{1}{2}$$
