King Saud University
Department of Mathematics
M-203[Final Examination]
(Differential and Integral Calculus)
(Summer-Semester 1431/32)

Max. Marks: 50 Time: 3 hrs

Marking Scheme: **Q.No:1**[3+4+5+4], **Q.No:2**[4+4+4+4], **Q.No:3**[4+4+4+6]

Q. No: 1 (a) Determine whether or not the sequence $\left\{(2n+5)^{1/n}\right\}$ converges and if it converges, find its limit.

(b) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)}{n}$ converges absolutely, converges conditionally, or diverges.

(c) Find the interval of convergence and the radius of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{2^n(x-3)^n}{n^2}.$$

(d) Find the Maclaurin series of $f(x)=\sin x$ and use its first three non-zero terms to approximate the integral $\int_0^1 \sin(x^2)dx$ to three decimal places.

Q. No: 2 (a) Evaluate the integral $\int_0^1\int_x^1 \sin(y^2)dydx$.

(b) Find the surface area of the plane $3x+2y+z=6$ that lies inside the first octant.

(c) Find the volume and the centroid of the region bounded by the plane $z=0$ and the paraboloid $z=9-x^2-y^2$.

(d) Evaluate the integral by changing it to spherical coordinates

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}dzdydx.$$

Q. No: 3 (a) Show that the line integral $\int_{(2,-2)}^{(-1,0)} 2xy^3dx+3y^2x^2dy$ is independent of path, and find its value.

(b) Use Green's theorem to evaluate the line integra $\oint_C(-y^2+e^x)dx+(\tan^{-1}y)dy$, where C is the boundary of the region between the parabolas $y=x^2$ and $x=y^2$.

(c) Use the Divergence theorem to calculate the surface integral $\iint_S \vec{F}\cdot\vec{n}ds$ where $\vec{F}(x,y,z)=x^3\vec{i}+y^3\vec{j}+z^2\vec{k}$, and S is the surface of the region bounded by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=2$.

(d) Verify Stokes's theorem for the surface S, where S is the portion of the paraboloid $z=4-x^2-y^2$ above the $xy-plane$ and the vector field $\vec{F}(x,y,z)=zx\vec{i}+2y\vec{j}+z^3\vec{k}$.

M-203

Summer Semester (Final Exam. 1431/1432)

Time: 3 Hours Solutions M.M: 50

Q # 1(a) Determine whether or not the sequence $\{(2n+5)^{\frac{1}{n}}\}$ converges and if it converges find its limit. Marks: [5]

Soln. Let $y = (2n+5)^{\frac{1}{n}}$

$\therefore \lim y = \lim_{n\to\infty} (2n+5)^{\frac{1}{n}}$ (∞^0 form) (1)

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n}\ln(2n+5)$ $(0\cdot\infty)$ (1)

$= \lim_{n\to\infty} \frac{\ln(2n+5)}{n}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{2n+5}(2)}{1} = 0$

$\lim_{n\to\infty} y = e^0 = 1$

$\therefore \lim_{n\to\infty} (2n+5)^{\frac{1}{n}} = 1$: Conv. (1)

(b) Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$ converges absolutely, converges conditionally or diverges. [Marks: 4]

Soln. By AST, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$ converges

because $\lim_{n\to\infty} \frac{\ln(n)}{n}$ $\frac{\infty}{\infty}$ form

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ $\left(1\frac{1}{2}\right)$

But, $\sum \frac{\ln(n)}{n}$ by integral test

$\lim_{t\to\infty} \int_1^{\infty} \frac{\ln(x)}{x}\,dx$ Put $\ln x = u \Rightarrow \frac{1}{x}dx = du$

$= \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^{\infty} = \infty$; Div. $\int u\,du = \frac{u^2}{2}$ $\left(1\frac{1}{2}\right)$

Hence, it is conditionally convergent (1)

Q# 1(c) Find the interval of conv. and radius of conv. of the power series $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$. [Marks: 5]

Soln. $\lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{(n+1)^2} \times \frac{n^2}{2^n (x-3)^n} \right|$

$= 2|x-3|$ (1)

For abs. conv.: $2|x-3| < 1$

$\Leftrightarrow |x-3| < \frac{1}{2}$

$\Leftrightarrow -\frac{1}{2} < x-3 < \frac{1}{2}$

$\Leftrightarrow \frac{5}{2} < x < \frac{7}{2}$ (1)

For $x = \frac{5}{2}$, we have $\sum_{n=1}^{\infty} \frac{2^n \left(\frac{5}{2}-3\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{n^2}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ which is conv. by AST. (1)

For $x = 7/2$, we have $\sum_{n=1}^{\infty} \frac{2^n \left(\frac{7}{2}-3\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is conv. by p-series for $p=2$ (1)

Interval of conv.: $[5/2, 7/2]$ and
Radius of conv.: $r = \frac{1}{2}$ (1)

(d) Find the Maclaurin series for $f(x) = \sin x$ and use the first three non-zero terms to app. $\int \sin(x^2)\,dx$ to three decimal places. [Marks: 4]

Soln. $f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}(0) = 0$

$\vdots$

$\therefore \quad f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ (2)

$\therefore \quad \int_0^1 \sin(x^2)\,dx = \int_0^1 \left[x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right]dx$

$= \left[\frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \cdots\right]_0^1$ (1)

$= \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)}$

$\simeq 0.31028 \simeq 0.3103$

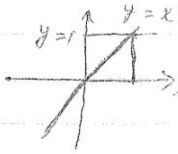
$\simeq \underline{0.310}$ (1)

Q # 2(a) Evaluate the Integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

[Marks - 4]

Soln. We reverse the order of the integral as

(2) $\int_0^1 \int_0^y \sin(y^2)\,dx\,dy.$



$= \int_0^1 \left[\sin y^2 . x\right]_0^y dy$

$= \int_0^1 y \sin(y^2)\,dy$

$= \frac{1}{2}\int_0^1 \sin t\,dt$

(1) Put $y^2 = t \Rightarrow$ $2y\,dy = dt$ $\therefore y\,dy = \frac{1}{2}dt$ If $y = 0$, $t = 0$ and If $y = 1$, $t = 1$

$= \frac{1}{2}\left[-\cos t\right]_0^1$

$= \underline{\frac{1}{2}\left[1 - \cos 1\right]}$ (1)

Q # 2(b) Find the Surface area of the plane $3x + 2y + z = 6$ that lies in the first octant [Marks: 4]

Soln. we have $z = 6 - 3x - 2y = g(x, y)$

$\therefore g_x = -3$ and $g_y = -2$ (1)

$\therefore$ Surface Area $A = \iint_R \sqrt{1 + (-3)^2 + (-2)^2}\, dA$

$= \iint_R \sqrt{14}\, dA$

(2) $= \int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{14}\, dy\, dx$

$= \sqrt{14} \int_0^2 [y]_0^{3-\frac{3}{2}x}\, dx$

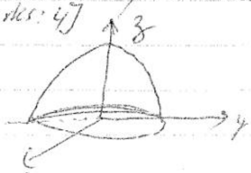
$= \sqrt{14} \int_0^2 \left(3 - \frac{3}{2}x\right) dx$

$= \sqrt{14} \left[3x - \frac{3}{2}\frac{x^2}{2}\right]_0^2$

$= \sqrt{14} \left[3(2) - \frac{3}{4}(2)^2\right]$

$= \sqrt{14}\, [6 - 3] = \underline{3\sqrt{14}}$ (1)

Q # 2(c) Find the Volume ~~and the centroid~~ of the region bounded by the plane $z = 0$ and the paraboloid $z = 9 - x^2 - y^2$ [Marks: 4]



Soln. Volume $V = \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r\, dz\, dr\, d\theta$ (3)

$= \int_0^{2\pi} \int_0^3 r(9 - r^2)\, dr\, d\theta$

$= \int_0^{2\pi} \left[9\frac{r^2}{2} - \frac{r^4}{4}\right]_0^3 = 2\pi \left[\frac{81}{2} - \frac{81}{4}\right]$

$= \frac{81}{2}\pi$ (1)

(5)

Q # 2(d) Evaluate the integral by changing into Spherical Coordinates:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

[Marks: 4]

Soln. $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{(\rho^2)^{3/2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ (3)

Put $\rho^3 = t \Rightarrow 3\rho^2 d\rho = dt$

$= \frac{1}{3}\int e^t\,dt = \frac{1}{3}e^t\Big]_0^1$

$= \frac{1}{3}[e-1]$

and $\int_0^{\pi}\sin\phi\,d\phi = [-\cos\phi]_0^{\pi}$

$= 1+1 = 2$

$\therefore\ = 2\pi\left(\frac{1}{3}\right)(e-1)2$

$= \frac{4\pi}{3}(e-1)$ (1)

Q #3(c) Show that the line integral $\int_{(2,-2)}^{(-1,0)} 2xy^3dx + 3y^2x^2dy$ is independent of path and find its value. [Marks: 4]

Soln. $\vec{F}(x,y) = 2xy^3\,\hat{i} + 3y^2x^2\,\hat{j} = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$

$\therefore f_x(x,y) = 2xy^3$ — (1) and $f_y(x,y) = 3y^2x^2$ — (2)

Integrating, we get $f(x,y) = 2\frac{x^2}{2}y^3 + g(y)$ (1) — (3)

$f(x,y) = 3\frac{y^3}{3}x^2 + f(x)$ — (4)

Comparing (3) and (4), we get $f(x,y) = x^2y^3 + c$ (1)

$\therefore \int_{(2,-2)}^{(-1,0)} 2xy^3dx + 3y^2x^2dy = \left[x^2y^3+c\right]_{(2,-2)}^{(-1,0)}$

$= 32$ (2)

Q# 3(b) Use Green's theorem to evaluate the line integral $\oint_C (-y^2+e^x)\,dx + (\tan^{-1} y)\,dy$, where C is the boundary of the region between the parabolas $y = x^2$ and $x = y^2$. [Marks: 4]

Soln. $M(x,y) = -y^2+e^x$ and $N(x,y) = \tan^{-1} y$

$$\frac{\partial M}{\partial y} = -2y \text{ and } \frac{\partial N}{\partial x} = 0 \quad (1)$$

$\therefore$ By Green's theorem, we have

$$\oint_C (-y^2+e^x)\,dx + (\tan^{-1} y)\,dy = \int_0^1\int_{x^2}^{\sqrt{x}} 2y\,dy\,dx \quad (2)$$

$$= \int_0^1 \left[2\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 (x - x^4)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{5-2}{10}$$

$$= \frac{3}{10} \quad (1)$$

Q# 3(c) Use the Divergence theorem to calculate the surface integral $\iint_S \vec{F}\cdot\vec{n}\,ds$, where $\vec{F}(x,y,z) = x^3\vec{i}+y^3\vec{j}+z^2\vec{k}$, and S is the surface of the region bounded by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=2$. [Marks: 4]

Soln. By Divergence theorem $\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_Q \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right) dv \quad (1)$

$$= \iiint_Q (3x^2+3y^2+2z)\,dv$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (3r^2+2z)\,r\,dz\,dr\,d\theta \quad (2)$$

$$= \int_0^{2\pi}\int_0^3 (6r^3+4r)\,dr\,d\theta = 2\pi\left[6\frac{r^4}{4}+4\frac{r^2}{2}\right]_0^3 = 2\pi\left(\frac{27}{2}\ldots\right.$$

$$= 279\pi \quad (1)$$

(7)

Q#3(d) Verify Stokes' theorem for the surface S, where S is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane and the vector field

$\vec{F}(x, y, z) = zx\,\vec{i} + 2y\,\vec{j} + z^3\,\vec{k}$. [Marks: 6]

Soln. We verify $\oint \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot \vec{n}\,ds$ (1) —(1)

First, we find $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds$

$$= \iint_{R_{xy}} (-M_1 g_x - N_1 g_y + P_1)\,dA$$

we have $z = 4 - x^2 - y^2 = g(x, y)$ $\therefore$ $g_x = -2x$ and $g_y = -2y$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx & 2y & z^3 \end{vmatrix} = 0\vec{i} + x\vec{j} + 0\vec{k} = M_1\vec{i} + N_1\vec{j} + P_1\vec{k} \quad (1)$$

$$= \iint_{R_{xy}} 2xy\,dA = 2\int_0^{2\pi}\int_0^2 (r\cos\theta\, r\sin\theta)\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 \cancel{d\theta} \cos\theta\sin\theta\,d\theta$$

$$= 8\int_0^{2\pi}\cos\theta\sin\theta\,d\theta$$

$$= 4\int_0^{2\pi}\sin 2\theta\,d\theta$$

$$= 4\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} = 0 \quad (1)$$

Now, we find $\oint_C \vec{F}\cdot d\vec{r}$, we have $C: x = 2\cos t,\ y = 2\sin t$ (1) and $z = 0$, $0 \le t \le 2\pi$

$$= \int_C zx\,dx + 2y\,dy + z^3\,dz$$

(1) $= \int_0^{2\pi} 2\cdot 2\sin t\; 2\cos t\,dt = 4\int_0^{2\pi}\sin 2t\,dt$

$4\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} = 0$ (1)

$\therefore$ L.H.S of (1) = R.H.S of (1).