King Saud University: Math. Dept. M-254
Semester I (1st Midterm Exam) 1438-1439 H
Max Marks=25 Time Allowed: 90 Mins.

Questions : (5 + 5 + 5 + 5 + 5)

Q1: Which of the following iterations

$$(i) \quad x_{n+1} = e^{x_n} - x_n - 1, \quad n \geq 0 \quad (ii) \quad x_{n+1} = \ln(2x_n + 1), \quad n \geq 0$$

is most suitable to approximate the root of the equation $e^x - 2x = 1$ in the interval $[1, 2]$? Starting with $x_0 = 1.5$, find the second approximation x_2 of the root. Also, compute the error bound for the approximation.

Q2: Successive approximations x_n to the desired root $\sqrt{3}$ are generated by the scheme

$$x_{n+1} = \frac{1}{2}x_n + \frac{3}{2x_n}, \qquad n \geq 0.$$

Use Newton's method to find the second approximation x_2 of the root, starting with $x_0 = 2$. Show that the order of convergence of Newton's method is at least quadratic.

Q3: Use Secant method to find the second approximation, using $x_0 = 1$ and $x_1 = 2$, of the value of x that produces the point on the graph of $y = \frac{1}{x}$ that is closest to the point $(2, 1)$.

Q4: Show that $\alpha = 1$ is the root for the equation $x^4 - 8x^3 + 18x^2 = 16x - 5$. Use quadratic convergent iterative method to find the first approximation of α starting with $x_0 = 0.5$. Compute absolute error.

Q5: Find the first approximation for the nonlinear system

$$\begin{aligned} y^2(1 - x) &= x^3 \\ x^2 + y^2 &= 1 \end{aligned}$$

using Newton's method, starting with initial approximation $(x_0, y_0)^T = (1, 1)^T$.

Solution of the Midterm I Examination

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Question 1: Which of the following iterations

$$(i) \quad x_{n+1} = e^{x_n} - x_n - 1, \quad n \geq 0 \quad (ii) \quad x_{n+1} = \ln(2x_n + 1), \quad n \geq 0$$

is most suitable to approximate the root of the equation $e^x - 2x = 1$ in the interval $[1, 2]$? Starting with $x_0 = 1.5$, find the second approximation x_2 of the root. Also, compute the error bound for the approximation.

Solution. Since $f(x) = e^x - 2x - 1$, we observe that

$$f(1).f(2) = (-0.2817)(2.3891) < 0,$$

then the solution we seek is in the interval $[1, 2]$.

For the first scheme, we are given $g(x) = e^x - x - 1$.

For this $g(x) = e^x - x - 1$, we have $g'(x) = e^x - 1$, which is greater than unity throughout the interval $[1, 2]$. So by the Fixed-Point Theorem, this iteration will fail to converge.

For the second scheme, we are given $g(x) = \ln(2x + 1)$.

For this $g(x) = \ln(2x + 1)$, we have $g'(x) = 2/(2x + 1) < 1$, for all x in the given interval $[1, 2]$. Also, g is increasing function of x, and $g(1) = \ln(3) = 1.0986123$ and $g(2) = \ln(5) = 1.6094379$ both lie in the interval $[1, 2]$. Thus $g(x) \in [1, 2]$, for all $x \in [1, 2]$, so from Fixed-Point Theorem, this g(x) has a unique fixed-point.

For finding the second approximation of the root lying in the interval $[1, 2]$, we will use the following suitable scheme

$$x_{n+1} = \ln(2x_n + 1), \qquad n \geq 0.$$

Using the given initial approximation $x_0 = 1.5$, we get the first approximation as

$$x_1 = g(x_0) = \ln(2x_0 + 1) = \ln(2(1.5) + 1) = \ln(4) = 1.386294,$$

and similarly, the second approximation is

$$x_2 = g(x_1) = \ln(2x_1 + 1) = \ln(2(1.386294) + 1) = 1.327761.$$

To compute the error bound, we will use the following formula:

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0|.$$

Since $a = 1$, $b = 2$ are given, and the value of k can be found as follows

$$\begin{aligned} k_1 &= |g'(1)| = |2/3| = 0.66667 \\ k_2 &= |g'(2)| = |2/5| = 0.40 \end{aligned}$$

which give $k = \max\{k_1, k_2\} = 0.66667$, therefore, the error bound for our approximation will be as follows:

$$|\alpha - x_2| \leq \frac{k^2}{1-k}|x_1 - x_0|,$$

and it gives

$$|\alpha - x_2| \leq \frac{(0.66667)^2}{1 - 0.66667}|1.386294 - 1.5|$$

or

$$|\alpha - x_3| \leq (1.33336)(0.113706) = 0.151611.$$

Question 2: Successive approximations x_n to the desired root $\sqrt{3}$ are generated by the scheme

$$x_{n+1} = \frac{1}{2}x_n + \frac{3}{2x_n}, \qquad n \geq 0.$$

Use Newton's method to find the second approximation x_2 of the root, starting with $x_0 = 2$. Show that the order of convergence of Newton's method is at least quadratic.

Solution. Given

$$x = \frac{1}{2}x + \frac{3}{2x} = g(x),$$

and

$$f(x) = x - g(x) = \frac{1}{2}x - \frac{3}{2x}.$$

So

$$f(x_n) = \frac{1}{2}x_n - \frac{3}{2x_n} \quad \text{and} \quad f'(x_n) = \frac{1}{2} + \frac{3}{2x_n^2}.$$

Using these functions values in the Newton's iterative formula, we have

$$x_{n+1} = x_n - \frac{(\frac{x_n}{2} - \frac{3}{2x_n})}{(\frac{1}{2} + \frac{3}{2x_n^2})}.$$

Finding the first approximation of the root using the initial approximation $x_0 = 2$, we get

$$x_1 = x_0 - \frac{(\frac{x_0}{2} - \frac{3}{2x_0})}{(\frac{1}{2} + \frac{3}{2x_0^2})} = 1.7143.$$

Similarly, the other approximations can be obtained as

$$x_2 = x_1 - \frac{(\frac{x_1}{2} - \frac{3}{2x_1})}{(\frac{1}{2} + \frac{3}{2x_1^2})} = 1.7319.$$

Since

$$g(x) = x - \frac{(\frac{x}{2} - \frac{3}{2x})}{(\frac{1}{2} + \frac{3}{2x^2})} = \frac{6x}{x^2 + 3},$$

so

$$g'(x) = \frac{18 - 6x^2}{(x^2 + 3)^2},$$

and

$$g'(\sqrt{3}) = \frac{18 - 6(3)}{(3 + 3)^2} = \frac{0}{36} = 0.$$

Thus at least quadratic.

Question 3: Use Secant method to find the second approximation, using $x_0 = 1$ and $x_1 = 2$, of the value of x that produces the point on the graph of $y = \dfrac{1}{x}$ that is closest to the point $(2, 1)$.

Solution. The distance between an arbitrary point $(x, 1/x)$ on the graph of $y = 1/x$ and the point $(2, 1)$ is

$$d(x) = \sqrt{(x-2)^2 + (1/x - 1)^2} = \sqrt{x^2 - 4x + 4 + 1/x^2 - 2/x + 1}.$$

Because a derivative is needed to find the critical point of d, it is easier to work with the square of this function

$$F(x) = [d(x)]^2 = x^2 - 4x + 4 + 1/x^2 - 2/x + 1,$$

whose minimum will occur at the same value of x as the minimum of $d(x)$. To minimize $F(x)$, we need x so that

$$F'(x) = 2x - 4 - 2/x^3 + 2/x^2 = 0, \quad \text{gives}, \quad f(x) = 2x - 4 - 2/x^3 + 2/x^2.$$

Applying Secant iterative formula to find the approximation of this equation, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(2x_n - 4 - 2/x_n^3 + 2/x_n^2)}{(2x_n - 4 - 2/x_n^3 + 2/x_n^2) - (2x_{n-1} - 4 - 2/x_{n-1}^3 + 2/x_{n-1}^2)}, \qquad n \geq 1.$$

Finding the second approximation using the initial approximations $x_0 = 1$ and $x_1 = 2$, we get

$$x_2 = 2 - 1/9 = 1.8889,$$

and

$$x_3 = 1.8667.$$

The point on the graph that is closest to $(2, 1)$ has the approximate coordinates $(1.8667, 0.5356)$.

Question 4: Show that $\alpha = 1$ is the root for the equation $x^4 - 8x^3 + 18x^2 = 16x - 5$. Use quadratic convergent iterative method to find the first approximation of α starting with $x_0 = 0.5$. Compute absolute error.

Solution. Since

$$\begin{array}{lllllll} f(x) & = & x^4 - 8x^3 + 18x^2 - 16x + 5, & f(1) & = & 0, \\ f'(x) & = & 4x^3 - 24x^2 + 36x - 16, & f'(1) & = & 0, \\ f''(x) & = & 12x^2 - 48x + 36, & f''(1) & = & 0, \\ f'''(x) & = & 24x - 48, & f'''(1) & = & -24 \neq 0, \end{array}$$

$$m = 3.$$

So using Modified Newton's method, we have

$$x_1 = 0.5 - 3\frac{0.5625}{-3.5} = 0.9821.$$

The absolute error is

$$|\alpha - x_1| = |1 - 0.9821| = 0.0179.$$

Question 5: Find the first approximation for the nonlinear system

$$\begin{aligned} y^2(1-x) &= x^3 \\ x^2+y^2 &= 1 \end{aligned}$$

using Newton's method, starting with initial approximation $(x_0, y_0)^T = (1, 1)^T$.

Solution. Given

$$\begin{aligned} f_1(x,y) &= y^2(1-x)-x^3, \quad f_{1x} = -y^2-3x^2, \quad f_{1y} = 2y(1-x), \\ f_2(x,y) &= x^2+y^2-1, \qquad f_{2x} = 2x, \qquad f_{2y} = 2y. \end{aligned}$$

At the given initial approximation $x_0 = 1$ and $y_0 = 1$, we have

$$\begin{aligned} f_1(1,-1) &= -1, \quad \frac{\partial f_1}{\partial x} = f_{1x} = -4, \quad \frac{\partial f_1}{\partial y} = f_{1y} = 0, \\ f_2(1,1) &= 1, \quad \frac{\partial f_1}{\partial x} = f_{2x} = 2, \quad \frac{\partial f_2}{\partial y} = f_{2y} = 2. \end{aligned}$$

The Jacobian matrix J at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{-8}\begin{pmatrix} 2 & 0 \\ -2 & -4 \end{pmatrix},$$

is the inverse of the Jacobian matrix.Now to find the first approximation we have to solve the following equation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{-8}\begin{pmatrix} 2 & 0 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.75 \end{pmatrix},$$

the required first approximation. •