King Saud University
Department Of Mathematics.
M-203 [Final Examination]
(Differential and Integral Calculus)
(II-Semester 1431/1432)

Max. Marks: 50 Time: 3 hrs

Marking Scheme: Q.No:1[3+3+5], Q.No:2[4+4+4], Q.No:3[4+4+5] , Q.No:4[4,5,5]

Q. No: 1 (a) Discuss the convergence of the sequence $\left\{\dfrac{3+\cos^2 n}{2^n}\right\}_1^\infty$.

(b) Determine whether the series $\sum_{n=0}^{\infty}(-1)^n e^{-n}$ is **absolutely convergent**, **conditionally convergent**, or **divergent**.

(c) Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=1}^{\infty}\dfrac{(2x-5)^n}{n^2}$.

Q. No: 2 (a) Find the **Taylor series** for $f(x)=\ln x$ **at** $x=1$ and use it to find the Taylor series for $(x-1)\ln x$ **at** $x=1$.

(b) Evaluate the integral $\int_0^1\int_y^1 \dfrac{1}{1+x^4}\,dx\,dy$ **by reversing the order of integration.**

(c) Find the **surface area** of the surface of the paraboloid $z=5-x^2-y^2$ cut off by the plane $z=1$.

Q. No: 3 (a) Find the **mass** of the triangular **lamina** with vertices (0,0), (1,0), and (0,2) if the density function is $\delta(x,y)=1+3x+y$.

(b) Use **cylindrical coordinates** to evaluate the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} yz\,dz\,dy\,dx$.

(c) Show that the line integral $\int_{(1,1,-1)}^{(1,1,1)} (6x^5+yz)dx+(4y^3+xz)dy+(2z+xy)dz$

is independent of path, and find its value.

Q.No:4 (a) Use **Green's theorem** to evaluate the line integral

$$\oint_C [(\tan^{-1}x)+y^2]dx+[e^y-x^2]dy .$$

where C is the boundary of the region bounded by the graphs of $y=\sqrt{9-x^2}$, $y=\sqrt{1-x^2}$, and the x-axis.

(b) Find the **flux** of the vector field $\vec{F}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$ through the surface S, where S is the solid bounded by the paraboloid $z=9-x^2-y^2$ and $z\geq 0$.

(c) Use the **Stokes's** to evaluate integral $\oint_C \bar{F}\bullet d\bar{r}$, if $\vec{F}(x,y,z)=2z\vec{i}+x\vec{j}+y^2\vec{k}$ where **S** is the surface of the paraboloid $z=4-x^2-y^2$ and C is the trace of S in the xy-plane.

Q. No: 1 (a) Discuss the convergence of the sequence $\left\{\frac{3+\cos^2 n}{2^n}\right\}_1^\infty$.

Solution: (a) Here $a_m = \frac{3+\cos^2 n}{2^2}$, **note**

$$0 \le \cos^2 n \le 1 \Rightarrow 3 \le 3+\cos^2 n \le 4 \Rightarrow \frac{3}{2^2} \le \frac{3+\cos^2 n}{2^2} \le \frac{4}{2^n}$$

$$0 = \lim_{n\to\infty} \frac{3}{2^n} \le \lim_{n\to\infty} \frac{3+\cos^2 n}{2^n} \le \lim_{n\to\infty} \frac{4}{2^2} = 0 \Rightarrow \lim_{n\to\infty} \frac{3+\cos^2 n}{2^n} = 0 \Rightarrow Convergent .$$

(b) Determine whether the series $\sum_{n=0}^{\infty}(-1)^n e^{-n}$ **is** absolutely convergent, conditionally convergent, or divergent.

Solution: (b) Let us first check absolute convergence

$$\left| \sum_{n=0}^{\infty}(-1)^n e^{-n} \right| = \sum_{n=0}^{\infty}\frac{1}{e^n}$$

It is a geometric series with common ratio $r = \frac{1}{e} < 1$.

Hence it is absolutely convergent.

(c) Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=1}^{\infty}\frac{(2x-5)^n}{n^2}$.

Solution: (c) $\sum_{n=1}^{\infty}\frac{(2x-5)^n}{n^2} = \sum_{n=1}^{\infty} u_n$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(2x-5)^{n+1}}{(n+1)^2} \times \frac{n^2}{(2x-5)^n}\right| = \frac{n^2}{n^2+2n+1}|2x-5|$$

$$\lim_{n\to\infty}\frac{n^2}{n^2+2n+1}|2x-5| = |2x-5|$$

So if $|2x-5| < 1 \Rightarrow -1 < 2x-5 < 1 \Rightarrow 2 < x < 3$ the given series is absolutely convergent. It only remains to check at the points x=2 and x=3.

Convergence at x=2

$$\sum_{n=1}^{\infty}\frac{[2(2)-5]^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

It is convergent a alternating Series.

Convergence at x=3

$$\sum_{n=1}^{\infty}\frac{[2(3)-5]^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

It is a convergent p-series

Hence the **interval of convergens** is $2 \le x \le 3$ and

Radius of convergence is $\rho = \frac{3-2}{2} = \frac{1}{2}$.

Q. No: 2 (a) Find the Taylor series for $f(x) = \ln x$ **at** $x = 1$ and use it to find the Taylor series for

$(x-1)\ln x$ **at** $x=1$ **.**

Solution: (a) Required Taylor series is for $f(x)$ is

$$f(1)+(x-1)f^{/}(1)+\frac{(x-1)^2}{2!}f^{//}(1)+.......+\frac{(x-1)^n}{n!}f^{(n)}(1)+.....\;........(1)$$

$$f(x)=\ln x \Rightarrow f(1)=0$$

$$f^{/}(x)=\frac{1}{x}\Rightarrow f^{/}(1)=1$$

$$f^{//}(x)=-\frac{1}{x^2}\Rightarrow f^{//}(1)=-(1!)$$

$$f^{///}(x)=+\frac{2}{x^3}\Rightarrow f^{//}(1)=+2!$$

$$f^{iv}(x)=-\frac{2.3}{x^4}\Rightarrow f^{iv}(1)=-(3!)\;........f^{(n)}(x)=-\frac{2.3......(n-1)}{x^n}\Rightarrow f^{(n)}(1)=-([n-1]!)$$

So series (1) becomes

$$\ln x=0+(x-1)+\frac{(x-1)^2}{2!}[-(1!)]+\frac{(x-1)^3}{3!}(2!)+.....+\frac{(x-1)^n}{n!}(-1)^{n-1}[(n-1)!]....$$

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+.....+(-1)^{n-1}\frac{(x-1)^n}{n}....$$

So the Taylor series for

$$(x-1)\ln x=(x-1)^2-\frac{(x-1)^3}{2}+\frac{(x-1)^4}{3}+.....+(-1)^{n-1}\frac{(x-1)^{n+1}}{n}....$$

(b) Evaluate the integral $\int_0^1\int_y^1 \frac{1}{1+x^4}dxdy$ **by reversing the order of integration.**

Solution: (b) Here $0\le y\le 1$**,** $y\le x\le 1$**.**

When we reverse the order we get $0\le x\le 1$ **and** $0\le y\le x$ **.**

$$\int_0^1\int_y^1 \frac{1}{1+x^4}dxdy=\int_0^1\int_0^x \frac{1}{1+x^4}dydx=\int_0^1 \frac{x}{1+x^4}dx\ .$$

Now put

$$u=x^2\Rightarrow \frac{1}{2}du=xdx$$

$$\Rightarrow \int_0^1 \frac{x}{1+(x^2)^2}dx=\frac{1}{2}\int_0^1 \frac{1}{1+u^2}du=\frac{1}{2}\left[\tan^{-1}(u)\right]_0^1=\frac{1}{2}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]=\frac{\pi}{8}.$$

(c) Find the surface area of the surface of the paraboloid $z=5-x^2-y^2$ cut off by the plane $z=1$.

Solution: (c) Surface area= $\iint_R \sqrt{1+(f_x)^2+(f_y)^2}dA$

Here $z=f(x,y)=5-x^2-y^2\Rightarrow f_x=-2x, f_y=-2y$

And the region R is the circle $x^2+y^2=4.$

Using polar, $0\le r\le 2, 0\le\theta\le 2\pi$ **.**

Hence the surface area is $= \iint_R \sqrt{1+(f_x)^2+(f_y)^2}\,dA = \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,r\,dr\,d\theta$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^2 \left(1+4r^2\right)^{1/2} 8r\,dr\,d\theta$$

$$= \;=\frac{1}{8}\int_0^{2\pi}\left[\frac{(1+4r^2)^{3/2}}{3/2}\right]_0^2 d\theta = \frac{1}{6}(17^{3/2}-1)\pi.$$

Q. No: 3 (a) Find the mass triangular lamina with vertices (0,0), (1,0), and (0,2) if the density

functions is $\delta(x,y)=1+3x+y$.

Solution (a): Here $0\le x\le 1, 0\le y\le 2-2x$.

$$\text{Mass } =\int_0^1\int_0^{2-2x}(1+3x+y)\,dydx = \int_0^1\left[y+3xy+\frac{y^2}{2}\right]_0^{2-2x} dx = 4\int_0^1 (1-x^2)dx$$

$$= \;=4\left[x-\frac{x^3}{3}\right]_0^1 = \frac{8}{3}.$$

(b) Use cylindrical coordinate to evaluate the integral $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} yz\,dzdydx$.

Solution (b): $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} yz\,dzdydx = \int_0^{\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r^2\sin^2\theta\, z\,dzdrd\theta$

$$= \int_0^{\pi}\int_0^1 r^2\sin\theta\left[\frac{z^2}{2}\right]_0^{\sqrt{1-r^2}} drd\theta = \frac{1}{2}\int_0^{\pi}\int_0^1 r^2\sin\theta(1-r^2)drd\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^1 \sin\theta(r^2-r^4)drd\theta = \frac{1}{2}\int_0^{\pi}\sin\theta\left[\frac{r^3}{3}-\frac{r^5}{5}\right]_0^1 d\theta = \frac{2}{15}\left[-\cos\theta\right]_0^{\pi} = \frac{2}{15}.$$

(c) Show that the line integral $\int_{(1,1,-1)}^{(1,1,1)} (6x^5+yz)dx+(4y^3+xz)dy+(2z+xy)dz$ is

independent of path, and find its value.

Solution: (c) Here $M=f_x=6x^5+yz$, $N=f_y=4y^3+xz$, $P=f_z=2z+xy$

$\frac{\partial M}{\partial y}=z=\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z}=y=\frac{\partial p}{\partial x}$, $\frac{\partial N}{\partial z}=x=\frac{\partial P}{\partial y}$. Hence given integral independent of path.

$f_x=6x^5+yz \Rightarrow f(x,y,z)=x^6+xyz+C_1(y,z)$

$f_y=4y^3+xz \Rightarrow f(x,y,z)=y^4+xyz+C_2(x,z)$

$f_z=2z+xyz \Rightarrow f(x,y,z)=z^2+xyz+C_3(x,y)$

$\Rightarrow f(x,y,z)=x^6+y^4+z^2+xyz+C$

$\Rightarrow [f(x,y,z)]_{(1,1,-1)}^{(1,1,1)}=[x^6+y^4+z^2+xyz]_{(1,1,-1)}^{(1,1,1)}=2.$

Q.No:4 (a) Use Green's theorem to evaluate the line integral

$$\oint_C (\tan^{-1} x + y^2)dx + (e^y - x^2)dy$$

where C is the boundry of the region bounded by the graphs

$y = \sqrt{9-x^2}$, $y = \sqrt{1-x^2}$, and the x-axis.

Solution: (a) Here $M = \tan^{-1} x + y^2$ **and** $N = e^y - x^2$

$$\oint_C (\tan^{-1} x + y^2)dx + (e^y - x^2)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \iint_R (-2x - 2y)dA$$

$$= -2\int_0^{\pi}\int_1^3 r(\cos\theta + \sin\theta) r dr d\theta = -2\int_0^{\pi}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_1^3 d\theta = -\frac{52}{3}\int_0^{\pi}(\cos\theta+\sin\theta)d\theta$$
$$= -\frac{52}{3}\left[\sin\theta - \cos\theta\right]_0^{\pi} = -\frac{52}{3}\left[(0+1)-(0-1)\right] = -\frac{104}{3}.$$

(b) Find the flux of the vector field $\vec{F}(x,y,z) = x\,\vec{i} + y\vec{j} + z\,\vec{k}$ through the surface S, where S is the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and $z \geq 0$.

Solution: (b) Flux $\iint_S \overline{F} \circ \overline{n} dS = \iint_R [-(-2x)x - (-2y)y + z)]dA = \iint_R [2x^2 + 2y^2 + 9 - x^2 - y^2]dA$

$$= \iint_R [9 + x^2 + y^2]dA = \int_0^{2\pi}\int_0^3 (9+r^2) r dr d\theta = \int_0^{2\pi}\frac{1}{2}\int_0^3 (9+r^2)(2r)drd\theta$$
$$= \int_0^{2\pi}\frac{1}{2}\left[\frac{(9+r^2)^2}{2}\right]_0^3 d\theta = \frac{[18^2 - 9^2]}{4}\int_0^{2\pi} d\theta = \frac{(243)}{4}2\pi = \frac{243}{2}\pi.$$

(c) Use the Stokes's to evaluate integral $\oint_C \vec{F} \bullet d\vec{r}$, if $\vec{F}(x,y,z) = 2z\,\vec{i} + x\vec{j} + y^2\,\vec{k}$ where **S** is the surface of the paraboloid $z = 4 - x^2 - y^2$ and C is the trace of S in the xy-plane.

Solution: (c) By Stokes's theorem $\oint_C \overline{F} \circ d\overline{r} = \iint_S Curl(\overline{F}) \circ \overline{n} dS = \iint_R (-\overline{M}g_x - \overline{N}g_y + \overline{P})dA$

Here $z = g(x,y) = 4 - x^2 - y^2 \quad \Rightarrow g_x = -2x, \quad g_y = -2y.$

$$Curl(\overline{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & x & y^2 \end{vmatrix} = (2y)i - 2j + k = \overline{M}i + \overline{N}j + \overline{P}k$$

$$\iint_S Curl(\overline{F}) \circ \overline{n} dS = \iint_R (-\overline{M}g_x - \overline{N}g_y + \overline{P})dA = \iint_R [-(-2x)(2y) - (2y)(2) + 1]dA$$

$$= \iint_R [4xy + 4y + 1]dA = \int_0^{2\pi}\int_0^2 (4r^2\cos\theta\sin\theta + 4r\sin\theta + 1) r dr d\theta$$

$$= \int_0^{2\pi} \left(4\left[\frac{r^4}{4}\right]_0^2 \cos\theta\sin\theta + 4\left[\frac{r^3}{3}\right]_0^2 \sin\theta + \left[\frac{r^2}{2}\right]_0^2\right) d\theta = \int_0^{2\pi} \left(16\cos\theta\sin\theta + \frac{32}{3}\sin\theta + 2\right) d\theta$$

$$= 8\int_0^{2\pi} \sin(2\theta)\,d\theta + \frac{32}{3}\int_0^{2\pi} \sin\theta\,d\theta + 2\int_0^{2\pi} d\theta = 4\pi.$$